CAP REPRESENTATIONS OF G_2 AND THE SPIN L-FUNCTION OF $PGSp_6$

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Abstract. We give a construction of a family of CAP representations of the exceptional group G_2, whose existence is predicted by Arthur's conjecture. These are constructed by lifting certain cuspidal representations of $PGSp_6$. To show that the lifting is non-zero, we establish a Rankin-Selberg integral for the degree 8 Spin L-function of these cuspidal representations of $PGSp_6$.

1. Introduction

This paper is part of our project devoted to the construction and classification of **CAP** representations of the split exceptional group of type G_2. Let us recall the notion of **CAP** representations, which is due to Piatetski-Shapiro. Let F be a number field with adele ring $\mathbb{A}$ and let G denote a quasi-split reductive group over F. Then we have:

Definition: A cuspidal representation π of $G(\mathbb{A})$ is said to be **CAP** (cuspidal associated to parabolics) if there exists

(i) a parabolic subgroup $P = MN$ of G;

(ii) a cuspidal unitary representation σ of $M(\mathbb{A})$ (the Levi factor of P);

(iii) an unramified character χ of $M(\mathbb{A})$,

such that $\pi = \otimes_v \pi_v$ is *nearly equivalent* to the irreducible constituents of $\mathrm{Ind}_P^G \sigma \otimes \chi$ (normalized induction). In other words, for almost all v, π_v is isomorphic to the unique unramified constituent of the local induced representation. In this case, we say that π is **CAP** with respect to (P, σ, χ).

One of the main problems in the study of **CAP** representations is the determination of those triples (P, σ, χ) with respect to which **CAP** representations exist. Arthur's conjecture on square-integrable automorphic forms provides a precise determination of such triples, as well as a classification scheme for the representations associated to a given triple (P, σ, χ). In particular, it predicts that the **CAP** representations should be precisely the non-tempered cuspidal representations and all of their local components should be non-generic. For a detailed discussion of this, the reader can consult [GG,§1].

With the knowledge of Arthur's conjecture, the study of **CAP** representations can be divided into two steps:

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(A) For each triple (P, σ, χ) furnished by Arthur's conjecture, give a construction of **CAP** representations with respect to (P, σ, χ). After constructing some cuspidal representations, checking whether they are **CAP** is an *almost everywhere* issue: one needs to determine their unramified local components for almost all places v.

(B) Give a precise classification of the representations constructed in (A) in accordance with Arthur's conjecture and show that they exhaust the part of the cuspidal spectrum associated to the given triple (P, σ, χ). This would entail the understanding of *all* the local components of the constructed representations, the arrangement of these representations into packets as well as the determination of the multiplicity with which each representation occurs in the space of cusp forms.

After these preliminaries, we turn our attention to the case of G_2. The group G_2 has 3 parabolic subgroups up to conjugacy: the Borel subgroup B, the Heisenberg parabolic P and the so-called 3-step parabolic Q. Regarding steps (A) and (B) above, the following results are known:

- For the Borel subgroup B, both steps (A) and (B) were carried out completely in [GGJ] and [G1].
- For the Heisenberg parabolic P, (A) was done by Rallis-Schiffmann [RS], while (B) was completed in [GG].

Somewhat interestingly, for the parabolic Q, Arthur's conjecture predicts that there should be two subfamilies of **CAP** representations (this is already reflected in the structure of the residual spectrum; cf. [GGJ, Prop. 7.2]):

- the first subfamily is **CAP** with respect to $(Q, \tau_E, |\det|)$, where τ_E is a particular dihedral cuspidal representation of the Levi subgroup $L \cong GL_2$ associated to a non-Galois cubic field E. For this, (A) and parts of (B) were addressed in [GGJ] and [G1].
- The second subfamily is **CAP** with respect to $(Q, \tau, |\det|^{1/2})$, where now τ is essentially any cuspidal representation of PGL_2. This is the remaining family of **CAP** representations for which nothing is known as yet.

The purpose of the present paper is to carry out step (A) for this remaining family of representations:

Main Theorem

Let τ be a cuspidal representation of $PGL_2(\mathbb{A})$ such that $L(\tau, 1/2) \neq 0$. Then we construct a packet of cuspidal representations of $G_2(\mathbb{A})$, all of which are **CAP** with respect to $(Q, \tau, |\det|^{1/2})$.

It seems that our current knowledge is not sufficient for accomplishing the more in-depth step (B), so we shall defer this issue to the future.

It is typical to construct **CAP** representations of G by lifting from a smaller group. This is the case for our earlier work concerning the other families of **CAP** representations of G_2.

However, since G_2 is itself fairly small, one quickly runs out of smaller groups to consider. In this paper, we shall construct the desired representations of G_2 by lifting certain **CAP** representations on the larger group $PGSp_6$.

The representations of $PGSp_6$ which we start with were constructed by D. Ginzburg in his recent paper [Gi]. In Section 4, we will briefly recall Ginzburg's construction, modifying it a little since he presented the construction for simply-connected groups, while we work with adjoint (or similitude) groups. Ginzburg's construction involves representations of the groups $\tilde{SL}_2$ and $PGSp_{2n}$ for $n = 1, 2, 3, 4$. So we shall begin by recalling some facts about the structures and representations of these groups in Sections 2 and 3.

The lifting from $PGSp_6$ to G_2 is an exceptional theta correspondence; $G_2 \times PGSp_6$ being a dual pair in the split adjoint group of type E_7. Results from local theta correspondence show that the theta lift of Ginzburg's representations, if non-zero, will be **CAP** of the desired type. In Section 5, we shall show that the theta lift of Ginzburg's representations to G_2 is non-zero if and only if these representations possess a non-zero period integral over the subgroup

$$C_E = \mathrm{Res}_{E/F}(SL_2)/\mu_2 \quad \text{for some étale cubic algebra } E.$$

We shall then show that such a period is non-vanishing for some E of the form $F \times K$, with K a quadratic algebra.

To achieve this, we consider a Rankin-Selberg integral which picks up this period as a potential residue. The main work of this paper consists of showing that this Rankin-Selberg integral represents the (degree 8) Spin L-function of Ginzburg's representations. The analysis of this Rankin-Selberg integral occupies Sections 6-12. Note that since Ginzburg's representations are **CAP**, their Spin L-functions are products of well-understood L-functions (on GL_2 and $GL_2 \times GL_2$) and do have a pole at the point of interest. From this, we see that the residue of our Rankin-Selberg integral at the relevant point is non-zero.

At the end of the paper, we explain how the constructions we use fit into the framework of Arthur's conjectures. In particular, we describe the Arthur parameters for all representations and discuss the conjectural structure of local and global Arthur packets.

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2. **Preliminaries**

In this section we discuss the structure of the groups that will be used in our construction.

2.1. **Symplectic groups.** Let $(W, \langle -, - \rangle)$ denote a symplectic vector space of dimension $2n$ over a field F. We have the corresponding symplectic group

$$Sp_{2n} = Sp(W) = \{g \in GL(W) | < gw, gv >=< w, v > \forall v, w \in W\}.$$

This is a split simply-connected absolutely simple group of rank n. We denote its simple roots by $\alpha_1, \ldots, \alpha_n$, where α_n is the unique long simple root.

The similitude group GSp_{2n} is given by

$$GSp(W) = \{(g, \lambda) \in GL(W) \times \mathbb{G}_m | < gw, gv >= \lambda < w, v > \forall v, w \in W\}$$

The projection onto the first factor defines an embedding $GSp_{2n} \hookrightarrow GL_{2n}$, while projection to the second factor gives the similitude map $s : GSp_{2n} \longrightarrow \mathbb{G}_m$. Obviously Sp_{2n} is the kernel of s and the group GSp_{2n} has a one dimensional center $Z_{2n} \cong \mathbb{G}_m$. The group $PGSp_{2n}$ is defined as GSp_{2n}/Z_{2n}. It is an adjoint group with trivial center. We often identify its representations with those representations of GSp_{2n} with trivial central character.

Let $W = W_1 \oplus W_2$ be the sum of two symplectic spaces. Then there exist a natural embedding

$$Sp(W_1) \times Sp(W_2) \hookrightarrow Sp(W).$$

For any subgroups H_1 of $GSp(W_1)$ and H_2 of $GSp(W_2)$, we denote

$$(H_1 \times H_2)^0 = \{(h_1, h_2) \in H_1 \times H_2 : s(h_1) = s(h_2)\}.$$

Then $(H_1 \times H_2)^0$ embeds as a subgroup of $GSp(W)$.

2.2. **Parabolic subgroups.** Any standard parabolic subgroup of a split group of type C_n has its Levi factor of type $A_{n_1-1} \times \ldots \times A_{n_k-1} \times C_l$ with $n_1 + ... + n_k + l = n$. We call such a parabolic subgroup of type $(n_1, \ldots, n_k; l)$. Since there is a natural bijection between parabolic subgroups of Sp_{2n}, GSp_{2n} and $PGSp_{2n}$, we use the same notations for the parabolic subgroups of these groups. Let us be more precise about the Levi factor in each case.

Let $P = MN$ be a parabolic subgroup of type $(n_1, \ldots, n_k; l)$. Then we have:

$$M \simeq \begin{cases} GL_{n_1} \times \ldots \times GL_{n_k} \times Sp_{2l} & G \simeq Sp_{2n} \\ GL_{n_1} \times \ldots \times GL_{n_k} \times GSp_{2l} & G \simeq GSp_{2n} \\ (GL_{n_1} \times \ldots \times GL_{n_k} \times GSp_{2l})/Z_{2n} & G \simeq PGSp_{2n} \end{cases}$$

In this paper we will consider symplectic groups of rank 1, 2, 3, 4. Here is a list of the parabolic subgroups which will be useful for this paper.

G	P	*type*
C_1	$B = TU$	$(1; 0)$
C_2	$P_4 = M_4U_4$	$(2; 0)$
	$Q_4 = L_4V_4$	$(1; 1)$
C_3	$P_6 = M_6U_6$	$(1, 2; 0)$
	$Q_6 = L_6V_6$	$(2; 1)$
C_4	$P_8 = M_8U_8$	$(1, 1, 2; 0)$
	$Q_8 = L_8V_8$	$(2; 2)$
	$R_8 = S_8W_8$	$(2, 2; 0)$

2.3. **Metaplectic groups.** If k is a local field, the group $Sp_{2n}(k)$ has a two-fold cover called the metaplectic group and denoted by $\widetilde{Sp}_{2n}(k)$. Similarly, the adelic group $Sp_{2n}(\mathbb{A})$ has a metaplectic double cover $\widetilde{Sp}_{2n}(\mathbb{A})$. In each case, for a subgroup H of Sp_{2n}, we denote by $\widetilde{H}$ the inverse image of H in $\widetilde{Sp}_{2n}$.

If $P = MN$ is a parabolic subgroup, then the two-fold covering splits naturally over $N(k)$. Thus, we may regard $N(k)$ as a subgroup of $\widetilde{Sp}_{2n}(k)$ so that $\widetilde{P}(k) = \widetilde{M}(k) \cdot N(k)$. If $M = GL_{a_1} \times ... \times GL_{a_i} \times Sp_{2b}$, then

$$\widetilde{M}(k) = \widetilde{GL}_{a_1}(k) \times_{\mu_2} \times .... \times_{\mu_2} \widetilde{GL}_{a_i}(k) \times_{\mu_2} \widetilde{Sp}_{2b}(k).$$

Here $\widetilde{GL}_a(k)$ is the double cover of $GL_a(k)$ defined by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 (\det(g_1), \det(g_2)_k)$$

where $(-,-)_k$ is the quadratic Hilbert symbol of k.

2.4. **Weil representations.** For each additive character ψ of k, there exists a Weil representation ω_ψ^{2n} of the Jacobi group

$$J_{2n}(k) = H_{2n+1}(k) \rtimes \widetilde{Sp}_{2n}(k),$$

where H_{2n+1} is the Heisenberg group of dimension $2n+1$. The restriction of ω_ψ^{2n} to $H_{2n+1}(k)$ is the unique irreducible smooth representation of $H_{2n+1}(k)$ with central character ψ. The representation ω_ψ^{2n} can be realized on the space $\mathcal{S}(k^n)$ of Schwarz-Bruhat functions on k^n.

Similarly, one has the Weil representation ω_ψ^{2n} of the adelic group $J_{2n}(\mathbb{A}) = H_{2n+1}(\mathbb{A}) \rtimes \widetilde{Sp}_{2n}(\mathbb{A})$, which can be realized on $\mathcal{S}(\mathbb{A}^n)$. It has an automorphic realization

$$\theta_\psi^{2n} : \mathcal{S}(\mathbb{A}^n) \longrightarrow \mathcal{A}(J_{2n}),$$

given by

$$\theta_\psi^{2n}(\phi)(g) = \sum_{x \in F^n} (\omega_\psi^{2n}(g)\phi)(x).$$

Here $\mathcal{A}(J_{2n})$ denotes the space of automorphic forms on J_{2n}, i.e. the space of Jacobi forms.

In the following, we will denote the space of automorphic forms on a group G by $\mathcal{A}(G)$ and the space of cusp forms by $\mathcal{A}_0(G)$.

2.5. **The group G_2.** Now we come to the split exceptional group of type G_2. We fix a based root system for G_2 and denote by α and β the short and long simple root respectively. The group G_2 has two maximal parabolic subgroups up to conjugacy:

(1) $P = MU$ is the Heisenberg parabolic subgroup. The Levi part M is isomorphic to GL_2 and generated by the short simple root α. The unipotent radical U is a Heisenberg group of dimension 5 with center Z.
(2) $Q = LV$ is the non-Heisenberg maximal parabolic. The Levi part L is isomorphic to GL_2 and generated by the long simple root β. The unipotent radical V is a three-step unipotent group. Hence we shall sometimes call Q the 3-step parabolic.

2.6. **Langlands quotients.** Given a parabolic subgroup $P = MN$ of G, a tempered representation σ of M and an unramified character λ of M in the positive Weyl chamber, we denote by $J_P(\sigma, \lambda)$ the associated Langlands quotient. It is the unique irreducible quotient of the induced representation $\mathrm{Ind}_P^G \sigma \otimes \lambda$, where the induction here and elsewhere is normalized. If P is a maximal parabolic and $s > 0$, we write $J_P(\sigma, s)$ in place of $J_P(\sigma, \delta_P^s)$.

2.7. **Fourier coefficients.** Let U be a unipotent subgroup of G and Ψ be a character of $U(\mathbb{A})$ trivial on $U(F)$. For any automorphic form f on $G(F)\backslash G(\mathbb{A})$, we define the Fourier coefficient of f with respect to (U, Ψ) by

$$f_{U,\Psi}(h) = \int_{U(F)\backslash U(\mathbb{A})} f(uh) \cdot \overline{\Psi(u)}\, du.$$

Fix once and for all an additive character $\psi = \prod_v \psi_v$ of $F\backslash\mathbb{A}$. For $a \in F^\times$, we let ψ_a denote the character $\psi_a(x) = \psi(ax)$. For each one of the groups $G = G_2$, $\tilde{SL}_2$, $PGSp_4$, $PGSp_6$ and $PGSp_8$, we will define a Fourier coefficient f_{U,Ψ_a} (depending on $a \in F^\times$ and the choice of ψ) of an automorphic form f on the group G.

- $G = G_2$ and U is the unipotent radical of the Heisenberg parabolic subgroup. The character Ψ_a on U is defined by
$$\Psi_a(\Pi_{\beta\in U} x_\beta(r_\beta)) = \psi(-r_\beta + a r_{2\alpha+\beta}).$$
- $G = \tilde{SL}_2$ and $U = N$. The character Ψ_a is defined by
$$\Psi_{2,a}(x_\alpha(r)) = \psi(ar)$$
- $G = PGSp_4$ and $U = U_4$. Define
$$\Psi_{4,a}(\Pi_{\beta\in U_4} x_\beta(r_\beta)) = \psi(r_{\alpha_2} - a r_{2\alpha_1+\alpha_2}).$$
- $G = PGSp_6$ and $U = U_6$. The character $\Psi_{6,a}$ on U_6 is defined by
$$\Psi_{6,a}(\Pi_{\beta\in U_6} x_\beta(r_\beta)) = \psi(r_{\alpha_1} + r_{\alpha_3} - a r_{2\alpha_2+\alpha_3}).$$
- $G = PGSp_8$ and $U = U_8$. The character $\Psi_{8,a}$ on U_8 is defined by
$$\Psi_{8,a}(\Pi_{\beta\in U_8} x_\beta(r_\beta)) = \psi(r_{\alpha_1} + r_{\alpha_4} - a r_{2\alpha_3+\alpha_4}).$$

For an automorphic subrepresentation π of $G(\mathbb{A})$, we set

$$\hat{F}(\pi) = \left\{ a \in F^\times : f_{U,\Psi_a} \neq 0 \quad \text{for some } f \in V_\pi \right\}.$$

It is not difficult to see that $\hat{F}(\pi)$ is the union of square classes in $F^\times$. Similarly for an admissible representation π_v of $G(F_v)$ we set

$$\hat{F}(\pi_v) = \left\{ a \in F_v^\times : (\pi_v)_{U,\Psi_a} \neq 0 \right\}.$$

2.8. **Periods.** We shall now define a class of period integrals on $PGSp_6$. For all étale cubic algebras E over F, set

$$GL_2(E)^0 = \left\{g \in GL_2(E) | \det(g) \in F^\times\right\}.$$

There is a natural conjugacy class of embeddings of $GL_2(E)^0$ into $GSp_6(F)$. We obtain such an embedding as follows. Consider the 2-dimensional E-vector space $E \oplus E$ with an alternating form $(-,-)$. By restriction of scalars, we view E^2 as 6-dimensional vector space over F equipped with the alternating from $Tr_{E/F}(-,-)$. It is easy to see that the group $GL_2(E)^0$ preserves the form $Tr_{E/F}(-,-)$ up to scalars, and thus we have an embedding $GL_2(E)^0 \hookrightarrow GSp_6$.

