Math 102
Winter '08
Homework #10

$\boxed{6.1}$

4 a) $F(x,y) = -1 + 4(e^x - x) - 5x\sin y + 6y^2$ @ $(x,y) = (0,0)$

$$\frac{dF}{dx} = 4e^x - 4 - 5\sin y$$

$$\frac{d^2F}{dx^2} = 4e^x \Big|_{(0,0)} = 4$$

$$\frac{dF}{dy} = -5x\cos y + 12y$$

$$\frac{d^2F}{dy^2} = 5x\sin y + 12 \Big|_{(0,0)} = 12$$

$$\frac{d^2F}{dy\,dx} = -5\cos y \Big|_{(0,0)} = -5$$

$\therefore F_{xx}(0,0) \cdot F_{yy}(0,0) - (F_{yx}(0,0))^2 = 4(12) - 25 > 0$

Since $F_{xx}(0,0)$ also > 0, $F(0,0)$ is a $\boxed{\text{local min}}$

b) $F(x,y) = (x^2 - 2x)\cos y = x^2\cos y - 2x\cos y$ @ $(1,\pi)$

$$\frac{dF}{dx} = 2x\cos y - 2\cos y$$

$$\frac{d^2F}{dx^2} = 2\cos y \Big|_{(1,\pi)} = 2\cos\pi = -2$$

$$\frac{dF}{dy} = -x^2\sin y + 2x\sin y$$

$$\frac{d^2F}{dy^2} = -x^2\cos y + 2x\cos y \Big|_{(1,\pi)} = -\cos\pi + 2\cos\pi = 1 - 2 = -1$$

$$\frac{d^2F}{dy\,dx} = -2x\sin y + 2\sin y \Big|_{(1,\pi)} = -2\sin\pi + 2\sin\pi = 0$$

$-2(-1) - (0)^2 = 2 > 0$, $\frac{d^2F}{dx^2} < 0 \Rightarrow F(1,\pi)$ is

$\boxed{\text{local max}}$

$\boxed{6.2}$

(2) $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$|2| = 2 > 0$

$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 + 1 > 0$

$$\begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = -\begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= -(1-(-2)) + (-2-1) + 2(5)$$

$$= -3 - 3 + 10 > 0$$

$\therefore$ A is P.D. by LB (III).

$B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

$|2| = 2 > 0$

$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 5 > 0$

$|B| = 4 > 0$

$\therefore$ B is P.D.

$C = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$

$|5| = 5 > 0$

$\begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} > 0$

$|C| = 16 > 0$

$\therefore$ C is P.D.

(4) Eigenvalues of A^2 are squares of eigenvalues of A.
Eigenvalues of A^{-1} are reciprocals of e'vals of A.
$\therefore$ A has all pos. e'vals $\Rightarrow A^2, A^{-1}$ have all pos e'vals
i.e. A pos. def $\Rightarrow A^2, A^{-1}$ pos def.

(8) A spd, C n.s

<u>Claim</u>: $B = C^T A C$ is s.p.d.

(i) Show B symmetric:

$$B^T = (C^T A C)^T = C^T A^T (C^T)^T = C^T \underbrace{A^T}_{A} C$$

but A sym, so $A^T = A$

$\therefore B^T = C^T A C = B$

i.e. B is symmetric ✓

(ii) Show B is pos def.

Let $x \in \mathbb{R}^n$. Then $x^T B x = x^T C^T A C x$

$= (Cx)^T A (Cx)$

> 0 since A pos def.

Since x was arbitrary, we have that

$x^T B x > 0 \quad \forall x \in \mathbb{R}^n$

$\therefore$ B is pos def. ✓

(i) & (ii) $\Rightarrow$ B is s.p.d. ☑

(30) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

a) $|A| = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 10.$

b) $\Lambda(A) = \{2, 5\}$

c) $A \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$

$\Rightarrow \left(A\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad A\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right) = \left(2\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad 5\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right)$

eigenvectors of A.

30, cont.

d) A is pos. def b/c $\lambda > 0 \ \forall \lambda \in \Lambda(A)$.

A is symmetric since

$$A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A. \checkmark$$

34 $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

pivot > 1 pivot > 1

∴ all pivots of A are > 1, but $\lambda = .5858$ is an e'value of A.

so the answer is $\boxed{\text{NO}}$.

