

EXERCISES MATH 202C - Fifth Assignment

1. (a) Calculate a Gröbner basis for the ideal $I = \langle x^2y - 1, xy^2 - x \rangle$ for lexicographical ordering.
 (b) Calculate a reduced Gröbner basis for the ideal I in (a).
2. Show that if f_m and f_{m+1} are homogeneous polynomials of degree m and $m + 1$ respectively in $k[x_1, \dots, x_n]$, and if $\gcd(f_m, f_{m+1}) = 1$, then $h = f_m + f_{m+1}$ is irreducible.
3. (a) Let $f, g \in k[x_1, \dots, x_n]$. Show that there exist well-defined (up to scalars) polynomials which are *gcd* and *lcm* of f and g (hint: use unique factorization property in $k[x_1, \dots, x_n]$). Moreover, show that $fg = \gcd(f, g)\text{lcm}(f, g)$.
 (b) Show that if $\langle h \rangle = \langle f \rangle \cap \langle g \rangle$, then $h = \text{lcm}(f, g)$.
 (c) Calculate $\gcd(f, g)$ for

$$f = x^4 + x^3y + x^3z^2 - x^2y^2 + x^2yz^2 - xy^3 - xy^2z^2 - y^3z^2,$$

$$g = x^4 + 2x^3z^2 - x^2y^2 + x^2z^4 - 2xy^2z^2 - y^2z^4.$$

- (d) Let $p = x^2 + xy + xz + yz$ and let $q = x^2 - xy - xz + yz$. Calculate a Gröbner basis for $\langle f, g \rangle \cap \langle p, q \rangle$.
4. Find the points in \mathbf{C}^3 on the variety defined by $x^3 - 4zy$, $x^2y - z^3$, $2xy - 4z - 1$ and $y^2 - z$, using Gröbner bases (*Hint*: After calculating a Gröbner basis (use the computer), it should be easy to calculate one of the coordinates of points of the variety).
5. Determine whether $f = x^3z - 2y^2$ is in the ideal I generated by $xz - y$, $xy + 2z^2$ and $y - z$.

REMARK: Some of these problems involve rather tedious calculations. You should use a computer package such as MAPLE or MATHEMATICA, which is available on our computers. E.g. to get into maple, just type *maple* at your UNIX computer and use the command *gbasis* (see attached notes, or pages 487 ff in the book by Cox, Little and O'Shea (see my web page), of which our library has an electronic version); to get out of maple, type *quit*. Ask me if you have any problems.