We shall restrict our attention to the algebras of the form $F \times K$, where K is a quadratic algebra over F. Such K's are parametrized by $a \in F^\times/F^{\times 2}$. We write

$$C_K = GL_2(F\times K)^0/Z_6 = \{(g_1,g_2) \in GL_2(F)\times GL_2(K) : \det(g_1) = \det(g_2)\}/\Delta\mathbb{G}_m.$$

Then we have seen that C_K can be embedded as a subgroup of $PGSp_6$. For the purpose of the computations later on, it is necessary to describe an embedding explicitly in this case.

We consider the $F \times K$-module:

$$W = Fe_1 \oplus Kf_1 \oplus Kf_2 \oplus Fe_1$$

equipped with the $(F \times K)$-valued alternating form

$$\langle e_1, e_2\rangle_{F\times K} = (1,0) \quad \text{and} \quad \langle f_1, f_2\rangle_{F\times K} = (0, \frac{1}{2}) \in F\times K.$$

The group $GL_2(F) \times GL_2(K)$ acts naturally on W and the subgroup $(GL_2(F) \times GL_2(K))^0$ preserves $\langle -,-\rangle_{F\times K}$ up to scalars. Regarding W as an F-vector space, we then have the alternating form $\langle -,-\rangle = Tr\langle -,-\rangle_{F\times K}$ given by

$$\langle ae_1 + \alpha f_1, be_2 + \beta f_2\rangle = ab + \frac{1}{2}Tr_{K/F}(\alpha\beta).$$

Writing $K = F\cdot 1 + F\cdot\sqrt{a}$, with $\sqrt{a}$ a trace zero element, we fix the following F-basis for W:

$$e_1,\ f_1,\ \sqrt{a}f_1,\ \frac{1}{\sqrt{a}}f_2,\ f_2,\ e_2.$$

With respect to this basis, the alternating form $\langle -,-\rangle$ is represented by the standard skew-symmetric matrix,

$$\begin{pmatrix} &&&&&1\\ &&&&1&\\ &&&1&&\\ &&-1&&&\\ &-1&&&&\\ -1&&&&& \end{pmatrix}.$$

We shall represent elements of $GSp(W)$ as matrices with respect to this basis. It is then easy to write down the embedding

$$(GL_2(F)\times GL_2(K))^0 \hookrightarrow GSp_6$$

explicitly. Note in particular that when $K = F \times F$ is split, this embedding is *not* a standard embedding of $SL_2 \times SL_2 \times SL_2 \hookrightarrow Sp_6$.

For a cusp form f of $PGSp_6$, we may now define the period integral

$$P_K(f) = \int_{C_K(F)\backslash C_K(\mathbb{A})} f(h)\, dh$$

and for a cuspidal representation π, set

$$\tilde{F}(\pi) = \left\{a \in F^\times : \; P_{K_a}(f) \neq 0 \quad \text{for some } f \in \pi\right\}.$$

Again, $\tilde{F}(\pi)$ is the union of square classes in $F^\times$. We shall see later that for a cuspidal representation π of $PGSp_6(\mathbb{A})$, one has

$$\tilde{F}(\pi) \subset \hat{F}(\pi).$$

3. **Waldspurger and Saito-Kurokawa Lifts**

In this section, we recall certain results about the square-integrable automorphic forms of various groups. Throughout, we fix a non-trivial character ψ of $\mathbb{A}/F$.

3.1. **Local Waldspurger packets.** Let F_v be a local field. and fix a non-trivial additive character ψ_v of F_v. For any irreducible infinite-dimensional (unitary) representation τ_v of $PGL_2(F_v)$, Waldspurger defined a local packet $\tilde{A}_{\tau_v}$ of irreducible representations of $\widetilde{SL}_2(F_v)$. The packet contains two elements σ_v^+ and σ_v^- if τ_v is a discrete series representation and it contains a single representation σ_v^+ otherwise. The packets themselves are intrinsically defined, but their parametrization by representations of PGL_2 depends on the choice of ψ_v, since it relies on the theta correspondences between $\widetilde{SL}_2$ and various forms of SO_3. Since we have fixed ψ_v, we shall suppress this dependence on ψ_v from our notations.

As an example, if $\tau_v = \mathrm{Ind}_B^{PGL_2} \mu$, then

$$\sigma_v^+ = \mathrm{Ind}_{\widetilde{B}}^{\widetilde{SL}_2} \chi_\psi \cdot \mu,$$

where χ_ψ is the character on $\widetilde{T}$ defined in [W1, Pg. 4].

3.2. **Global Waldspurger packets.** With the local packets defined and given an irreducible cuspidal representation $\tau = \otimes \tau_v$ of $PGL_2(\mathbb{A})$, one can define the global packet

$$\left\{\otimes \sigma_v^{\epsilon_v} : \; \sigma_v^{\epsilon_v} \in \widetilde{A}_{\tau_\nu}\right\}.$$

Waldspurger proved that a representation $\sigma = \otimes \sigma_\nu^{\epsilon_\nu}$ in the global packet occurs in the space of cusp forms on $\widetilde{SL}_2$ if and only if

$$\prod_\nu \epsilon_\nu = \epsilon(\tau, 1/2) \in \{\pm 1\}.$$

We denote by $\tilde{A}_\tau$ the subset of the global packet which consists of those representations satisfying this condition. Thus, if S_τ denotes the set of places ν such that τ_ν is discrete series, then

$$\#\tilde{A}_\tau = 2^{\#S_\tau - 1} \quad \text{if } \#S_\tau \geq 1.$$

On the other hand, if S_τ is empty, then $\tilde{A}_\tau$ contains one element if the epsilon factor $\epsilon(\tau, 1/2) = 1$, and is empty otherwise. As τ varies over all cuspidal representations of PGL_2, the representations in $\tilde{A}_\tau$ exhaust the cuspidal spectrum of $\widetilde{SL}_2$ orthogonal to the elementary Weil representations.

3.3. **Local Saito-Kurokawa lift.** In [PS], Piatetskii-Shapiro investigated the theta lift from $\widetilde{SL}_2$ to $PGSp_4 \simeq SO_5$, using the dual pair $\widetilde{SL}_2 \times SO_5 \subset \widetilde{Sp}_{10}$ and the Weil representation ω_ψ^{10}. This is the so-called Saito-Kurokawa lift. His local results can be summarized as follows (cf. also [Sch] and [G2])

Proposition 3.1. *Let τ_v be an irreducible representation of $PGL_2(F_v)$ and let $\sigma_v \in \tilde{A}_{\tau_\nu}$. Let $\theta_{\psi_v}(\sigma_v)$ be the local theta lift of σ_v. Then we have:*

i) $\theta_{\psi_v}(\sigma_v)$ is a non-zero irreducible representation.

ii) If σ_v is supercuspidal, then $\theta_{\psi_v}(\sigma_v)$ is supercuspidal iff $1 \notin \hat{F}(\sigma_v)$. If $1 \in \hat{F}(\sigma_v)$ then

$$\theta_{\psi_v}(\sigma_v) = J_{P_4}(\tau_v, 1/6).$$

For $\sigma_v^\pm \in \tilde{A}_{\tau_v}$, we denote

$$SK^\pm(\tau_v) = SK(\sigma_v^\pm) = \theta_{\psi_v}(\sigma_v^\pm).$$

Thus, each irreducible unitary infinite-dimensional representation τ_v of $PGL_2(F_v)$ determines a packet of representations $\{SK^\pm(\tau_v)\}$ on $PGSp_4(F_v)$. The parametrization of the Saito-Kurokawa packets by representations of PGL_2 is independent of the choice of ψ_v.

3.4. **Global Saito-Kurokawa lift.** Let τ be an irreducible cuspidal representation of $PGL_2(\mathbb{A})$ and suppose that σ belongs to $\tilde{A}_\tau$, so that σ is a cuspidal representation of $\widetilde{SL}_2(\mathbb{A})$. Consider the global theta lift $\theta_\psi(\sigma)$ of σ to $PGSp_4(\mathbb{A}) \simeq SO_5(\mathbb{A})$. In the following theorem we summarize the results of [PS] about this lift. We refer the reader to [G2] for a more detailed discussion and proof of this theorem.

Theorem 3.2. *Let σ belong to $\tilde{A}_\tau$. Then*

i) $\theta_\psi(\sigma)$ is non-zero irreducible and is contained in the space of square-integrable automorphic forms of $PGSp_4$.

ii) If $1 \notin \hat{F}(\sigma)$ then $\theta_\psi(\sigma)$ is contained in the space of cusp forms.

iii) If $1 \in \hat{F}(\sigma)$ then $\theta_\psi(\sigma)$ is not orthogonal to the residual spectrum. Such a σ exists iff $L(\tau, 1/2) \neq 0$.

iv) $\theta_\psi(\sigma)$ is **CAP** *with respect to $(P_4, \tau, \delta_{P_4}^{1/6})$.*

v) $\hat{F}(\theta_\psi(\sigma)) = \hat{F}(\sigma)$.

As in the local case, we set

$$SK(\sigma) = \theta_\psi(\sigma)$$

for $\sigma \in \tilde{A}_\tau$.

4. Ginzburg's Representations

In [Gi], Ginzburg gave a construction of a large class of **CAP** representations on classical groups, starting from cuspidal representations of smaller classical groups. His method of construction generalizes the theta correspondence, but instead of using the Weil or minimal representations, he makes use of other small representations, some of which may even be cuspidal!

Here, we will be concerned with just two examples of his far more general construction. In these examples, representations of $GSp(V)$ are lifted to representations of $GSp(W)$ using the embedding

$$(GSp(W) \times GSp(V))^0 \hookrightarrow GSp(W \oplus V)$$

and restricting certain small representations of $GSp(W \oplus V)$ to this subgroup. Specifically, we shall consider the dual pairs

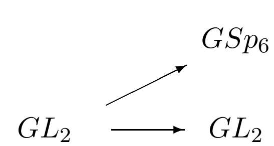
$$\begin{cases} (GL_2 \times GL_2)^0 \hookrightarrow GSp_4 \\ (GSp_6 \times GL_2)^0 \hookrightarrow GSp_8. \end{cases}$$

In particular, the second embedding above is given by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_6 \mapsto \begin{pmatrix} A & & B \\ & I_2 & \\ C & & D \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 \mapsto \begin{pmatrix} I_3 & & & \\ & a & b & \\ & c & d & \\ & & & I_3 \end{pmatrix}.$$

Observe that we have a tower of liftings:



4.1. Local lifting from PGL_2 to PGL_2.

Let τ_1 be an irreducible representation of $PGL_2(F_v)$. Take the Saito-Kurokawa representation $SK^{\pm}(\tau_1)$ of $PGSp_4(F_v)$ and consider its restriction to $(GL_2 \times GL_2)^0/\Delta\mathbb{G}_m$. In the p-adic case, the study of this local restriction was initiated by D. Prasad in [P] and completed in our recent paper [GG2], whereas in the archimedean case, it was studied by Savin [S2]. The results are:

Proposition 4.1. *Let τ_1, τ_2 and τ_3 be irreducible infinite dimensional unitary representations of $PGL_2(F_v)$.*

i) If $\tau_2 \neq \tau_3$, then
$$\mathrm{Hom}_{(GL_2\times GL_2)^0}(SK^{\pm}(\tau_1), \tau_2 \otimes \tau_3) = 0.$$
ii) If F_v is non-archimedean, then
$$\mathrm{Hom}_{(GL_2\times GL_2)^0}(SK^{\epsilon}(\tau_1), \tau_2 \otimes \tau_2) \neq 0$$
if and only if
$$\epsilon(\tau_1 \otimes \tau_2 \otimes \tau_2, 1/2) = \epsilon,$$
in which case
$$\dim \mathrm{Hom}_{(GL_2\times GL_2)^0}(SK^{\epsilon}(\tau_1), \tau_2 \otimes \tau_2) = 1.$$
iii) If F_v is archimedean, then we have:
$$\mathrm{Hom}_{(GL_2\times GL_2)^0}(SK^{-}(\tau_1), \tau_2 \otimes \tau_2) \neq 0$$
if and only if
$$\epsilon(\tau_1 \otimes \tau_2 \otimes \tau_2, 1/2) = -1.$$
On the other hand,
$$\mathrm{Hom}_{(GL_2\times GL_2)^0}(SK^{+}(\tau_1), \tau_2 \otimes \tau_2) \neq 0$$
implies that
$$\epsilon(\tau_1 \otimes \tau_2 \otimes \tau_2, 1/2) = 1.$$
The converse holds if τ_1 is a principal series representation (e.g. when $F_v = \mathbb{C}$).

4.2. **Global lifting from PGL_2 to PGL_2.** Now we come to the global situation. Let τ_1 and τ_2 be cuspidal representations of $PGL_2(\mathbb{A})$ and let σ be in $\tilde{A}_{\tau_1}$. Then we have the square-integrable automorphic representation $SK(\sigma)$ on $PGSp_4$.

Given $f \in SK(\sigma)$ and $\varphi \in \tau_2$, we define a function on $GL_2(\mathbb{A})$ by:
$$\theta_{SK(\sigma)}(f, \varphi)(g) = \int_{SL_2(F)\backslash SL_2(\mathbb{A})} f(g, h\cdot s(g)) \cdot \varphi(h \cdot s(g))\, dh,$$
where $s(g)$ is any element of GL_2 such that $(g, s(g)) \in (GL_2 \times GL_2)^0$; for example, we may take $s(g) = g$. It is easy to check that the value $\theta_{SK(\sigma)}(f, \varphi)(g)$ is independent of the choice of $s(g)$ and $\theta_{SK(\sigma)}(f, \varphi)$ is a function on $PGL_2(F)\backslash PGL_2(\mathbb{A})$.

Let $\Theta_{r(\sigma)}(\tau_2)$ denote the $PGL_2(\mathbb{A})$-submodule of $\mathcal{A}(PGL_2)$ spanned by the $\theta_{r(\sigma)}(f, \varphi)$'s. Then we have a surjective map
$$\theta_{SK(\sigma)} : SK(\sigma) \otimes \tau_2 \longrightarrow \Theta_{SK(\sigma)}(\tau_2)$$
which is $(GL_2 \times GL_2)^0$-equivariant. Here, $(GL_2 \times GL_2)^0$ acts on $\Theta_{SK(\sigma)}(\tau_2)$ via projection onto the second factor, and on τ_2 via the projection onto the first factor. Since τ_2 is isomorphic to $\tau_2^{\vee}$, the above map induces a $(GL_2 \times GL_2)^0$-equivariant map

$$L_{\sigma,\tau_2} : SK(\sigma) \longrightarrow \tau_2 \boxtimes \Theta_{SK(\sigma)}(\tau_2).$$

Now we have the following result of [GG2]:

Proposition 4.2. *Let $\sigma \in \tilde{A}_{\tau_1}$ and let τ_2 be a cuspidal representation of $PGL_2(\mathbb{A})$. Then we have:*

(i) There is at most one $\sigma \in \tilde{A}_{\tau_1}$ such that $\Theta_{SK(\sigma)}(\tau_2)$ is non-zero, namely when $\sigma = \otimes_v \sigma_v^{\epsilon_v}$ with $\epsilon_v = \epsilon(\tau_1 \otimes \tau_1 \otimes \tau_2, 1/2)$.

(ii) For the particular σ in (i), if $\Theta_{SK(\sigma)}(\tau_2) \neq 0$, then $\Theta_{SK(\sigma)}(\tau_2) = \tau_2$.