$\boxed{6.3}$

2 a) $AA^T = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 17 & 34 \\ 34 & 68 \end{pmatrix}$

$\det(AA^T - \lambda I) = 0 \iff \begin{vmatrix} 17-\lambda & 34 \\ 34 & 68-\lambda \end{vmatrix} = 0 \iff (17-\lambda)(68-\lambda) - 1156 = 0$

$\iff \lambda = 0, 85.$

$\lambda = 0 = \sigma_2^2$ $\left(\begin{array}{cc|c} 17 & 34 & 0 \\ 34 & 68 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 17 & 34 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

∴ $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is evec corr to $\lambda = 0$. $\left\|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\| = \sqrt{4+1} = \sqrt{5}$

∴ $u_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

$\lambda = 85 = \sigma_1^2$ $\left(\begin{array}{cc|c} -68 & 34 & 0 \\ 34 & -17 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -68 & 34 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -2x_1 + x_2 = 0$

$\Rightarrow x_2 = 2x_1$

∴ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is evec corr to $\lambda = 85$.

∴ $u_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

(b) Find V: cols. are e'vectors of A^TA

$$A^TA = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix}$$

$\lambda = 0 = \sigma_2^2$: $\left(\begin{array}{cc|c} 5 & 20 & 0 \\ 20 & 80 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 5 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + 4x_2 = 0$

$\therefore \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ is e'vector corr. to $\lambda = 0$ $\quad \left\| \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\| = \sqrt{17}$

$\therefore V_2 = \begin{pmatrix} -4/\sqrt{17} \\ 1/\sqrt{17} \end{pmatrix}$

$\lambda = 85 = \sigma^2$: $\left(\begin{array}{cc|c} -80 & 20 & 0 \\ 20 & -5 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -80 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -4x_1 + x_2 = 0$

$\therefore \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is e'vector corr. to $\lambda = 85$

$\therefore V_1 = \begin{pmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix}$.

So $A = U\Sigma V^T$

$$= \underbrace{\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ -4/\sqrt{17} & 1/\sqrt{17} \end{pmatrix}}_{V^T}.$$

(c) Rank $A = 1$

$\therefore \mathcal{C}(A) = \text{sp}(\text{col } 1 \text{ of } U) = \text{sp}\left\{\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\right\}$

$\mathcal{N}(A^T) = \text{sp}(\text{col } 2 \text{ of } U) = \text{sp}\left\{\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\right\}$.

$\mathcal{C}(A^T) = \mathcal{R}(A) = \text{sp}(\text{col } 1 \text{ of } V) = \text{sp}\left\{\begin{pmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix}\right\}$.

$\mathcal{N}(A) = \text{sp}(\text{col } 2 \text{ of } V) = \text{sp}\left\{\begin{pmatrix} -4/\sqrt{17} \\ 1/\sqrt{17} \end{pmatrix}\right\}$.

(10) A 2×2 symmetric w/ unit e'vectors u_1 & u_2
A has e'values $\lambda_1=3$ & $\lambda_2=-2$
$\therefore$ A diag'ble (distinct e'values)

<u>and</u> If $A=U\Sigma V^T$ is SVD of A, then

$$A \text{ symmetric} \Rightarrow A^T=A$$
$$\Rightarrow (U\Sigma V^T)^T=(U\Sigma V^T)$$
$$\Rightarrow V\Sigma^T U^T = U\Sigma V^T$$
$$\Rightarrow U=V \quad \text{since } \Sigma^T=\Sigma \text{ is invertible}$$
U^T, V^T both invertible.

$$\therefore\ A = \underbrace{\left(u_1 | u_2\right)}_{U}\underbrace{\begin{pmatrix}3 & 0\\ 0 & -2\end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix}u_1^T\\ \hline u_2^T\end{pmatrix}}_{V^T}$$

is SVD of A.

(19) (A) <u>Claim</u>: If A has indpt cols, $A^+=(A^TA)^{-1}A^T$

<u>Pf</u> Let $A=U\Sigma V^T$ be the SVD of A

$m\begin{pmatrix} \overset{n}{\ } \end{pmatrix}$

Suppose A is $m\times n$ $\Rightarrow$ Σ is also $m\times n$ $= \begin{pmatrix}\sigma_1 & & 0\\ 0 & \ddots & \sigma_n\\ & 0 & \end{pmatrix}$

Since A has LI cols, $m\geq n$.

By defn, $A^+=V\Sigma^+U^T$ where $\Sigma^+=\begin{pmatrix}1/\sigma_1 & & 0 & \vdots & \\ & \ddots & & & 0\\ 0 & & 1/\sigma_n & \vdots & \end{pmatrix}$.

<u>WTS</u>: $A^+=(A^TA)^{-1}A^T$

Let's compute the RHS using the SVD, and show that it equals $V\Sigma^+U^T$.

$$(A^TA)^{-1}A^T = [(U\Sigma V^T)^T U\Sigma V^T]^{-1}(U\Sigma V^T)^T$$
$$= [V\Sigma^T \underbrace{\cancel{U^T U}}_{I}\Sigma V^T]^{-1} V\Sigma^T U^T$$
$$= [V(\Sigma^T\Sigma)V^T]^{-1} V\Sigma^T U^T$$

#19 cont. At this point, note that $\Sigma^T\Sigma$ is diagonal and since A has full col. rank, $\sigma_1 \geq \cdots \geq \sigma_n > 0$ and $\therefore$ all of the diag entries of $\Sigma^T\Sigma$ are nonzero

$\Rightarrow \Sigma^T\Sigma$ is invertible.