(iii) For the particular σ in (i),

$$\Theta_{SK(\sigma)}(\tau_2) \neq 0 \Longleftrightarrow L(\tau_1 \times Sym^2\tau_2, 1/2) \neq 0.$$

4.3. **Lifting from PGL_2 to $PGSp_6$.** Following [Gi], for any two irreducible cuspidal representations τ_1 and τ_2 of $PGL_2(\mathbb{A})$ and $\sigma \in \tilde{A}_{\tau_1}$, we construct a square integrable representation $\Sigma(\sigma, \tau_2)$ of $PGSp_6(\mathbb{A})$. The construction consists of two steps:

(i) For any $\sigma \in \tilde{A}_\tau$, construct an irreducible square integrable representation $\Pi(\sigma)$ of $PGSp_8$ whose space is spanned by the residues of the Eisenstein series $E(g, s, f_\sigma)$ associated to the induced representation $\mathrm{Ind}_{Q_8}^{GSp_8} \delta_{Q_8}^s \cdot (\tau_1 \otimes r(\sigma))$ at the point $s = 3/14$.

(ii) Using the dual pair $Sp_6 \times SL_2$ inside Sp_8 and the representation $\Pi(\sigma)$ as a kernel, one can define a lifting from PGL_2 to $PGSp_6$ as follows. Given a cuspidal representation τ_2 of PGL_2, let

$$\Sigma(\sigma, \tau_2) = \Theta_{\Pi(\sigma)}(\tau_2)$$

be the span of all the functions on GSp_6 of the form

$$\theta_{\Pi(\sigma)}(f, \varphi)(g) = \int_{SL_2(F)\backslash SL_2(\mathbb{A})} f(g, hs(g)) \cdot \varphi(hs(g))dh,$$

where $s(g)$ is any element of GL_2 such that $(g, s(g)) \in (GSp_6 \times GL_2)^0$, and as φ and f run over all elements of τ_2 and $\Pi(\sigma)$ respectively.

Again, it is not difficult to check that $\theta_{\Pi(\sigma)}(f, \varphi)$ is independent of the choice of the section s and is a function on $PGSp_6(F)\backslash PGSp_6(\mathbb{A})$. Moreover, the (surjective) map

$$\theta_{\Pi(\sigma)} : \Pi(\sigma) \otimes \tau_2 \longrightarrow \Sigma(\sigma, \tau_2)$$

is $(GL_2 \times GSp_6)^0$ equivariant.

In the following theorem, we summarize the results of Ginzburg (refined by Prop. 4.2) concerning this lift.

Theorem 4.3. *Let τ_1 and τ_2 be cuspidal representations of $PGL_2(\mathbb{A})$ and $\sigma \in \tilde{A}_{\tau_1}$.*

i) $\Sigma(\sigma, \tau_2)$ is a non-zero (possibly reducible) submodule in the space of square-integrable automorphic forms on $PGSp_6(\mathbb{A})$.

ii) (Tower property) $\Sigma(\sigma,\tau_2)$ *is contained in the space of cusp forms if and only if* $\Theta_{SK(\sigma)}(\tau_2) = 0$*. In particular, there exists at most one* $\sigma \in \tilde{A}_{\tau_1}$ *such that* $\Sigma(\sigma,\tau_2)$ *is a non-cuspidal representation. This distinguished* $\sigma = \otimes_v \sigma_v^{\epsilon_v}$ *is specified by* $\epsilon_v = \epsilon(\tau_{1,v} \times \tau_{2,v} \times \tau_{2,v}, 1/2)$ *for all* v*.*

iii) For the distinguished σ *in (ii),* $\Sigma(\sigma,\tau_2)$ *is non-cuspidal if and only if* $L(\tau_1 \times Sym^2(\tau_2), 1/2) \neq 0$*.*

iv) Any irreducible summand of $\Sigma(\sigma,\tau_2)$ *is nearly equivalent to a constituent of* $Ind_{Q_6}^{PGSp_6}\delta_{Q_6}^{1/10}\cdot(\tau_1 \otimes \tau_2)$*.*

v) We have:

$$\hat{F}(\Sigma(\sigma,\tau_2)) = \hat{F}(\Pi(\sigma)) = \hat{F}(SK(\sigma)) = \hat{F}(\sigma).$$

Proof. To be honest, Ginzburg only dealt with the isometry groups in [Gi]. However, all the statements of the theorem are logical consequences of the corresponding results in the isometry case, except for part (iv). Though part (iv) is not a logical consequence of the corresponding result in the isometry case, the same proof goes through (with only changes in notations) to show the desired result. We omit the details. □

4.4. **Remarks.** The above result is less precise than the analogous global theorem concerning lifting from PGL_2 to PGL_2. For example, while we know the unramified local components of any irreducible constituent of $\Sigma(\sigma,\tau_2)$, we do not know if $\Sigma(\sigma,\tau_2)$ is itself irreducible. Nor do we know what are the possible ramified local components. The reason is because we do not as yet understand the local lifting as completely and precisely as in the earlier case. This local analog of the lifting is defined by using the Langlands quotients

$$\begin{cases} \Pi^+(\tau) = J_{R_8}(\tau \otimes \tau, |\det|^{3/2} \otimes |\det|^{1/2}), \\ \Pi^-(\tau) = J_{P_8}(\tau \otimes SK^-(\tau), 3/14), \end{cases}$$

and restricting each of these to the subgroup $(GL_2 \times GSp_6)^0$ of GSp_8. However, we have not examined this local restriction problem, because we do not understand the representations $\Pi^{\pm}(\tau)$ well enough. In any case, in (13.7), we describe some of our speculations regarding this local problem.

5. An Exceptional Theta Correspondence

The dual pair $G_2 \times PGSp_6 \hookrightarrow E_7$ and the minimal representation $\Pi = \otimes_v \Pi_v$ of $E_7(\mathbb{A})$ enable us to define a theta lift

$$\Theta_{E_7}^{C_3} : \Pi \otimes \mathcal{A}_0(GSp_6) \longrightarrow \mathcal{A}(G_2).$$

Similarly, one also has a local lifting of representations. This exceptional theta lifting has been studied in various papers, such as [GJ] and [GS].

5.1. **Local lifting.** In [GS], a precise conjecture was formulated for the local lifting from G_2 to $PGSp_6$. On the level of L-packets, the local theta lifting should realize the Langlands functorial lifting associated to the inclusion

$$\iota : G_2(\mathbb{C}) \longrightarrow \mathrm{Spin}_7(\mathbb{C})$$

of dual groups. The conjecture in [GS] describes precisely how the elements in the associated L-packets should match up. For our purpose here, we only need the following consequence, which follows from results of Magaard-Savin [MS]:

Proposition 5.1. *Let $\pi(s)$ (resp. $\pi(s')$) be the unramfied representation of $G_2(F_v)$ (resp. $PGSp_6(F_v)$) with Satake parameter $s \in G_2(\mathbb{C})$ (resp. $s' \in \mathrm{Spin}_7(\mathbb{C})$). If*

$$\mathrm{Hom}_{G_2 \times PGSp_6}(\Pi_v, \pi(s) \otimes \pi(s')) \neq 0,$$

then (up to conjugacy)

$$s' = \iota(s).$$

5.2. **Global lifting.** Now we come to the global lifting. The result we need is:

Proposition 5.2. *Let Σ be a cuspidal representation of $PGSp_6(\mathbb{A})$. Then $\Theta_{E_7}^{C_3}(\Sigma)$ is contained in the space of cusp forms on G_2. Moreover,*

$$\hat{F}(\Theta_{E_7}^{C_3}(\Sigma)) = \tilde{F}(\Sigma),$$

so that $\Theta_{E_7}^{C_3}(\Sigma)$ is non-zero if $\tilde{F}(\Sigma)$ is non-empty.

Proof. The proposition is proved by fairly standard computations. To show that $\Theta_{E_7}^{C_3}(\Sigma)$ is cuspidal, one computes its constant terms along the two maximal parabolic subgroups of G_2. By a computation analogous to that in [G3, §8], one sees that these constant terms vanish if and only if Σ has non-zero period integral along the subgroup $(GL_2 \times GSp_4)^0/\Delta\mathbb{G}_m \subset PGSp_6$. It is known by [AGR] that this period integral is zero for any cuspidal representation Σ. This proves the cuspidality of $\Theta_{E_7}^{C_3}(\Sigma)$.

Since $\Theta_{E_7}^{C_3}(\Sigma)$ is cuspidal, it is non-zero if and only if it has a non-zero ψ_E-th Fourier coefficient along U for some étale cubic algebra E (by [G1, Thm. 3.1]). By a computation similar to the proof of [G3, Prop. 9.3], one sees that the ψ_E-th Fourier coefficient of $\Theta_{E_7}^{C_3}(\Sigma)$ is non-zero if and only if Σ has non-zero period integral over the subgroup

$$C_E = \{g \in GL_2(E) : \det(g) \in F^\times\}/\mathbb{G}_m \subset PGSp_6.$$

This shows that

$$\hat{F}(\Theta_{E_7}^{C_3}(\Sigma)) = \tilde{F}(\Sigma).$$

The proposition is proved. □

Let us apply the above results to the case when $\Sigma = \otimes_v \Sigma_v$ is an irreducible constituent of the representation $\Sigma(\sigma, \tau_2)$. We know the local unramified component Σ_v for almost all v: Σ_v is the unique unramified constituent of the induced representation $\mathrm{Ind}_{Q_6}^{PGSp_6} \delta_{Q_6}^{1/10} \cdot (\tau_1 \otimes \tau_2)$. Let s' be the Satake parameter of such an unramified Σ_v. Then one sees that s' lies in the image of ι iff $\tau_1 \cong \tau_2$. Thus, $\Theta_{E_7}^{C_3}(\Sigma) = 0$ unless $\tau_1 \cong \tau_2$.

On the other hand, Prop. 5.1 implies:

Proposition 5.3. *Assume that $\tau_1 \cong \tau_2 \cong \tau$. If $\Theta_{E_7}^{C_3}(\Sigma(\sigma, \tau)) \neq 0$, then its irreducible constituents are* **CAP** *with respect to $(Q, \tau, |\det|^{1/2})$.*

6. A Rankin-Selberg Integral

In the following sections, our goal is to show that when $\tau_1 = \tau_2 = \tau$ and $L(\tau, 1/2) \neq 0$, the representations $\Sigma(\sigma, \tau)$ of $PGSp_6$ has non-zero theta lift to G_2. By Prop. 5.3, these non-zero theta lifts give the desired **CAP** representations of G_2. In view of Prop. 5.2, it suffices to show that $\Sigma(\sigma, \tau)$ has non-zero period integral along a subgroup

$$C_K = (GL_2 \times Res_{K/F} GL_2)^0 / \Delta\mathbb{G}_m \subset PGSp_6.$$

Let $a \in F^\times$ correspond to the quadratic algebra K. For simplicity, we set

$$H = GL_2 \quad \text{and} \quad H_K = Res_{K/F} GL_2.$$

We fix the embedding

$$\iota_K : C_K \hookrightarrow PGSp_6$$

as in (2.8). In particular, the simple root of H is mapped to the highest root of $PGSp_6$.

6.1. **An Eisenstein series on C_K.** The group C_K contains a maximal parabolic subgroup

$$P_K = (B \times H_K)^0 / \Delta\mathbb{G}_m.$$

The Borel subgroup of H_K is denoted by $B_K = T_K N_K$. Consider the induced representation

$$I_{C_K}(s) = \mathrm{Ind}_{P_K}^{C_K} \delta_{P_K}^s \quad \text{(normalized induction)}.$$

For any flat smooth section $f_s \in I_{C_K}(s)$, let $E(g, f, s)$ be the associated normalized Eisenstein series:

$$E(g, f, s) = \zeta(2s+1) \cdot \sum_{\gamma \in P_K(F) \backslash C_K(F)} f_s(\gamma g)$$

which converges absolutely when $Re(s) > 1/2$ and has meromorphic continuation to the whole of $\mathbb{C}$. When f is K-finite, this is the classical result in the theory of Eisenstein series. For a general smooth section, the meromorphic continuation is a recent result of Lapid [L]. In any case, $E(g, f, s)$ has a pole of order at most one at $s = 1/2$. This pole is attained by the spherical section and its residue there is a constant function on $C_K(\mathbb{A})$.

6.2. **A Rankin-Selberg integral.** For $\varphi \in \Sigma(\sigma, \tau_2)$ and $f_s \in I_{C_K}(s)$, we consider the following integral:

$$J_K(f, \varphi, s) = \int_{C_K(F) \backslash C_K(\mathbb{A})} \varphi(g) \cdot E(g, f, s)\, dg. \tag{6.1}$$

Since φ is cuspidal, this integral defines a meromorphic function on the whole complex plane. Our main result is:

Theorem 6.2. *Let $\sigma \in \tilde{A}_{\tau_1}$ and suppose that $a \in \hat{F}(\sigma)$. Then for $\varphi \in \Sigma(\sigma, \tau_2)$, one has*

$$\begin{aligned} J_K(f, \varphi, s) &= d_S(f_S, \varphi_S, s) \cdot L^S(s, \Sigma(\sigma, \tau_2), \mathrm{Spin}) \\ &= d_S(f_S, \varphi_S, s) \cdot L^S(\tau_1 \otimes \tau_2, s + 1/2) \cdot L^S(\tau_2, s) \cdot L^S(\tau_2, s+1). \end{aligned}$$

Here, S is a finite set of places of F and $L^S(s, \Sigma(\sigma, \tau_2), \mathrm{Spin})$ is the partial Spin L-function of $\Sigma(\sigma, \tau_2)$. Moreover, d_S is a meromorphic function of $\mathbb{C}$. Given $s_0 \in \mathbb{C}$, there exist φ_S and

f_S such that $d_S(f_S,\varphi_S,s_0)$ is finite and non-zero. If $a\notin\hat{F}(\sigma)$, then $J(\varphi,f,s)$ is identically zero.

Taking residue of both sides at $s=1/2$, we obtain

Corollary 6.3. *$a\in\tilde{F}(\Sigma(\sigma,\tau_2))$ if and only if*

- $a\in\hat{F}(\sigma)$
- $\tau_1\simeq\tau_2$
- $L(\tau_2,1/2)\neq 0$.

The main application of Theorem 6.2 is:

Theorem 6.4. *Let τ be a cuspidal representation of PGL_2 such that $L(\tau,1/2)\neq 0$. For $\sigma\in\tilde{A}(\tau)$, let*

$$\pi(\sigma)=\Theta_{E_7}^{C_3}(\Sigma(\sigma,\tau)),$$

so that $\pi(\sigma)$ is a cuspidal representation of $G_2(\mathbb{A})$. Then $\pi(\sigma)$ is non-zero and any irreducible summand of $\pi(\sigma)$ is **CAP** *with respect to $(Q,\tau,\delta_Q^{1/10})$. Moreover, $\hat{F}(\pi(\sigma))=\hat{F}(\sigma)$.*

Proof. The above corollary implies that

$$\tilde{F}(\Sigma(\sigma,\tau))=\hat{F}(\sigma)\neq\emptyset.$$

Thus, Prop. 5.2 implies that $\pi(\sigma)\neq 0$ and Prop. 5.3 implies that $\pi(\sigma)$ is **CAP** with respect to $(Q,\tau,\delta_Q^{1/10})$.

Moreover, by Prop. 5.2 and the above corollary, we have:

$$\hat{F}(\pi(\sigma))=\tilde{F}(\Sigma(\sigma,\tau))=\hat{F}(\sigma).$$

□

7. Unfolding of Zeta Integral

The proof of Theorem 6.2 will occupy Sections 7 to 12. In this section, we shall unfold the Rankin-Selberg integral. Let us set

$$J_K^*(f,\varphi,s)=\zeta(2s+1)^{-1}\cdot J_K(f,\varphi,s).$$

Proposition 7.1. *If φ is a cusp form on $PGSp_6(\mathbb{A})$, then for $Re(s)\gg 0$,*

$$J_K^*(f,\varphi,s)=\int_{N(\mathbb{A})\times N_K(\mathbb{A})\backslash C_K(\mathbb{A})}\varphi_{\Psi_{6,a}}(g)\cdot f_s(g)\,dg.$$

Proof. Assume that $Re(s)$ is sufficiently large. Unfolding the Eisenstein series and collapsing the sum, we get

$$J_K^*(f,\varphi,s)=\int_{(T(F)\times H_K(F))^0N(\mathbb{A})\backslash C_K(\mathbb{A})}\varphi_Z(g)\cdot f_s(g)\,dg,$$

where $Z = N$ is the root subgroup associated to the highest root of $PGSp_6$ and φ_Z denotes the constant term of φ along Z.

Consider the Fourier expansion of φ_Z along the unipotent radical U_1 of the Heisenberg parabolic subgroup of $PGSp_6$. We have:

$$\varphi_Z(g) = \sum_{\Psi \neq 1} \varphi_{U_1,\Psi}(g)$$

where the sum is over non-trivial characters of $Z(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})$. Let us fix one such character Ψ_0 given by

$$\Psi_0(u) = \psi(r_{\alpha_1}).$$

Now if K is a field, the group $((T \times H_K)^0/\Delta\mathbb{G}_m)(F)$ acts transitively on the non-trivial characters of $U_1(F)Z(\mathbb{A})\backslash U_1(\mathbb{A})$. If $K = F \times F$, there are 3 orbits, but exactly one of them is open, namely the orbit of Ψ_0. The two degenerate orbits will not contribute to the integral because of the cuspidality of φ. Thus, we may focus on the orbit of Ψ_0. The stabilizer of Ψ_0 in $(T \times H_K)^0/\Delta\mathbb{G}_m$ is the subgroup $T^{\Delta}(F) \cdot N_K(F)$ where

$$T^{\Delta}(F) = \{\begin{pmatrix} t & \\ & t^{-1}\lambda \end{pmatrix} \times \begin{pmatrix} t & \\ & t^{-1}\lambda \end{pmatrix} : t,\, \lambda \in F^{\times}\}/\Delta\mathbb{G}_m.$$

Hence, we obtain

$$J_K^* = \int\limits_{T^{\Delta}(F)(N(\mathbb{A})\times N_K(\mathbb{A}))\backslash C_K(\mathbb{A})} \varphi_{U_1\cdot N_K,\Psi_0}(g) \cdot f_s(g)\, dg.$$

Now, consider the Fourier expansion of $\varphi_{U_1\cdot N_K,\Psi_0}$ along $V_4/N_K \cong \mathbb{G}_a$. The group $T^{\Delta}(F)$ acts simply transitively on the non-trivial characters of of $N_K(\mathbb{A})V_4(F)\backslash V_4(\mathbb{A})$. Thus, on collapsing the sum over $T^{\Delta}(F)$, we obtain

$$J_K^* = \int\limits_{N(\mathbb{A})\times N_K(\mathbb{A})\backslash C_K(\mathbb{A})} \varphi_{\Psi_{6,a}}(g) \cdot f_s(g)\, dg.$$

□

8. **Factorizability of Zeta Integral**

For a general automorphic form φ of $PGSp_6$, the Fourier coefficient $\varphi_{\Psi_{6,a}}$ need not be factorizable. However, we shall only be interested in those automorphic forms φ in $\Sigma(\sigma, \tau_2)$. Recall that we have a surjective $(GSp_6 \times GL_2)^0$-equivariant map

$$\theta_{\Pi(\sigma)} : \Pi(\sigma) \otimes \tau_2 \longrightarrow \Sigma(\sigma, \tau_2).$$

This section is devoted to the proof of the following proposition:

Proposition 8.1. *The linear functional on $\Pi(\sigma) \otimes \tau_2$ defined by:*

$$\beta \otimes \phi \mapsto \theta_{\Pi(\sigma)}(\beta, \phi)_{\Psi_{6,a}}(1)$$

is factorizable. More precisely, for every place v, there exists a functional

$$L_v : \Pi(\sigma_v) \otimes \tau_2 \longrightarrow \mathbb{C}$$

such that if $\beta = \otimes_v \beta_v$ and $\phi = \otimes_v \phi_v$ are pure tensors, then

$$\theta_{\Pi(\sigma)}(\beta, \phi)_{\Psi_{6,a}}(g) = \prod_v L_v((g_v, s(g_v))(\beta_v \otimes \phi_v)).$$

Proof. Let $\varphi = \theta_{\Pi(\sigma)}(\beta, \phi)$. By the result of Ginzburg,

$$\varphi_{\Psi_{6,a}}(g) = \int_{N'(\mathbb{A})\backslash SL_2(\mathbb{A})} \beta_{\Psi_{8,a}}(w_2 w_3 x_{\alpha_2+\alpha_3}(1)(g, hs(g)) \cdot \phi_{N',\psi}(hs(g))\, dh,$$

where $\phi_{N',\psi}$ is the Whittaker coefficient of ϕ with respect to the character ψ. We know that the function $\phi_{N',\psi}$ is factorizable (by the uniqueness of Whittaker models):

$$\phi_{N',\psi}(h) = \prod_v W_{\tau_2,,v}(h_v \phi_v)$$

with

$$W_{\tau_2,v} \in \mathrm{Hom}_{N'(F_v)}(\tau_2, \mathbb{C}_{\psi_v}).$$

Thus, it remains to investigate the factorizability of the functional

$$D \in Hom_{U_8(\mathbb{A})}(\Pi(\sigma), \mathbb{C}_{\Psi_{8,a}})$$

given by

$$D(\beta) = \beta_{\Psi_{8,a}}(1).$$

Observe that the functional D factors through the constant term map of $\Pi(\sigma)$ along the maximal parabolic Q_8, whose Levi factor is $(GL_2 \times GSp_4)/\Delta\mathbb{G}_m$. This constant term map is a Q_8-equivariant surjection

$$\Pi(\sigma) \longrightarrow \tau_1 |\det|^2 \boxtimes SK(\sigma)|\det|^{-1}.$$

Now, one has locally:

$$\dim \mathrm{Hom}_{Q_8}(\Pi(\sigma_v), \tau_{1,v}|\det|^2 \boxtimes SK(\sigma_v)|\det|^{-1}) = 1,$$

so that $C = \otimes_v C_v$ for a non-zero

$$C_v \in \mathrm{Hom}_{Q_8}(\Pi(\sigma_v), \tau_{1,v}|\det|^2 \boxtimes SK(\sigma_v)|\det|^{-1}).$$

Thus, we have:

$$D = W \circ C$$

where

$$W \in \mathrm{Hom}_{N(\mathbb{A})\times U_4(\mathbb{A})}(\tau_1|\det|^2 \boxtimes SK(\sigma)|\det|^{-1}, \psi \boxtimes \Psi_{4,a}).$$

Since the Hom spaces

$$\mathrm{Hom}_{N'(F_v)}(\tau_{1,v}|\det|^2, \mathbb{C}_{\psi_v}) \quad \text{and} \quad \mathrm{Hom}_{U_4(F_v)}(SK(\sigma_v)|\det|^{-1}, \mathbb{C}_{\Psi_{4,a}})$$

(and thus their adelic analogs) are known to be one dimensional, we may write

$$W = W_{\tau_1} \boxtimes W_{SK(\sigma)}$$

where $W_{\tau_1} = \otimes_v W_{\tau_1,v}$ and $W_{SK(\sigma)} = \otimes_v W_{SK(\sigma),v}$ are factorizable elements of the above adelic Hom spaces.

Hence we conclude that $D = \otimes D_v$ with

$$D_v = (W_{\tau_1,v} \boxtimes W_{SK(\sigma),v}) \circ C_v.$$

In conclusion, we see that if $\varphi = \theta_{\Pi(\sigma)}(\beta,\phi)$, then

$$\varphi_{\Psi_{6,a}}(g) = \prod_v L_v((g_v, s(g_v)) \cdot (\beta_v \otimes \phi_v))$$

where

$$L_v((g_v,s(g_v))\cdot(\beta_v\otimes\phi_v)) = \int_{N'(F_v)\backslash SL_2(F_v)} D_v\left(w_2w_3x_{\alpha_2+\alpha_3}(1)(g_v,hs(g_v))\beta_v\right)\cdot W_{\tau_2,v}(hs(g_v)\cdot\phi_v)\,dh.$$

□

Corollary 8.2. *Let $\varphi = \theta_{\Pi(\sigma)}(\beta,\phi)$ be a function in the space of $\Sigma(\sigma,\tau_2)$. Then*

$$J_K^*(f,\varphi,s) = \prod_v J_{K_v}^*(\beta_v,\phi_v,f_v,s),$$

where

$$J_{K_v}^*(\beta_v,\phi_v,f_v,s) = \tag{8.3}$$

$$\int_{(N(F_v)\times N_K(F_v))\backslash C_K(F_v)} \int_{N'(F_v)\backslash SL_2(F_v)} D_v(w_2w_3x_{\alpha_2+\alpha_3}(1)(g,hs(g))\beta_v)\cdot W_{\tau_2,v}(hs(g)\cdot\phi_v)\cdot f(g)\,dh\,dg.$$

By standard estimates, the integral defining $J_{K_v}(\beta_v,\phi_v,f_v,s)$ converges absolutely when $Re(s) \gg 0$.

In fact, the double integral in (8.3) can be more elegantly expressed as a single integral as follows. Consider the embedding

$$\iota_K : G_K = ((GL_2 \times Res_{K/F}GL_2) \times GL_2)^0/\Delta\mathbb{G}_m \hookrightarrow (GSp_6 \times GL_2)^0/\Delta\mathbb{G}_m \hookrightarrow PGSp_8.$$

Note that via projection onto the first two factors, we have

$$1 \longrightarrow SL_2 \longrightarrow G_K \xrightarrow{p_1} C_K \longrightarrow 1$$

Similarly, via projection onto the last factor, we have

$$1 \longrightarrow SL_2 \times Res_{K/F}SL_2 \longrightarrow G_K \xrightarrow{p_2} PGL_2 \longrightarrow 1.$$

Letting U_K denote the unipotent radical of the standard Borel subgroup of G_K, we see that

$$J_{K_v}^*(\beta_v,\phi_v,f_v,s) = \int_{U_K(F_v)\backslash G_K(F_v)} D_v(w_2w_3x_{\alpha_2+\alpha_3}(1)g\cdot\beta_v)\cdot W_{\tau_2,v}(p_2(g)\phi_v)\cdot f_s(p_1(g))\,dg. \tag{8.4}$$

9. **Some Explicit Formulas**

Now we come to the local unramified computation. We want to show:

Proposition 9.1. *Let v be a place of F at which all relevant data are unramified, i.e. the representations $\tau_{1,v}$ and $\tau_{2,v}$ are unramified, K_v is unramified over F_v and ψ_v has conductor $\mathcal{O}_{F_v}$. Let $\beta_{0,v}$, $\phi_{0,v}$ and $f_{0,v}$ be the spherical vectors in $\Pi(\sigma_v)$, $\tau_{2,v}$ and $I_{C_{K_v}}(s)$ respectively. Then we have:*

$$J^*_{K_v}(\beta_{0,v}, \phi_{0,v}, f_{0,v}, s) = L(\tau_{1,v} \otimes \tau_{2,v}, s + 1/2) \cdot L(\tau_{2,v}, s) \cdot L(\tau_{2,v}, s + 1).$$

Since everything is local in the next few sections, we shall suppress v from the notations. Thus, F will denote a p-adic field with uniformizer ϖ and residue field of size q. We also assume that $p \neq 2$.

To prove the Proposition, we shall apply Iwasawa decomposition to the equation (8.4). This gives rise to an integral over the maximal split (diagonal) torus of $G_K(F)$. An important point to note here is that under the embedding $\iota_K : G_K \hookrightarrow PGSp_8$, the diagonal split torus of G_K need not sit in the diagonal split torus of $PGSp_8$. Indeed, this occurs precisely when $K = F \times F$ is split; thus the local computation in the split case is more complicated than the non-split case.

Let us fix some notations for the maximal split torus of GSp_8. An element of a split torus of GSp_8 has the form

$$\begin{pmatrix} diag(a, b, c, d) & \\ & \lambda \cdot diag(d, c, b, a)^{-1} \end{pmatrix}.$$

We denote it by $(a, b, c, d; \lambda)$ and note that λ is the similitude factor of this element. So, for example,

$$\delta_{P_8}(a, b, c, d; \lambda) = (ab)^2\lambda^{-2}.$$

Moreover, the spherical section $f_{0,s}$ of $\mathrm{Ind}_{B_K}^{C_K} \delta^s_{B_K}$, pulled back to a function on G_K via the projection p_1, is given by:

$$f_{0,s}(p_1(a, b, c, d; \lambda)) = \left|\frac{a^2}{\lambda}\right|^{s+1/2}.$$

Before we can begin the unramified local computations, we need to derive an explicit formula for the linear functional D_v on $PGSp_8$. Suppose that the representation τ_1 of PGL_2 that we started with is the unramified representation $\pi(\chi, \chi^{-1})$ for an unramified character χ of $F^\times$. To simplify notations, we shall write χ for $\chi(\varpi)$ in the following.

By the proof of Proposition 8.1, we see that

$$D = (W_{\tau_1} \otimes W_{SK(\sigma)}) \circ C$$

with

$$C : \Pi(\sigma) \longrightarrow \tau_1|\det|^2 \boxtimes SK(\sigma)|\det|^{-1}.$$

a surjective Q_8-equivariant map. Since $\Pi(\sigma)$ is generated by β_0 as a Q_8-module, we have:

$$C(\beta_0) = \xi_0 \otimes \eta_0 \neq 0,$$

where ξ_0 and η_0 are the spherical vectors in $\tau_1|\det|^2$ and $SK(\sigma)|\det|^{-1}$ respectively. So for $(g_1, g_2) \in GL_2 \times GSp_4$,

$$D((g_1, g_2)\beta_0) = W_{\tau_1}(g_1\xi_0) \cdot W_{SK(\sigma)}(g_2\eta_0).$$

We shall write for simplicity $W_{\tau_1}(g_1)$ in place of $W_{\tau_1}(g_1\xi_0)$. Note that W_{τ_1} is an unramified Whittaker function for $\tau_1|\det|^2$ and is thus completely determined by its restriction to the maximal split torus of GL_2. We record the well-known formula:

$$W_{\tau_1}\begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix} = q^{-5k/2} \cdot \frac{\chi^{-(k+1)} - \chi^{k+1}}{\chi^{-1} - \chi} \quad \text{if } k \geq 0, \tag{9.2}$$

and is zero if $k < 0$.

Now we need to give an explicit formula for $W_{SK(\sigma)}$, where we recall that

$$W_{SK(\sigma)} \in \mathrm{Hom}_{U_4}(SK(\sigma)|\det|^{-1}, \mathbb{C}_{\Psi_{4,a}}).$$

Writing $W_{SK(\sigma)}(g_2)$ for $W_{SK(\sigma)}(g_2\eta_0)$, we note that the function $W_{SK(\sigma)}$ is determined by its restriction to the Levi subgroup

$$M_4 \cong GL_2 \times GL_1$$

of the Siegel parabolic $P_4 \subset GSp_4$. Using the fact that $SK(\sigma)$ is the theta lift of σ on $\tilde{SL}_2$, one can express the functional $W_{SK(\sigma)}$ in terms of an appropriate Whittaker functional of σ. The result is as follows. Let

$$p : GSp_4 \longrightarrow PGSp_4 \cong SO_5$$

be the natural projection. Then we have:

$$W_{SK(\sigma)}(m) = |s(m)|^{-2} \int_{N\backslash \tilde{SL}_2} \omega_\psi(p(m), h) 1_K(1,0)(1,0,a) \cdot \overline{\tilde{W}(h)}\, dh \tag{9.3}$$

where

- $s : GSp_4 \longrightarrow \mathbb{G}_m$ is the similitude factor.
- ω_ψ is the Weil representation for the dual pair $\tilde{SL}_2 \times SO_5$, and we are using a mixed model for the Weil representation adapted to the parabolic P_4.
- $\tilde{W}$ is the unramified ψ_a-Whittaker function on $\tilde{SL}_2$ associated to the representation σ of $\tilde{SL}_2$. It is given explicitly by:

$$\tilde{W}\begin{pmatrix} \varpi^m & \\ & \varpi^{-m} \end{pmatrix} = \gamma_\psi(\varpi^m)^{-1} \cdot q^{-m} \left(\frac{\chi^{m+1} - \chi^{-(m+1)}}{\chi - \chi^{-1}} + \epsilon q^{-1/2} \frac{\chi^m - \chi^{-m}}{\chi - \chi^{-1}} \right) \tag{9.4}$$

if $m \geq 0$, and is zero if $m < 0$. Here,

$$\epsilon = \begin{cases} +, & \text{if } a \text{ is not a square;} \\ -, & \text{if } a \text{ is a square,} \end{cases}$$

and $\gamma_\psi(a)$ is the standard Weil index.