Also, recall that $V^{-1} = V^T$.

$$\therefore \quad (A^TA)^{-1}A^T = (V^T)^{-1}(\Sigma^T\Sigma)^{-1}\overbrace{\cancel{V^{-1}V}}^{I}\,\Sigma^T U^T$$

$$= V\underbrace{(\Sigma^T\Sigma)^{-1}\Sigma^T}\,U^T$$

So, it remains to show this equals Σ^+.

$$\Sigma^T\Sigma = \underset{n\times m}{\begin{pmatrix} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{pmatrix}} \underset{m\times n}{\begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ & 0 & \end{pmatrix}} = \underset{n\times n}{\begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix}}$$

$$\Rightarrow (\Sigma^T\Sigma)^{-1} = \underset{n\times n}{\begin{pmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n^2 \end{pmatrix}}$$

$$\Rightarrow (\Sigma^T\Sigma)^{-1}\Sigma^T = \underset{n\times n}{\begin{pmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n^2 \end{pmatrix}} \underset{n\times m}{\left(\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right)}$$

$$= \left(\begin{array}{ccc|c} 1/\sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1/\sigma_n & \end{array}\right) = \Sigma^+ \qquad \boxed{\checkmark}$$

$x^+ \equiv A^+b$.

<u>Claim</u>: ① $x^+ \in \text{ROWSP}(A)$
② $A^TA\,x^+ = A^Tb$

$\Big\}$ see back of book for pf.

App. B #6

(A) A has e'vals $\lambda = 0,0,0,0,0$.

Note geo. mult. of $(\lambda=0) = \dim(\text{Nul } A) = 2$.

$\therefore$ we'll have 2 Jordan blocks.

The largest block has order $\underline{\underline{3}}$ since $(A-0I)^3 = 0$.

- $y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \in \text{Nul}(A-0I)^3$ but $y \notin \text{Nul } A$.

  So we form a chain of length 3:

  $$Ay = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad A(Ay) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A(A(Ay)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

  $\therefore \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is chain of length 3 corr. to 3×3 Jordan block.

- $y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \text{Nul}(A-0I)^2$ but $y \notin \text{Nul } A$

  $$Ay = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A(Ay) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

  $\therefore \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is chain of length 2 corr. to 2×2 Jordan block.

So $A = M^{-1} J M$ where

$$J = \left(\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & & \\ \hline & 0 & & 0 & 1 \\ & & & & 0 \end{array}\right), \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

☑

6, cont.

A good reference for this is "Matrix Analysis" by Horn & Johnson, pp. 132-134

Solve $\frac{du}{dt} = Au$.

$A = M^{-1} J M$. Let $u = Mv$. Then $\frac{du}{dt} = M \frac{dv}{dt}$.

$\therefore M \frac{dv}{dt} = Au = AMv$

$$\Rightarrow \boxed{\frac{dv}{dt}} = \boxed{M^{-1} A M v} = \boxed{Jv} \qquad (v = M^{-1} u).$$

Let $v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_5(t) \end{pmatrix}$

Then
$$\begin{pmatrix} v_1'(t) \\ \vdots \\ v_5'(t) \end{pmatrix} = \left(\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & & \\ \hline & & & 0 & 1 \\ & 0 & & & 0 \end{array}\right) \begin{pmatrix} v_1(t) \\ \vdots \\ v_5(t) \end{pmatrix} \Rightarrow \begin{cases} v_1'(t) = v_2(t) \\ v_2'(t) = v_3(t) \\ v_3'(t) = 0 \end{cases}$$

$$\begin{cases} v_4'(t) = v_5(t) \\ v_5'(t) = 0 \end{cases}$$

So $v_5(t) = c_5$ $\qquad$ $v_3(t) = c_3$

$v_4(t) = c_5 t + c_4$ $\qquad$ $v_2(t) = c_3 t + c_2$

$v_1(t) = c_1 + c_2 t + \frac{c_3}{2} t^2$

$$\therefore \underbrace{\begin{pmatrix} u_1(t) \\ \vdots \\ u_5(t) \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} c_1 + c_2 t + \frac{c_3}{2} t^2 \\ c_3 t + c_2 \\ c_3 \\ c_5 t + c_4 \\ c_5 \end{pmatrix}}_{v}$$

$$= \begin{pmatrix} c_1 + c_2 t + \frac{c_3}{2} t^2 \\ (c_3 + c_5) t + c_2 + c_4 \\ c_3 t + c_2 \\ c_3 + c_5 \\ c_3 \end{pmatrix} = u(t).$$