In particular, we may compute the value of $W_{SK(\sigma)}(c, d; \lambda)$, where $(c, d; \lambda)$ is an element in the diagonal split torus of GSp_4. We first note that

$$p(c, d; \lambda) = diag(cd/\lambda, c/d, 1, d/c, \lambda/dc) \in SO_5.$$

So we obtain:

$$W_{SK(\sigma)}(c, d; \lambda) = |\lambda|^{-2} \cdot \int_{N(F)\backslash SL_2(F)} \omega_\psi(h, (cd/\lambda, c/d, 1, d/c, \lambda/dc))\phi(1,0)(1,0,a) \cdot \overline{\tilde{W}(h)}\, dh.$$

Applying Iwasawa decomposition for SL_2, we get:

$$\begin{aligned} (9.5) \qquad & W_{SK(\sigma)}(c, d; \lambda) \\ =& |\lambda|^{-2} \cdot \int_{F^\times} \omega_\psi((t, t^{-1}), (cd/\lambda, c/d, 1, d/c, \lambda/cd)\phi(1,0)(1,0,a) \cdot \overline{\tilde{W}(t, t^{-1})} \cdot |t|^{-2}\, dt \\ =& |\lambda|^{-2} \cdot \int_{F^\times} |t|^{3/2} \cdot |cd/\lambda| \cdot \chi_\psi(t) \cdot \phi(t^{-1}cd/\lambda, 0)(td/c, 0, atc/d) \cdot \overline{\tilde{W}(t, t^{-1})} \cdot |t|^{-2}\, dt \\ =& |cd/\lambda^3| \cdot \int_{|cd/\lambda| \leq |t| \leq min(|c/d|, |d/c|)} |t|^{-1/2} \cdot \chi_\psi(t) \cdot \overline{\tilde{W}(t, t^{-1})}\, dt. \end{aligned}$$

10. **The unramified computation: K is a field**

In this section, we will carry out the unramified local computation in the case when K is a field, which is actually easier than that for the split case (because the maximal split diagonal torus of G_K is contained in the diagonal split torus of $PGSp_8$ under the embedding ι_K). This is a brute-force computation based on the explicit formulas developed in the previous section. We apologize to the reader for not giving a more conceptual approach.

Recall that we are suppressing v from the notations. We shall apply Iwasawa decomposition to G_K in equation (8.4), which leads to an integral over the maximal split torus of G_K. Under the embedding $\iota_K : G_K \hookrightarrow PGSp_8$, an element of the maximal split torus of G_K is in the diagonal torus of $PGSp_8$ and can be represented by:

$$t = (r, 1, 1, s; \lambda).$$

Moreover,

$$\delta_{B_K}(t)^{-1} = \left|\frac{r^2}{\lambda}\right|^{-1} \left|\frac{s^2}{\lambda}\right|^{-1} \left|\frac{1}{\lambda}\right|^{-2}.$$

So we obtain:

$$J_K^*(z) = \int_{(F^\times)^3} D(w_2 w_3 x_{\alpha_2+\alpha_3}(1)(r,1,1,s;\lambda)) \quad W_{\tau_2}(s,s^{-1}\lambda)\cdot\left|\frac{r^2}{\lambda}\right|^{z+1/2}\cdot\left|\frac{r^2}{\lambda}\right|^{-1}\left|\frac{s^2}{\lambda}\right|^{-1}\left|\frac{1}{\lambda}\right|^{-2} dr\,ds\,d\lambda \tag{10.1}$$

Conjugating the torus element to the left and applying Iwasawa decomposition where necessary, we obtain:

$$D(w_2 w_3 x_{\alpha_2+\alpha_3}(1)(r,1,1,s;\lambda)) = \begin{cases} L((r,s,1,1;\lambda)), & \text{if } |s|\le 1;\\ L((r,1,s,1;\lambda)), & \text{if } |s|>1.\end{cases}$$

Recalling from the previous section that

$$D(a,b,c,d;\lambda) = W_{\tau_1}(a,b)\cdot W_{SK(\sigma)}(c,d;\lambda).$$

we see that (10.1) equals

$$\int_{|s|\le 1} W_{\tau_1}(r,s)\cdot W_{SK(\sigma)}(1,1;\lambda)\cdot W_{\tau_2}(s,s^{-1}\lambda)\cdot\left|\frac{r}{s}\right|^{2z-1}\left|\frac{s^2}{\lambda}\right|^{z-3/2}\left|\frac{1}{\lambda}\right|^{-2} dr\,ds\,d\lambda+ \tag{10.2}$$

$$+\int_{|s|>1} W_{\tau_1}(r,1)\cdot W_{SK(\sigma)}(s,1;\lambda)\cdot W_{\tau_2}(s,s^{-1}\lambda)\cdot |r|^{2z-1}\left|\frac{1}{\lambda}\right|^{z-5/2}\left|\frac{s^2}{\lambda}\right|^{-1} dr\,ds\,d\lambda. \tag{10.3}$$

Recall that we have already given the value of $W_{SK(\sigma)}$ on a torus element in (9.5) and those for W_{τ_1} and W_{τ_2} are well-known.

Let us first compute (10.2). Note that

$$\int_{F^\times} W_{\tau_1}(r,s)\cdot\left|\frac{r}{s}\right|^{2z-1} dr = |s|^4\cdot L(\tau_1, 2z+3/2).$$

Thus, after performing the integration over r and replacing $W_{SK(\sigma)}(1,1,\lambda)$ by the explicit formula (9.5) (which introduces an extra integral over a new variable t), we see that (10.2) equals $L(\tau_1, 2z+3/2)\times$

$$\int_{|t|\le 1}\int_{|1/\lambda|\le|t|}\int_{|s|\le 1} W_{\tau_2}(s,s^{-1}\lambda)\cdot\left|s^2\right|^{z+1/2}\cdot|1/\lambda|^{z-1/2}\cdot|t|^{-1/2}\cdot\chi_\psi(t)\cdot\overline{\tilde{W}(t,t^{-1})}\,ds\,d\lambda\,dt$$

Set

$$ord(s) = m, \quad ord(1/\lambda) = k, \quad ord(t) = l.$$

Then (10.2) equals $L(\tau_1, 2z + 3/2)\times$

$$\sum_{l=0}^{\infty} q^{l/2}\chi_\psi(\varpi^l)\overline{\tilde{W}(\varpi^l,\varpi^{-l})}\cdot \sum_{k=l}^{\infty} q^{-k(z-1/2)} \cdot \sum_{m=0}^{\infty} q^{-2m(z+1/2)} W_{\tau_2}(\varpi^{2k+m},1). \tag{10.4}$$

Next we compute (10.3). As before, we have

$$\int_{F^\times} W_{\tau_1}(r,1)\cdot |r|^{2z-1} dr = L(\tau_1, 2z+3/2).$$

Thus, after performing the integral over r and replacing $W_{SK(\sigma)}(s,1;\lambda)$ by the explicit formula (9.5) (which introduces an integral over a new variable t), we see that (10.3) equals $L(\tau_1, 2z+3/2)\times$

$$\int_{|s|>1} W_{\tau_2}(s^2/\lambda, 1)\cdot \left|\frac{1}{\lambda}\right|^{z-1/2}\left|\frac{1}{s}\right| \cdot \int_{|s/\lambda|\le|t|\le|s^{-1}|} |t|^{-1/2}\chi_\psi(t)\cdot \overline{\tilde{W}(t,t^{-1})}dt\, ds\, d\lambda$$

$$= \int_{|1/s|<1}\left(\int_{|t|\le|1/s|}\left(\int_{\left|\frac{s}{\lambda}\right|\le|t|} W_{\tau_2}(s\cdot\frac{s}{\lambda},1)\cdot\left|\frac{s}{\lambda}\right|^{z-1/2} d\frac{s}{\lambda}\right)\cdot |t|^{-1/2}\chi_\psi(t)\cdot\overline{\tilde{W}(t,t^{-1})}\,dt\right)\cdot\left|\frac{1}{s}\right|^{z+1/2} ds.$$

Set

$$ord(s/\lambda) = m, \quad ord(t) = k, \quad ord(1/s) = l.$$

Then (10.3) equals

$$\sum_{l=1}^{\infty} q^{-l(z+1/2)}\cdot \sum_{k=l}^{\infty} q^{k/2}\,\chi_\psi(\varpi^k)\,\overline{\tilde{W}(\varpi^k,\varpi^{-k})}\cdot \sum_{m=k}^{\infty} W_{\tau_2}(\varpi^{m-l},1)\,q^{-m(z-1/2)}. \tag{10.5}$$

It remains to evaluate and combine (10.4) and (10.5). For this, we use the formula for $\tilde{W}$ and W_{τ_2} (described in the previous section). After a direct and messy computation, omitted here but included in the Appendix, we get the desired identity:

$$J_K(z) = L(\tau_2, z)\cdot L(\tau_2, z+1)\cdot L(\tau_1\otimes\tau_2, z+1/2).$$

11. The unramified computation: $K \simeq F\times F$

In this section, we compute $J_K^*(\beta_0,\phi_0,f_0,s)$ for the case $K\simeq F\times F$.

As before, we apply the Iwasawa decomposition for G_K so that (8.4) becomes an integral over the maximal split torus of G_K. The maximal split torus of G_K is equal to

$$(T(F)\times T_K(F)\times T'(F))^0/\Delta\mathbb{G}_m \subset (GL_2(F)\times GL_2(K)\times GL_2(F))^0/\Delta\mathbb{G}_m$$

and an element of it can be uniquely represented as:

$$\begin{pmatrix} y & \\ & y^{-1}\lambda \end{pmatrix} \times \begin{pmatrix} (r,1) & \\ & (r^{-1},1)\lambda \end{pmatrix} \times \begin{pmatrix} s & \\ & s^{-1}\lambda \end{pmatrix}$$

with $y, r, s, \lambda \in F^{\times}$. We denote this element by $t(y, r, 1, s; \lambda)$. Thus, we have:

$$J_K^*(z) = \int D(w_2 w_3 x_{\alpha_2+\alpha_3}(1)\iota_K(t(y,r,1,s;\lambda)) \cdot W_{\tau_2}(s, s^{-1}\lambda) \cdot \delta_{B_K}(t(y,r,1,s;\lambda))^{-1}\, dy\, dr\, ds\, d\lambda$$

where

$$\delta_{B_K}(t(y,r,1,s;\lambda))^{-1} = \left|\frac{y^2}{\lambda}\right|^{-1} \left|\frac{s^2}{\lambda}\right|^{-1} \left|\frac{r^2}{\lambda^2}\right|^{-1}.$$

The complication in the split case arises because the element $\iota_K(t(y,r,1,s;\lambda))$ is not contained in the diagonal torus of $PGSp_8$. More precisely, if we set

$$\alpha(r) = \begin{pmatrix} (r+1)/2 & (r-1)/2 \\ (r-1)/2 & (r+1)/2 \end{pmatrix},$$

then

$$\iota_K(t(y,r,1,s;\lambda)) = \begin{pmatrix} y &&&&& \\ & \alpha(r) &&&& \\ && s &&& \\ &&& s^{-1}\lambda && \\ &&&& \alpha(r)^*\lambda & \\ &&&&& y^{-1}\lambda \end{pmatrix} \in PGSp_8(F).$$

To perform the integral, we need to apply the Iwasawa decomposition for the group GL_2 to the element $\alpha(r)$:

$$\alpha(r) = \begin{cases} \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ (r-1)/2 & (r+1)/2 \end{pmatrix}, & \text{if } |r| \le 1; \\ \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ (1-r^{-1})/2 & (r^{-1}+1)/2 \end{pmatrix}, & \text{if } |r| > 1. \end{cases}$$

Thus one can write $\iota_K(t(y,r,1,s;\lambda)) \in PGSp_8$ in one of the following forms:

$$\iota_K(t(y,r,1,s;\lambda)) = \begin{cases} (y,r,1,s;\lambda) \cdot x_{\alpha_3}(-1/r) \cdot k, & \text{if } |r| \le 1; \\ (y,1,r,s;\lambda) \cdot x_{\alpha_3}(r) \cdot k & \text{if } |r| > 1, \end{cases}$$

with $k \in PGSp_8(\mathcal{O}_F)$. As a consequence, we obtain

Lemma 11.1.

$$D(w_2 w_3 x_{\alpha_2+\alpha_3}(1)(y, \alpha(r), s; \lambda)) = \begin{cases} W_{\tau_1}(y,s) \cdot W_{SK(\sigma)}((r,1;\lambda) x_{\alpha_3}(-r^{-1})), & \textit{if } |s| \le |r| \le 1; \\ W_{\tau_1}(y,s) \cdot W_{SK(\sigma)}((1,r;\lambda) x_{\alpha_3}(r)), & \textit{if } |s| \le 1 < |r|; \\ W_{\tau_1}(y,r) \cdot W_{SK(\sigma)}((s,1;\lambda) x_{\alpha_3}(-s^{-1})), & \textit{if } |r| < |s| \textit{ and } |r| \le 1; \\ W_{\tau_1}(y,1) \cdot W_{SK(\sigma)}((s,r;\lambda) x_{\alpha_3}(r/s)), & \textit{if } 1 < |s| \textit{ and } 1 < |r|. \end{cases}$$

So $J_K^*(z)$ is a sum of 4 integrals over the 4 domains in the Lemma.

We compute $W_{SK(\sigma)}((c,d;\lambda)x_{\alpha_3}(x))$ using the formula (9.3) and the equality

$$\omega_\psi((cd/\lambda, c/d, 1, d/c, \lambda/cd)x_\alpha(x), (t, t^{-1}))\phi(1,0)(1,0,a) = |cd/\lambda| \cdot |t|^{3/2} \cdot \phi(cd/(\lambda t), 0)(td/c - x^2tc/d, -txc/d, tc/d). \tag{11.2}$$

So for the 4 different cases in Lemma 11.1, we obtain

$$W_{SK(\sigma)}((r,1;\lambda)x_{\alpha_3}(-r^{-1})) = |r/\lambda^3| \cdot \int_{|r/\lambda|\le|t|\le 1} |t|^{-1/2} \cdot \chi_\psi(t) \cdot \overline{\tilde{W}(t,t^{-1})}\, dt$$

$$W_{SK(\sigma)}((1,r;\lambda)x_{\alpha_3}(r)) = |r/\lambda^3| \cdot \int_{|r/\lambda|\le|t|\le 1} |t|^{-1/2} \cdot \chi_\psi(t) \cdot \overline{\tilde{W}(t,t^{-1})}\, dt$$

$$W_{SK(\sigma)}((s,1;\lambda)x_{\alpha_3}(-s^{-1})) = |s/\lambda^3| \cdot \int_{|s/\lambda|\le|t|\le min\{1,|s^{-1}|\}} |t|^{-1/2} \cdot \chi_\psi(t) \cdot \overline{\tilde{W}(t,t^{-1})} dt$$

$$W_{SK(\sigma)}((s,r;\lambda)x_{\alpha_3}(r/s)) = |sr/\lambda^3| \cdot \int_{|rs/\lambda|\le|t|\le min\{1,|r/s|\}} |t|^{-1/2} \cdot \chi_\psi(t) \cdot \overline{\tilde{W}(t,t^{-1})} dt.$$

Putting the formulas above together and performing the integral over y, we see that $J_K^*(z)$ equals $L(\tau_1, 2z+3/2)\times$

$$\int_{|s|\le|r|\le 1} \left|\frac{1}{\lambda}\right|^{z-1/2} |s|^{2z+1}|r|^{-1} W_{\tau_2}(s, s^{-1}\lambda) \int_{|r/\lambda|\le|t|\le 1} |t|^{-1/2}\chi_\psi(t)\overline{\tilde{W}(t,t^{-1})}\, dt \tag{11.3}$$

$$+ \int_{|s|\le 1<|r|} \left|\frac{1}{\lambda}\right|^{z-1/2} |s|^{2z+1}|r|^{-1} W_{\tau_2}(s, s^{-1}\lambda) \int_{|r/\lambda|\le|t|\le 1} |t|^{-1/2}\chi_\psi(t)\overline{\tilde{W}(t,t^{-1})}\, dt \tag{11.4}$$

$$+ \int_{|r|\le 1, |r|<|s|} \left|\frac{1}{\lambda}\right|^{z-1/2} |r|^{2z+1}|s|^{-1} W_{\tau_2}(s, s^{-1}\lambda) \int_{|s/\lambda|\le|t|\le min(1,|s|^{-1})} |t|^{-1/2}\chi_\psi(t)\overline{\tilde{W}(t,t^{-1})}\, dt \tag{11.5}$$

$$+ \int_{|r|>1, |s|>1} \left|\frac{1}{\lambda}\right|^{z-1/2} |sr|^{-1} W_{\tau_2}(s, s^{-1}\lambda) \int_{|rs/\lambda|\le|t|\le min(1,|r/s|)} |t|^{-1/2}\chi_\psi(t)\overline{\tilde{W}(t,t^{-1})}\, dt. \tag{11.6}$$

Our next goal is to compute these four terms. Combining the first two terms, we get $L(\tau_1, 2z+3/2)\times$

$$\int\limits_{|s|\le 1, |s|\le |r|} \left|\frac{1}{\lambda}\right|^{z-1/2} |s|^{2z+1}|r|^{-1} W_{\tau_2}(s, s^{-1}\lambda) \int\limits_{|r/\lambda|\le |t|\le 1} |t|^{-1/2}\chi_\psi(t)\overline{\tilde{W}(t,t^{-1})}dt =$$

$$(11.7) \quad \int\limits_{|t|\le 1} |t|^{-1/2}\chi_\psi(t)\overline{\tilde{W}(t,t^{-1})}\,dt \int\limits_{|s|\le 1} |s|^{2z+1} \int\limits_{|r^{-1}|\le |s^{-1}|} |r^{-1}|^{z+1/2} \int\limits_{|r/\lambda|\le |t|} \left|\frac{r}{\lambda}\right|^{z-1/2} W_{\tau_2}(r^{-1}\cdot r/\lambda \cdot s^2, 1)\, d\frac{r}{\lambda}\, dr^{-1}\, ds\, dt.$$

Put

$$ord(r/\lambda) = k, \quad ord(r^{-1}) = n, \quad ord(s) = m, \quad ord(t) = l.$$

Then (11.7) equals

$$(11.8) \quad \sum_{l=0}^{\infty} q^{l/2}\,\chi_\psi(\varpi^l)\overline{\tilde{W}(\varpi^l, \varpi^{-l})} \sum_{m=0}^{\infty} q^{-m(2z+1)} \sum_{n=-m}^{\infty} q^{-n(z+1/2)} \sum_{k=l}^{\infty} q^{-k(z-1/2)} W_{\tau_2}(\varpi^{k+2m+n}, 1).$$

To compute (11.5), we rewrite (11.5) as:

$$\int\limits_{|t|\le 1} |t|^{-1/2}\chi_\psi(t)\cdot\overline{\tilde{W}(t,t^{-1})} \int\limits_{|r|\le 1} |r|^{2z+1} \int\limits_{|r|<|s|\le |t^{-1}|} |s|^{-(z+1/2)} \int\limits_{|s/\lambda|\le |t|} |s/\lambda|^{z-1/2} W_{\tau_2}(s\cdot s/\lambda, 1) d\frac{s}{\lambda}\, ds\, dr\, dt.$$

Put

$$ord(s/\lambda) = k, \quad ord(s) = n, \quad ord(r) = m, \quad ord(t) = l.$$

Then (11.5) equals

$$(11.9) \quad \sum_{l=0}^{\infty} q^{l/2}\chi_\psi(p^l)\cdot\overline{\tilde{W}(\varpi^l, \varpi^{-l})}\cdot \sum_{m=0}^{\infty} q^{-m(2z+1)} \sum_{n=-l}^{m-1} q^{n(z+1/2)} \sum_{k=l}^{\infty} q^{-k(z-1/2)} W_{\tau_2}(\varpi^{k+n}, 1).$$

To compute (11.6), we rewrite (11.6) as

$$\int\limits_{|r|>1, |s|>1} \left|\frac{1}{\lambda}\right|^{z-1/2} |sr|^{-1} W_{\tau_2}(s, s^{-1}\lambda) \int\limits_{|rs/\lambda|\le |t|\le min(1, |r/s|)} |t|^{-1/2}\chi_\psi(t)\overline{\tilde{W}(t,t^{-1})}dt\, dr\, ds\, d\lambda =$$

$$\int_{|t|\leq 1} |t|^{2z}|t|^{-1/2}\chi_\psi(t)\overline{\tilde{W}(t,t^{-1})} \int_{|\frac{1}{r}|<1} \left|\frac{1}{r}\right|^{2z+1} \int_{|\frac{t}{r}|<|\frac{ts}{r}|\leq 1} \left|\frac{ts}{r}\right|^{-2z} \int_{|\frac{s^2}{\lambda}|\leq|\frac{ts}{r}|} |\frac{s^2}{\lambda}|^{(z-1/2)} W_{\tau_2}(\frac{s^2}{\lambda},1) d\frac{s^2}{\lambda}\, d\frac{ts}{r}\, d\frac{1}{r}\, dt.$$

Put

$$ord(t)=l, \quad ord(\frac{1}{r})=m, \quad ord(\frac{ts}{r})=n, \quad ord(\frac{s^2}{\lambda})=k.$$

Then (11.6) equals

(11.10)

$$\sum_{l=0}^{\infty} q^{-2lz}q^{l/2}\,\chi_\psi(\varpi^l)\cdot\overline{\tilde{W}(\varpi^l,\varpi^{-l})}\cdot\sum_{m=1}^{\infty} q^{-m(2z+1)}\sum_{n=0}^{l+m-1} q^{-2nz}\sum_{k=l}^{\infty} q^{-k(z-1/2)}W_{\tau_2}(\varpi^k,1).$$

Using the explicit formulas for W_{τ_2} and $\tilde{W}$, a direct but messy computation shows that on summing (11.8), (11.9) and (11.10) and canceling $L(\tau_1,2z+3/2)$ and $\zeta(2z+1)^{-1}$, we get (miraculously)

$$J_K(z)=L(\tau_2,z)\cdot L(\tau_2,z+1)\cdot L(\tau_1\otimes\tau_2,z+1/2),$$

as desired.

This finishes the proof of Proposition 9.1.

12. **The Ramified Factor**

After the proof of Proposition 9.1, we are almost done with the proof of Theorem 6.2. Indeed, Proposition 9.1 implies that when $Re(s)\gg 0$, we have:

$$J_K(f,\varphi,s)=d_S(f_S,\varphi_S,s)\cdot L^S(\tau_1\otimes\tau_2,s+1/2)\cdot L^S(\tau_2,s)\cdot L^S(\tau_2,s+1),$$

where the ramified factor d_S is given by:

$$d_S=\prod_{v\in S} J_{K_v}(f_v,\varphi_v,s).$$

Since $J_K(f,\varphi,s)$ and the partial L-functions on the right-hand-side have meromorphic continuation to $\mathbb{C}$, we deduce that $d_S(f_S,\varphi_S,s)$ also has meromorphic continuation to $\mathbb{C}$ when f_S and φ_S are smooth vectors, and the above identity holds for all $s\in\mathbb{C}$. To finish the proof of Theorem 6.2, we need to show:

Proposition 12.1. *For any given s_0, one can find f_S and φ_S such that $d_S(f_S,,\varphi_S,s_0)\neq 0$.*

The rest of this section is devoted to the proof of this proposition. It suffices to check the proposition for each J_{K_v} for $v\in S$. Since the setting is local, we shall write F for F_v

in the rest of the section. We also apologize for the fact that the symbol K denotes both a quadratic extension of F and a maximal compact subgroup of $PGSp_6(F)$.

We first assume that $Re(s)$ is sufficiently large, so that the integral expression of J_K converges. Applying Iwasawa decompostion for the parabolic subgroup P_K, we obtain

$$J_K(f,\varphi,s) = \int_{(K\cap P_K(F))\backslash K} f(k)\cdot\left(\int_{(\Delta\mathbb{G}_m N_K\backslash(T\times H_K)^0)(F)} L(gk\cdot\varphi)\cdot\delta_{P_K}(g)^{s-1}\,dg\right)dk$$

Setting

$$G(\varphi,s) = \int_{(\Delta\mathbb{G}_m N_K\backslash(T\times H_K)^0)(F)} L(g\varphi)\cdot\delta_{P_K}(g)^{s-1}\,dg, \tag{12.2}$$

we have:

Lemma 12.3. *For any smooth vector φ, $G(\varphi,s)$ has meromorphic continuation to $\mathbb{C}$.*

Proof. In the p-adic case, this follows immediately from the meromorphcity of $J_K(f,\varphi,s)$ since any smooth φ is also K-finite. In the archimedean case, a well-known result of Dixmier and Malliavin [DM] implies that any smooth φ is a finite linear combination of vectors of the form

$$f*\phi = \int_K f(k)\cdot k\phi\,dk$$

with f a smooth function on K. However, one has:

$$G(f*\phi,s) = J_K(f',\varphi,s)$$

where

$$f'(k) = \int_{K\cap P_K(F)} f(pk)dp.$$

This proves the lemma. □

Thus, to prove the proposition, it suffices to show that $G(\varphi,s)$ does not vanish at s_0 for some choice of smooth vector φ.

Let U_1 be the unipotent radical of the Heisenberg parabolic of $PGSp_6$ with the center Z. As we have observed in the proof of Prop. 7.1, the group $\Delta\mathbb{G}_m\backslash(T\times H_K)^0$ acts transitively on the non-trivial character of U_1/Z and the stabilizer of the character

$$\Psi_0(v) = \psi(\langle x_{-\alpha_1}, v\rangle)$$

is the subgroup $T^\Delta\cdot N_K$. So we obtain an injective map

$$i:\Delta\mathbb{G}_m T^\Delta N_K\backslash(T\times H_K)^0 \hookrightarrow \overline{U_1}/\overline{Z}$$

given by

$$g\mapsto g^{-1}\cdot x_{-\alpha_1},$$

whose image is open and dense. Moreover, because

$$\Delta\mathbb{G}_m\backslash(T \times H_K)^0 \cong T^{\Delta} \ltimes SL_2(K),$$

with $SL_2(K)$ naturally a subgroup of H_K, the restriction of i to $SL_2(K)$ induces a bijection of $N_K\backslash SL_2(K)$ with the image of i. Thus, for $Re(s) \gg 0$, we may write:

$$\begin{aligned} G(\varphi, s) &= \int_{T^{\Delta}} \int_{N_K\backslash SL_2(K)} L(ht \cdot \varphi) \cdot \delta_{P_K}(t)^{s-1}\, dh\, dt \\ &= \int_{N_K\backslash SL_2(K)} \left(\int_{T^{\Delta}} L(th \cdot \varphi) \cdot \delta_{B_K}(t)^{-1} \cdot \delta_{P_K}(t))^{s-1}\, dt \right) dh \end{aligned}$$

where we recall that B_K is the Borel subgroup of $(T \times H_K)^0$. In addition, when the right-$SL_2(K)$-invariant measure on $N_K\backslash SL_2(K)$ is pushed forward by i, one obtains the restriction of the Haar measure of $\overline{U_1}/\overline{Z}$. Thus, we may write:

$$G(\varphi, s) = \int_{\overline{Z}\backslash\overline{U_1}} H(h_x \cdot \varphi, s)\, dx$$

where h_x denotes the unique element in $N_K\backslash SL_2(K)$ such that $i(h_x) = x$ (as long as x is in the image of i) and

$$H(\varphi, s) := \int_{T^{\Delta}} L(t \cdot \varphi) \cdot \delta_{B_K}(t)^{-1} \cdot \delta_{P_K}(t)^{s-1}\, dt.$$

Moreover, for $x \in U_1$,

$$L(gx \cdot \varphi) = \psi(\langle g^{-1} \cdot x_{-\alpha_1}, x\rangle) \cdot L(g \cdot \varphi)$$

and so, for a Schwartz function f on U_1, one has

$$L(h_x \cdot (f * \varphi)) = \widehat{f'}(x) \cdot L(h_x \cdot \varphi)$$

where $f'(x) = \int_Z f(zx)\, dz$ and $\widehat{f'}$ is the Fourier transform of f'. Thus

$$G(f * \varphi, s) = \int_{\overline{Z}\backslash\overline{U_1}} \widehat{f'}(x) \cdot H(h_x \cdot \varphi, s)\, dx.$$

Since $\widehat{f'}$ can be an arbitrary Schwarz function on $\overline{U_1}/\overline{Z}$, we see that, for any Schwarz function f_1 on $\overline{U_1}/\overline{Z}$, the integral

$$\int_{\overline{Z}\backslash\overline{U_1}} f_1(x) \cdot H(h_x \cdot \varphi, s)\, dx$$

has meromorphic continuation to $\mathbb{C}$.

Lemma 12.4. *For any smooth vector φ, $H(\varphi, s)$ has meromorphic continuation to $\mathbb{C}$.*

Proof. To prove the meromorphicity of $H(\varphi, s)$, one may assume that $\varphi = f * \phi$ where f is a smooth function with compact support on $SL_2(K)$; this is clear for the p-adic case and follows from the Dixmier-Malliavin theorem [DM] in the archimedean case. Then we have

$$\begin{aligned} H(f*\phi,s) &= \int_{T^\Delta}\int_{N_K\backslash SL_2(K)} f'(h)\cdot L(ht\cdot\phi)\cdot\delta_{P_K}(t)^{s-1}\,dh\,dt \\ &= \int_{N_K\backslash SL_2(K)} f'(h)\cdot H(h\phi,s)\,dh \end{aligned}$$

where f' is the smooth function with compact support on $N_K\backslash SL_2(K)$ defined by $f'(h) = \int_{N_K} f(nh)\,dn$. Now via the embedding i, we may regard f' as a smooth function with compact support on the open dense subset $i(N_K\backslash SL_2(K))$ of $\overline{Z}\backslash\overline{U_1}$ and extending by zero, we may regard f' as a an element of $C_c^\infty(\overline{Z}\backslash\overline{U_1})$. We thus have

$$H(f*\phi,s) = \int_{\overline{Z}\backslash\overline{U_1}} f'(x)\cdot H(h_x\cdot\phi,s)\,dx$$

for $f'\in C_c^\infty(\overline{Z}\backslash\overline{U_1})$. As we observed right before the lemma, this has meromorphic continuation to $\mathbb{C}$. □

In view of the lemma and the discussion preceding it, it suffices to show that $H(\varphi,s)$ does not vanish at $s=s_0$ for some smooth vector φ.

As in the proof of Prop. 7.1, the group T^Δ acts by the adjoint action on V_4 preserving the subgroup N_K. Indeed, we can fix isomorphism $T^\Delta\cong F^\times$ and $V_4/N_K\cong F$ so that the action is simply given by multiplication. Thus, T^Δ acts simply transitively on the non-trivial characters of V_4/N_K. In particular, for a Schwartz function f on V_4 and $t\in T^\Delta$, one has

$$L(t\cdot(f*\varphi)) = \widehat{f'}(t)\cdot L(t\cdot\varphi)$$

where $f'(v)=\int_{N_K} f(nv)\,dn$ and $\widehat{f'}$ is its Fourier transform.

Now we have:

Lemma 12.5. *For each $s_0\in\mathbb{C}$, there exists φ such that $H(\varphi,s_0)\neq 0$.*

Proof. For $f\in\mathcal{S}(V_4)$ and $Re(s)\gg 0$, one has

$$H(f*\varphi,s) = \int_{T^\Delta}\widehat{f'}(t)\cdot L(t\cdot\varphi)\cdot\delta_{P_K}(t)^{s-1}\,dt.$$

Thus, if we pick φ such that $L(\varphi)\neq 0$ and f so that $\widehat{f'}$ is non-negative and supported in a sufficiently small relatively compact neighbourhood of $t=1$, then one can ensure that the integral above converges for all $s\in\mathbb{C}$ and is non-zero at a given point s_0. This proves the lemma. □

This completes the proof of Proposition 12.1 and thus of Theorem 6.2.

13. **Arthur's Conjecture**

In this section, let us explain how the cuspidal representations constructed in this paper fit into the framework of Arthur's conjecture, which describes the discrete spectrum $L^2_{disc}(G(F)\backslash G(\mathbb{A}))$ of a reductive linear algebraic group G defined over a number field F.

Assume for simplicity that G is split. According to Arthur, the discrete spectrum possesses a decomposition

$$L^2_{disc}(G(F)\backslash G(\mathbb{A})) = \widehat{\bigoplus}_{\psi} L^2_{\psi},$$

where the Hilbert space direct sum runs over equivalence classes of A-parameters ψ, i.e. admissible maps

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

where L_F denotes the conjectural Langlands group of F and $\hat{G}$ is the complex dual group of G. For any ψ, the space L^2_{ψ} is a direct sum of nearly equivalent representations, which we now describe.

13.1. **Local A-packets.** The global A-parameter ψ gives rise to a local A-parameter

$$\psi_v : L_{F_v} \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

for each place v of F. Denote by S_{ψ_v} the group of components of $Z_{\hat{G}}(Im(\psi_v))/Z_{\hat{G}}$. To each irreducible representation η_v of S_{ψ_v}, Arthur conjecturally attached a unitarizable admissible (possibly reducible, possibly zero) representation π_{η_v} of $G(F_v)$. The set

$$A_{\psi_v} = \left\{ \pi_{\eta_v} \, : \, \eta_v \in \widehat{S}_{\psi_v} \right\}$$

is called a local A-packet.

It is required that for almost all v, if η_v is the trivial character, then π_{η_v} is irreducible and unramified with Satake parameter given by

$$s_{\psi_v} = \psi_v \left(Fr_v \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right),$$

where Fr_v is a Frobenius element at v and q_v is the cardinality of the residue field at v.

13.2. **Global A-packets and multiplicity formula.** With the local packets A_{ψ_v} at hand, we may define the global A-packet by:

$$A_{\psi} = \{ \pi = \otimes_v \pi_{\eta_v} : \pi_{\eta_v} \in A_{\psi_v}, \eta_v = 1_v \text{ for almost all } v \}.$$

We stress again that the representation $\pi \in A_{\psi}$ may be reducible.

The global A-packet is a set of nearly equivalent representations of $G(\mathbb{A})$ and L^2_{ψ} is the sum of the elements of A_{ψ} with some multiplicities. This multiplicity is precisely given as follows. Arthur attached to ψ a quadratic character ϵ_{ψ} of $S_{\psi} = Z_{\hat{G}}(Im(\psi))/Z_{\hat{G}}$. Now if $\pi = \otimes_v \pi_{\eta_v} \in A_{\psi}$, set

$$m(\pi) = \frac{1}{\# S_{\psi}} \cdot \left(\sum_{s \in S_{\psi}} \epsilon_{\psi}(s) \cdot \eta(s) \right)$$

with $\eta = \otimes_v \eta_v$. Then one should have:

$$L^2_\psi \cong \bigoplus_{\pi \in A_\psi} m(\pi)\pi.$$

13.3. **Arthur Parameters.** In this paper, given a cuspidal representation τ of $PGL_2(\mathbb{A})$, with associated global Waldspurger packet $\tilde{A}_\tau$ on $\widetilde{SL}_2$, we have constructed various sets of nearly equivalent automorphic representations

$$\begin{cases} \pi(\sigma) \text{ of } G_2(\mathbb{A}); \\ SK(\sigma) \text{ of } PGSp_4(\mathbb{A}); \\ \Sigma(\sigma,\tau) \text{ of } PGSp_6(\mathbb{A}); \\ \Pi(\sigma) \text{ of } PGSp_8(\mathbb{A}), \end{cases}$$

as σ varies over $\tilde{A}_\tau$. How can these representations be viewed in the framework of Arthur's conjecture? In the following, we shall write down the conjectural Arthur parameters of these representations and explain how the constructions of these representations fits into Arthur's scheme. The parameters for the group G will be denoted by $\psi_{G,\tau}$.

In the first place, the cuspidal representation τ of $PGL_2(\mathbb{A})$ conjecturally corresponds to an irreducible map

$$\phi_\tau : L_F \longrightarrow SL_2(\mathbb{C}).$$

Let us consider the following A-parameters:

- For the group G_2;

$$\psi_{G_2,\tau} : L_F \times SL_2(\mathbb{C}) \xrightarrow{\phi_\tau \times id} SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{i_s \times i_l} G_2(\mathbb{C})$$

  Here i_s and i_l are embeddings $SL_2(\mathbb{C}) \longrightarrow G_2(\mathbb{C})$ corresponding to the short and long root subgroup respectively.

- For the group $PGSp_4$:

$$\psi_{PGSp_4,\tau} : L_F \times SL_2(\mathbb{C}) \xrightarrow{\phi_\tau \times id} SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{i_2} Sp_4(\mathbb{C}) \cong Spin_5(\mathbb{C})$$

- For the group $PGSp_6$:

$$\psi_{PGSp_6,\tau} : L_F \times SL_2(\mathbb{C}) \xrightarrow{\phi_\tau^\triangle \times id} SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{i_3} Spin_7(\mathbb{C})$$

  Here the map i_3 is the one which lies over the natural embedding $SO_4(\mathbb{C}) \times SO_3(C) \hookrightarrow SO_7(\mathbb{C})$.

- For the group $PGSp_8$:

$$\psi_{PGSp_8,\tau} : L_F \times SL_2(\mathbb{C}) \xrightarrow{\phi_\tau \times Sym^3} SL_2(\mathbb{C}) \times Sp_4(\mathbb{C}) \xrightarrow{i_4} Spin_9(\mathbb{C})$$

  Note that i_2 and i_4 are embeddings, while $i_s \times i_l$ and i_3 have kernels isomorphic to μ_2.

Observe that the parameter $\psi_{PGSp_6,\tau}$ factors through $\psi_{G_2,\tau}$, via the embedding of dual groups $G_2(\mathbb{C}) \hookrightarrow \mathrm{Spin}_7(\mathbb{C})$. This suggests that $L^2_{\psi_{G_2,\tau}}$ can be obtained as a lift of $L^2_{\psi_{PGSp_6,\tau}}$. This was the main motivation for our construction of $\pi(\sigma)$ as a theta lifting of $\Sigma(\sigma,\tau)$.

13.4. **Predictions of Arthur's conjecture.** Let us write down what Arthur's conjecture predicts for the A-parameters above. In fact, the A-parameters $\psi_{G,\tau}$ share many properties:

- The global component group is $S_{\psi_{G,\tau}} = \mathbb{Z}/2\mathbb{Z}$, whereas the local component groups are given by

$$S_{\psi_{G,\tau_v}} = \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ if } \phi_{\tau_v} \text{ is irreducible;} \\ 1, \text{ if } \phi_{\tau_v} \text{ is reducible.} \end{cases}$$

- Thus, the local packets have the form

$$A_{\psi_{G,\tau_v}} = \begin{cases} \{\pi^+_{G,\tau_v}, \pi^-_{G,\tau_v}\}, \text{ if } \tau_v \text{ is discrete series;} \\ \{\pi^+_{G,\tau_v}\}, \text{ otherwise.} \end{cases}$$

Here, π^+_{G,τ_v} is the representation indexed by the trivial character of S_{τ_v}. For almost all v, where the local parameter ψ_{G,τ_v} is unramified, we have:

$$\pi^+_{G,\tau_v} = \begin{cases} J_Q(\tau_v, 1/10), \text{ if } G = G_2; \\ J_{P_4}(\tau_v, 1/6), \text{ if } G = PGSp_4; \\ J_{Q_6}(\tau_v \boxtimes \tau_v, 1/10), \text{ if } G = PGSp_6; \\ J_{R_8}(\tau_v \boxtimes \tau_v, |\det|^{3/2} \boxtimes |\det|^{1/2}), \text{ if } G = PGSp_8. \end{cases}$$

- The epsilon character $\epsilon_{\psi_{G,\tau}}$ are given by:

$$\epsilon_{\psi_{G_2,\tau}} = \begin{cases} 1, \text{ if } \epsilon(\tau, Sym^3, 1/2) = 1; \\ sgn, \text{ otherwise.} \end{cases}$$

$$\epsilon_{\psi_{PGSp_4,\tau}} = \epsilon_{\psi_{PGSp_8,\tau}} = \begin{cases} 1, \text{ if } \epsilon(\tau, 1/2) = 1; \\ sgn, \text{ otherwise} \end{cases}$$

$$\epsilon_{\psi_{PGSp_6,\tau}} = \begin{cases} 1, \text{ if } \epsilon(\tau, 1/2) \cdot \epsilon(\tau, Sym^3, 1/2) = 1; \\ sgn, \text{ otherwise.} \end{cases}$$

- The multiplicity with which $\pi = \otimes_v \pi^{\epsilon_v}_{\tau,v} \in A_{\psi_{G,\tau}}$ occurs in $L^2_{\psi_{G,\tau}}$ can now be computed. For $\pi \in A_{\psi_{PGSp_4,\tau}}$ or $A_{\psi_{PGSp_8,\tau}}$,

$$m(\pi) = \begin{cases} 1, \text{ if } \epsilon(\tau, 1/2) = \epsilon_\pi := \prod_v \epsilon_v; \\ 0, \text{ if } \epsilon(\tau, 1/2) = -\epsilon_\pi. \end{cases}$$

For $\pi \in A_{\psi_{G_2,\tau}}$,

$$m(\pi) = \begin{cases} 1, \text{ if } \epsilon(\tau, Sym^3, 1/2) = \epsilon_\pi; \\ 0, \text{ if } \epsilon(\tau, Sym^3, 1/2) = -\epsilon_\pi. \end{cases}$$

For $\pi \in A_{\psi_{PGSp_6,\tau}}$

$$m(\pi) = \begin{cases} 1, & \text{if , } \epsilon(\tau, 1/2) \cdot \epsilon(\tau, Sym^3, 1/2) = \epsilon_\pi; \\ 0, & \text{if } \epsilon(\tau, 1/2) \cdot \epsilon(\tau, Sym^3, 1/2) = -\epsilon_\pi. \end{cases}$$

- Thus the global A-packet $A_{\psi_{G,\tau}}$ has $2^{\#S}$ elements, where S is the set of places where τ_v belongs to the discrete series. The spaces $L^2_{\psi_{G,\tau}}$ are direct sums of $2^{\#S-1}$ (possibly reducible) representations.

Now we shall explain how the results of this paper provide constructions of the A-packets described above.

13.5. **The packet** $A_{\psi_{PGSp_4,\tau_v}}$. This is the Saito-Kurokawa A-packet and it is well-known that one should take:

$$A_{\psi_{PGSp_4,\tau_v}} = \begin{cases} \{SK^+(\tau_v), SK^-(\tau_v)\}, & \text{if } \tau_v \text{ is discrete series;} \\ \{SK^+(\tau_v)\}, & \text{if } \tau_v \text{ is not a discrete series.} \end{cases}$$

Here, we recall that

$$SK^\pm(\tau_\nu) = \theta_\psi(\sigma_v^\pm), \quad \sigma_v^\pm \in \tilde{A}_{\tau_v}.$$

A more detailed discussion can be found in [G2].

13.6. **The packet** $A_{\psi_{PGSp_8,\tau_v}}$. As we explained above, the structure of this A-packet is identical to that of the Saito-Kurokawa packets. In the p-adic case, it is not unreasonable to define this packet by:

$$A_{\psi_{PGSp_8,\tau_v}} = \begin{cases} \{\Pi^+(\tau_v), \Pi^-(\tau_v)\}, & \text{if } \tau_v \text{ is discrete series;} \\ \{\Pi^+(\tau_v)\}, & \text{if } \tau_v \text{ is not discrete series.} \end{cases}$$

Here, we recall that

$$\Pi^+(\tau_v) = J_{R_8}(\tau_v \boxtimes \tau_v, |\det|^{3/2} \boxtimes |\det|^{1/2}) \quad \text{and} \quad \Pi^-(\tau) = J_{Q_8}(\tau_v \otimes SK^-(\tau_v), 3/14).$$

In a recent paper [Mo], Moeglin has defined A-packets for general classical p-adic groups, assuming a conjectural parametrization of supercuspidal representations. For the case at hand, the above definition of this local A-packet agrees with hers.

In the archimedean case, this A-packet has been constructed by Adams-Johnson [AJ]. It turns out that in the real case, if τ_v is a discrete series representation, the representation $\Pi^+(\tau_v)$ in the packet is reducible: it is the direct sum of the Langlands quotient as in the p-adic case and a holomorphic/anti-holomorphic discrete series.

The construction of Adams-Johnson uses cohomological induction. Let us describe their construction more precisely. Fix a maximal compact subgroup K of $PGSp_8(\mathbb{R})$ which determines a Cartan involution θ. In the complexified Lie algebra $\mathfrak{sp}_8(\mathbb{C})$, one has various θ-stable Siegel parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ such that the Levi subalgebra $\mathfrak{l}$ is the complexification of the Lie algebra of $U(p,q)$ with $p+q=4$. For an integral character λ of $U(p,q)$, one may thus define the cohomologically induced $(\mathfrak{sp}_8(\mathbb{C}),K)$-module $A_{\mathfrak{q}}(\lambda)$. For fixed λ, we denote this module by $\pi_\lambda(p,q)$. Since $\pi_\lambda(p,q)\cong\pi_\lambda(q,p)$, we have 3 different modules $\pi_\lambda(2,2)$, $\pi_\lambda(3,1)$ and $\pi_\lambda(4,0)$.

Now a discrete series representation τ_v of $PGL_2(\mathbb{R})$ determines a character λ and Adams-Johnson set:

$$\begin{cases}\Pi^+(\tau_v)=\pi_\lambda(2,2)\oplus\pi_\lambda(4,0)\\ \Pi^-(\tau_v)=\pi_\lambda(3,1).\end{cases}$$

We remark also that

$$\pi_\lambda(2,2)=J_{R_8}(\tau_v\boxtimes\tau_v,|\det|^{3/2}\boxtimes|\det|^{1/2})\quad\text{and}\quad\pi_\lambda(3,1)=J_{Q_8}(\tau_v\boxtimes SK^-(\tau_v),3/14),$$

whereas $\pi_\lambda(4,0)$ is a holomorphic/antiholomorphic discrete series.

Thus, we see that $\Pi^+(\tau_v)$ can be reducible in the real case, and the representation $\pi_\lambda(4,0)$ should also occur as a local component of some constituents of $L^2_{\Psi_{PGSp_8,\tau}}$. Such a constituent is necessarily cuspidal and thus is not captured by the residual Eisenstein series used in Ginzburg's construction. When τ corresponds to a holomorphic cusp form relative to $SL_2(\mathbb{Z})$, such a cuspidal representation in this A-packet has recently been constructed by Ikeda [I1] .

13.7. **The packet $A_{\psi_{PGSp_6,\tau_v}}$.** In this case, we wish to define the local A-packet by using the local analog of Ginzburg's lifting. Namely, we consider the correspondence of representations obtained by the restriction of $\Pi^\pm(\tau_v)$ from GSp_8 to $(GL_2\times GSp_6)^0$. If we let $\Theta_{\Pi^\pm(\tau_v)}(\tau_v)$ denote the direct sum of all irreducible representations π_v of $PGSp_6$ such that

$$\mathrm{Hom}_{(GL_2\times GSp_6)^0}(\Pi^\pm(\tau_v),\tau_v\boxtimes\pi_v)\neq 0$$

then we would like to propose the following

Conjecture 13.1. *i) The representation $\Theta_{\Pi^\pm(\tau_v)}(\tau_v)$ of $PGSp_6(F_v)$ is irreducible (or at least of finite length).*

ii) (tower property) If τ_v is supercuspidal, then $\Theta_{\Pi^{\pm(\tau_v)}(\tau_v)}$ is supercupidal iff $\Theta_{SK^\pm(\tau_v)}(\tau_v)$ is zero.

If this conjecture holds, we would like to set:

$$A_{\psi_{PGSp_6,\tau_v}}=\begin{cases}\{\Sigma^+(\tau_v),\Sigma^-(\tau_v)\},\ \text{if } \tau_v \text{ is discrete series;}\\ \{\Sigma^+(\tau_v)\},\ \text{if } \tau_v \text{ is not discrete series.}\end{cases}$$

with

$$\Sigma^\pm(\tau_v)=\Theta_{\Pi^{\pm\epsilon(\tau_v,Sym^3,1/2)}(\tau_v)}(\tau_v).$$

For us, the intriguing aspect of this definition is the shift in the labelling of the representations by the epsilon factor $\epsilon(\tau_v,Sym^3,1/2)$. This shift seems to be necessary. Firstly, in view of the above conjecture and Prop. 4.1, it has the desirable effect of making $\Sigma^-(\tau_v)$ supercuspidal when τ_v is supercuspidal. Secondly, this labelling of the representations is consistent with the predicted multiplicity formula for a representation $\Sigma = \otimes_v \Sigma^{\epsilon_v}(\tau_v)$ in the global packet. Indeed, we see from Arthur's multiplicity formula that

$$m(\Sigma) \neq 0 \Longleftrightarrow \prod_v \epsilon_v = \epsilon(\tau, 1/2) \cdot \epsilon(\tau, Sym^3, 1/2).$$

On the other hand, since

$$\Sigma = \otimes_v \Theta_{\Pi^{\epsilon_v \cdot \epsilon(\tau_v, Sym^3, 1/2)}(\tau_v)}(\tau_v),$$

this last condition implies that the representation

$$\bigotimes_v \Pi^{\epsilon_v \cdot \epsilon(\tau_v, Sym^3, 1/2)}(\tau_v).$$

is automorphic on $PGSp_8$ and so we may use it in Ginzburg's construction to yield π in the discrete spectrum of $PGSp_6$.

We also mention that the archimedean packets are among those constructed by Adams-Johnson. In view of our discussion of the packets on $PGSp_8$, it is clear that in order to exhaust the space $L^2_{\psi_{PGSp_6,\tau}}$ using Ginzburg's construction, it is necessary to use all the representations in $L^2_{\psi_{PGSp_8,\tau}}$ as theta kernels in the lifting from PGL_2 to $PGSp_6$. In particular, one needs to use the cuspidal representations constructed by Ikeda [I1]. This lifting problem was investigated in another recent paper of Ikeda [I2], but he was not able to show that the lifting is non-vanishing.

For the purpose of this paper, it is not necessary for us to use the subspace of $L^2_{\psi_{PGSp_6,\tau}}$ constructed using Ikeda's cuspidal representation of $PGSp_8$. This is because the real components of these representations are holomorphic/antiholomorphic discrete series of $PGSp_6(\mathbb{R})$, and under the archimedean theta correspondence, these discrete series lift to the compact G_2 but not the split G_2.

13.8. **The packets $A_{\psi_{G_2,\tau_v}}$.** As we mentioned before, the composite

$$L_F \times SL_2(\mathbb{C}) \xrightarrow{\psi_{G_2,\tau_v}} G_2(\mathbb{C}) \longrightarrow Spin_7(\mathbb{C})$$

is the parameter ψ_{PGSp_6,τ_v}. Since the theta correspondence for $G_2 \times PGSp_6$ is known to be functorial for the above inclusion of dual groups (at least for unramified representations), it is natural to construct the local A-packet on G_2 as the theta lift of the corresponding one on $PGSp_6$. In other words, we should set

$$\pi^+_{G_2,\tau_v} = \Theta(\Sigma^+(\tau_v)) = J_Q(\tau_v, 1/10) \quad \text{and} \quad \pi^-_{G_2,\tau_v} = \Theta(\Sigma^-(\tau_v)).$$

This should be essentially correct, except when τ_v is the Steinberg representation, as we explain next.

Suppose that v is finite and $\tau_v = St_\chi$ is the twisted Steinberg representation with χ a quadratic character. The local Arthur parameter in question is

$$. \ \psi_{St_\chi} : W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{\chi \times i_s \times i_l} G_2(\mathbb{C})$$

Consider also the parameter

$$\hat{\psi}_{St_\chi} : W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{\rho \times i_l \times i_s} G_2(\mathbb{C})$$

obtained from the previous one by switching the two SL_2's. The local Arthur packet corresponding to the parameter $\hat{\psi}_{St_\chi}$ has been determined in [GG] (and proven to be correct!). More precisely,

$$A_{\hat{\psi}_{St_\chi}} = \begin{cases} \{\pi_\epsilon \oplus J_P(St, 1/6), J_Q(St, 1/10)\}, & \text{if } \chi = 1; \\ \{\pi(\chi), J_P(St_\chi, 1/6)\}, & \text{if } \chi \neq 1. \end{cases}$$

Here π_ϵ and $\pi(\chi)$ are certain unipotent supercuspidal representations.

According to a conjecture which we first learned from Moeglin, the packet $A_{\psi_{St_\chi}}$ should be obtained from $A_{\hat{\psi}_{St_\chi}}$ by applying the Iwahori-Mastumoto involution D_{G_2}. In [M], Muic has computed the effect of this involution for the relevant representations. In particular, one has:

$$\begin{cases} D_{G_2}(J_P(St, 1/6)) = \pi'_r, \\ D_{G_2}(J_Q(St, 1/10)) = J_Q(St, 1/10), \\ D_{G_2}(J_P(St_\chi, 1/6)) = J_Q(St_\chi, 1/10), & \text{if } \chi \neq 1. \end{cases}$$

Here, π'_r is a discrete series representation which is a submodule of both $Ind_P^{G_2} St \cdot \delta_P^{1/6}$ and $Ind_Q^{G_2} St \cdot \delta_Q^{1/10}$.

It thus follows that we should have:

$$A_{\psi_{St_\chi}} = \begin{cases} \{\pi'_r \oplus \pi_\epsilon, J_Q(St, 1/10)\}, & \text{if } \chi = 1; \\ \{\pi(\chi), J_Q(St_\chi, 1/10)\}, & \text{if } \chi \neq 1. \end{cases}$$

The main point we want to make here is that according to the precise conjecture of Gross-Savin on the theta correspondence for $G_2 \times PGSp_6$ [GS], the representation π'_r does not appear as a theta lift from $PGSp_6$. This means that one cannot expect to obtain the whole packet $A_{\psi_{G_2}, St}$ as the theta lift of the packet $A_{\psi_{PGSp_6}, St}$.

14. **Appendix**

In this appendix, we complete the local unramified computations in the case when K is a field. Recall that

$$J_K^* = L(\tau_1, 2z + 3/2) \times ((10.4) + (10.5))$$

We must verify that this equals the desired L-function. Let us begin with (10.4):

$$\sum_{l=0}^{\infty} q^{l/2} \chi_\psi(\varpi^l) \overline{\tilde{W}(\varpi^l, \varpi^{-l})} \sum_{k=l}^{\infty} q^{-k(z-1/2)} \sum_{m=0}^{\infty} q^{-2m(z+1/2)} W_{\tau_2}(\varpi^{2m+k}, 1).$$

We suppose that $\tau_2 = \pi(\mu, \mu^{-1})$ and write μ in place of $\mu(\varpi)$. The innermost sum is

$$\sum_{m=0}^{\infty} q^{-(2m+k)/2} \frac{\mu^{-(2m+k+1)} - \mu^{2m+k+1}}{\mu^{-1} - \mu} q^{-2m(z+1/2)} =$$

$$= \frac{q^{-k/2}\mu^{-(k+1)}}{(\mu^{-1}-\mu)(1-\mu^{-2}q^{-(2z+2)})} - \frac{q^{-k/2}\mu^{k+1}}{(\mu^{-1}-\mu)(1-\mu^{2}q^{-(2z+2)})}.$$

The second inner summation then equals

$$\frac{\sum_{k=l}^{\infty} q^{-k/2}\mu^{-(k+1)}q^{-k(z-1/2)}}{(\mu^{-1}-\mu)(1-\mu^{-2}q^{-(2z+2)})} - \frac{\sum_{k=l}^{\infty} q^{-k/2}\mu^{k+1}q^{-k(z-1/2)}}{(\mu^{-1}-\mu)(1-\mu^{2}q^{-(2z+2)})}$$
$$= \frac{\mu^{-(l+1)}q^{-lz}}{(\mu^{-1}-\mu)(1-\mu^{-2}q^{-(2z+2)})(1-\mu^{-1}q^{-z})} - \frac{\mu^{l+1}q^{-lz}}{(\mu^{-1}-\mu)(1-\mu^{2}q^{-(2z+2)})(1-\mu q^{-z})}.$$

Let us write this expression as $J_1 - J_2$.

Recall now that

$$\chi_\psi(\varpi^l)\cdot\overline{\tilde{W}(\varpi^l,\varpi^{-l})} = q^{-l}\chi^l\frac{\chi+q^{-1/2}}{\chi-\chi^{-1}} - q^{-l}\chi^{-l}\frac{\chi^{-1}+q^{-1/2}}{\chi-\chi^{-1}}.$$

The outer summation has contributions from J_1 and J_2. The contribution from J_1 equals

$$\frac{1}{(\mu^{-1}-\mu)(1-\mu^{-2}q^{-(2z+2)})(1-\mu^{-1}q^{-z})}\times$$
$$\frac{\chi+q^{-1/2}}{\chi-\chi^{-1}}\left(\sum_{l=0}^{\infty} q^{l/2}\mu^{-(l+1)}q^{-lz}q^{-l}\chi^l\right) - \frac{\chi^{-1}+q^{-1/2}}{\chi-\chi^{-1}}\left(\sum_{l=0}^{\infty} q^{l/2}\mu^{-(l+1)}q^{-lz}q^{-l}\chi^{-l}\right)$$
$$= \frac{(\chi+q^{-1/2})\mu^{-1}}{(\chi-\chi^{-1})(1-\chi\mu^{-1}q^{-(z+1/2)})} - \frac{(\chi^{-1}+q^{-1/2})\mu^{-1}}{(\chi-\chi^{-1})(1-\chi^{-1}\mu^{-1}q^{-(z+1/2)})}$$
$$= \frac{\mu^{-1}(1+\mu^{-1}q^{-(z+1)})}{(1-\chi\mu^{-1}q^{-(z+1/2)})(1-\chi^{-1}\mu^{-1}q^{-(z+1/2)})}.$$

The total contribution from J_1 thus equals

$$\frac{\mu^{-1}(1+\mu^{-1}q^{-(z+1)})}{(\mu^{-1}-\mu)(1-\mu^{-2}q^{-(2z+2)})(1-\mu^{-1}q^{-z})(1-\chi\mu^{-1}q^{-(z+1/2)})(1-\chi^{-}\mu^{-1}q^{-(z+1/2)})}$$
$$= \frac{\mu^{-1}}{(\mu^{-1}-\mu)(1-\mu^{-1}q^{-(z+1)})(1-\mu^{-1}q^{-z})(1-\chi\mu^{-1}q^{-(z+1/2)})(1-\chi^{-}\mu^{-1}q^{-(z+1/2)})}.$$

Note that J_2 is essentially obtained from J_1 by changing μ to μ^{-1}. So the contribution from J_2 equals

$$\frac{\mu}{(\mu^{-1}-\mu)(1-\mu q^{-(z+1)})(1-\mu q^{-z})(1-\chi^{-1}\mu q^{-(z+1/2)})(1-\chi\mu q^{-(z+1/2)})}.$$

Taking the difference, we obtain:

$$J_1 - J_2 = L(\tau_2, z) \cdot L(\tau_2, z+1) \cdot L(\tau_1 \otimes \tau_2, z+1/2) \times \tag{14.1}$$

$$(1 - (\chi + \chi^{-1})q^{-(2z+1/2)} - (\chi + \chi^{-1})q^{-(2z+3/2)} + (\mu + \mu^{-1})q^{-(3z+1)} + (\mu + \mu^{-1})q^{-(3z+2)} - (\mu + \mu^{-1})(\chi + \chi^{-1})q^{-(3z+3/2)} - (\mu^2 + 1 + \mu^{-2})q^{-(4z+2)}).$$

Now we compute (10.5), which is given by:

$$\sum_{l=1}^{\infty} q^{-l(z+1/2)} \sum_{k=l}^{\infty} q^{k/2} \chi_\psi(\varpi^k) \overline{\tilde{W}(\varpi^k, \varpi^{-k})} \cdot \sum_{m=k}^{\infty} W_{\tau_2}(\varpi^{m-l}, 1) q^{-m(z-1/2)}$$

So the innermost summation equals

$$\sum_{m=k}^{\infty} q^{-(m-l)/2} \frac{\mu^{-(m-l+1)} - \mu^{m-l+1}}{\mu^{-1} - \mu} q^{-m(z-1/2)}$$
$$= \frac{q^{l/2}\mu^{l-1}\mu^{-k}q^{-kz}}{(\mu^{-1} - \mu)(1 - \mu^{-1}q^{-z})} - \frac{q^{l/2}\chi^{-(l-1)}\chi^k q^{-kz}}{(\mu^{-1} - \mu)(1 - \mu q^{-z})}.$$

We again write this as $J_1 - J_2$.

The contribution of J_1 to the next sum is

$$\frac{q^{l/2}\mu^{l-1}}{(\mu^{-1} - \mu)(1 - \mu^{-1}q^{-z})} \times$$
$$\sum_{k=l}^{\infty} q^{k/2}\mu^{-k}q^{-kz}q^{-k}\chi^k \frac{\chi + q^{-1/2}}{\chi - \chi^{-1}} - q^{k/2}\mu^{-k}q^{-kz}q^{-k}\chi^{-k} \frac{\chi^{-1} + q^{-1/2}}{\chi - \chi^{-1}}$$
$$= \frac{q^{-lz}\mu^{-1}}{(\chi - \chi^{-1})(\mu^{-1} - \mu)(1 - \mu^{-1}q^{-z}} \left(\frac{\chi^l(\chi + q^{-1/2})}{1 - \mu^{-1}\chi q^{-(z+1/2)}} - \frac{\chi^{-l}(\chi^{-1} + q^{-1/2})}{1 - \mu^{-1}\chi^{-1}q^{-(z+1/2)}} \right).$$

We denote it as $J_{11} - J_{12}$.

Similarly the contribution of J_2 to the next sum is

$$\frac{q^{-lz}\mu}{(\chi - \chi^{-1})(\mu^{-1} - \mu)(1 - \mu q^{-z})} \left(\frac{\chi^l(\chi + q^{-1/2})}{1 - \chi\mu q^{-(z+1/2)}} - \frac{\chi^{-l}(\chi^{-1} + q^{-1/2})}{1 - \chi^{-1}\mu q^{-(z+1/2)})} \right).$$

We write this as $J_{21} - J_{22}$.

Now $J_{11} - J_{21}$ equals

$$\frac{q^{-lz}\chi^l(\chi + q^{-1/2})}{(\mu^{-1} - \mu)(\chi - \chi^{-1})} \left(\frac{\mu^{-1}}{(1 - \mu^{-1}q^{-z})(1 - \mu^{-1}\chi q^{-(z+1/2)})} - \frac{\mu}{(1 - \mu q^{-z})(1 - \mu\chi q^{-(z+1/2)})} \right)$$

$$= \frac{q^{-lz}\chi^l(\chi + q^{-1/2})}{(\chi - \chi^{-1})} \times \frac{1 - \chi q^{-(2z+1/2)}}{(1-\mu^{-1}q^{-z})(1-\mu^{-1}\chi q^{-(z+1/2)})(1-\mu q^{-z})(1-\mu\chi q^{-(z+1/2)})}.$$

Similarly, $J_{22} - J_{12}$ equals

$$\frac{q^{-lz}\chi^{-l}(\chi^{-1} + q^{-1/2})}{(\mu^{-1}-\mu)(\chi - \chi^{-1})}\left(\frac{\mu}{(1-\mu q^{-z})(1-\mu\chi^{-1}q^{-(z+1/2)})} - \frac{\mu^{-1}}{(1-\mu^{-1}q^{-z})(1-\mu^{-1}\chi^{-1}q^{-(z+1/2)})}\right)$$

$$= -\frac{q^{-lz}\chi^{-l}(\chi^{-1} + q^{-1/2})}{(\chi - \chi^{-1})} \times \frac{1 - \chi^{-1} q^{-(2z+1/2)}}{(1-\mu^{-1}q^{-z})(1-\mu\chi^{-1} q^{-(z+1/2)})(1-\mu q^{-z})(1-\mu^{-1}\chi^{-1} q^{-(z+1/2)})}.$$

Now we do the last summation. The first part contributes

$$\frac{\sum_{l=1}^{\infty} q^{-lz}\chi^l q^{-l(z+1/2)}(\chi + q^{-1/2})(1-\chi q^{-(2z+1/2)})}{(\chi - \chi^{-1})(1-\mu^{-1}q^{-z})(1-\mu^{-1}\chi q^{-(z+1/2)})(1-\mu q^{-z})(1-\mu\chi q^{-(z+1/2)})}$$

$$= \frac{q^{-(2z+1/2)}\chi(\chi + q^{-1/2})}{(\chi - \chi^{-1})(1-\mu^{-1}q^{-z})(1-\mu^{-1}\chi q^{-(z+1/2)})(1-\mu q^{-z})(1-\mu\chi q^{-(z+1/2)})}.$$

The second part contributes

$$-\frac{\sum_{l=1}^{\infty} q^{-lz}\chi^{-l}(\chi^{-1} + q^{-1/2})(1-\chi^{-1} q^{-(2z+1/2)})}{(\chi - \chi^{-1})(1-\mu^{-1}q^{-z})(1-\mu\chi^{-1} q^{-(z+1/2)})(1-\mu q^{-z})(1-\mu^{-1}\chi^{-1} q^{-(z+1/2)})}$$

$$= -\frac{q^{-(2z+1/2)}\chi^{-1}(\chi^{-1} + q^{-1/2})}{(\chi - \chi^{-1})(1-\mu^{-1}q^{-z})(1-\mu\chi^{-1} q^{-(z+1/2)})(1-\mu q^{-z})(1-\mu^{-1}\chi^{-1} q^{-(z+1/2)})}.$$

The total second summand thus equals

$$L(\tau_2, z)L(\tau_1 \otimes \tau_2, z+1/2)\times$$
$$q^{-(2z+1/2)}(\chi + \chi^{-1} + q^{-1/2} - (\mu + \mu^{-1})q^{-(z+1/2)} - q^{-(2z+3/2)}).$$

Thus (10.4)+(10.5) equals

$$\zeta(2z+1) \cdot \frac{L(\tau_2, z)L(\tau_2, z+1)L(\tau_1 \otimes \tau_2, z+1/2)}{L(\tau, 2z+3/2)}.$$

In other words

$$J_K = L(\tau_2, z) \cdot L(\tau_2, z+1) \cdot L(\tau_1 \otimes \tau_2, z+1/2)$$

as desired!

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