Restriction of Representations of Classical Groups: the Gross-Prasad Conjecture

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References

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(These are to appear in Asterisque Vol 118)

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The period of ω over C is the integral

$$\int_C i^*(\omega).$$

Question: Is this nonzero? What is its value?

Special cases of such a question are significant in number theory, and are related to special values of L-functions.

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Let Π be the representation of G generated by a function f_0. Then regard $\mathcal{P}_C$ as a G-invariant linear functional on Π.

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Question: Is $\mathcal{P}_C$ nonzero on Π?

It is natural to first consider the representation theoretic question:

When is $\mathrm{Hom}_G(\Pi, \mathbb{C}) \neq 0$?

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  and fixed field $k_0 = k^{\sigma}$;
- V = a finite dimensional vector space over a field k;
- $\langle -, - \rangle$ a (σ, ϵ)-hermitian form (non-degenerate) on V $(\epsilon = \pm)$:

$$\begin{aligned} \langle \alpha v + \beta w, u \rangle &= \alpha \langle v, u \rangle + \beta \langle w, u \rangle \\ \langle u, v \rangle &= \epsilon \cdot \langle v, u \rangle^{\sigma}. \end{aligned}$$

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- Set

$$G(V) = Aut(V, \langle -, - \rangle)^0 .$$

  This is a classical group defined over the field k_0.

Table of Possibilities

The different possibilities for $G(V)$ are given in the following table.

(k, ϵ)	$G(V)$
$k = k_0,\ \epsilon = 1$	special orthogonal group $\mathrm{O}(V)$
$k = k_0,\ \epsilon = -1$	symplectic group $\mathrm{Sp}(V)$
k/k_0 quadratic field, $\epsilon = \pm 1$	unitary group $\mathrm{U}(V)$
$k = k_0 \times k_0$, $\epsilon = \pm 1$	general linear group $\mathrm{GL}(V_0)$

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- $V = \mathbb{C}^n$ and

$$\langle \underline{v}, \underline{w} \rangle = v_1 \cdot w_1^\sigma + ... + v_n \cdot w_n^\sigma.$$

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$$G(V) = \mathrm{U}(n) = \{g \in \mathrm{GL}_n(\mathbb{C}) : {}^t g^\sigma \cdot g = I\}$$

  is a compact Lie group.

Classification of Forms over $\mathbb{R}$ and $\mathbb{C}$

Proposition

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- *Over $\mathbb{C}$, there is a unique (σ, ϵ) hermitian space V of a given dimension.*
- *There is a unique symplectic space V of a given dimension over $\mathbb{R}$.*
- *Over $\mathbb{R}$, quadratic and hermitian spaces V are classified by their signatures (p, q), with $p + q = \dim V$.*

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- *Quadratic spaces V are classified by:*

$$(\dim V, \quad \mathrm{disc}(V) \in k_0^\times / k_0^{\times 2}, \quad \epsilon(V) = \pm 1)$$

- *Hermitian and skew-Hermitian spaces V are classified by:*

$$(\dim V, \quad \mathrm{disc}(V) \in k_0^\times / \mathbb{N} k^\times)$$

Pure Inner Forms

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- For a given V, with associated $G(V)$, we consider those V' with

  $$\dim V' = \dim V$$

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and in the quadratic case, we further require
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The collection of $G(V')$'s obtained in this way are called the pure inner forms of $G(V)$.
- It is possible that $V' \not\cong V$, but $G(V') \cong G(V)$. But we still regard $G(V)$ and $G(V')$ as different elements in the set of pure inner forms.

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- a unique pure inner form, if V is symplectic, or if $G(V) = \mathrm{GL}(V_0)$;
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In general, for a given group G, the set of pure inner forms of V are classified by the Galois cohomology set $H^1(k_0, G(V))$.

A Restriction Problem

Suppose that $W \subset V$, so that

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This counts the number of times the dual representation $\sigma^\vee$ occurs as quotient of π.

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  where ω is a (so-called) Weil representation.

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  where ω is a (so-called) Weil representation.

To be precise, one needs to work on the double cover of symplectic groups to talk about the Weil representation. Because of this technical complication, we will not consider this case henceforth.

A Classical Example

If $W \subset V$ are positive definite Hermitian spaces, then

$$G(W) = U(n-1) \subset G(V) = U(n).$$

The irreducible representations of $\mathrm{U}(n)$ are finite-dimensional.

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The irreducible representations of $\mathrm{U}(n)$ are finite-dimensional.

Given such a π, we consider its restriction to $U(n-1)$, and ask which irreducible representations of $U(n-1)$ occur, and with what multiplicity:

$$\pi|_{U(n-1)} = \bigoplus_{\sigma} m_{\pi,\sigma} \cdot \sigma.$$

Irreducible Representations of Connected Compact Lie Groups

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Theorem

The irreducible representations of $\mathrm{U}(n)$ are naturally parametrized by the set of n-tuples of integers

$$(a_1, a_2, ..., a_n)$$

satisfying

$$a_1 \geq a_2 ... \geq a_n$$

A Classical Branching Law

The restriction problem from $\mathrm{U}(n)$ to $\mathrm{U}(n-1)$ has a very nice answer:

Theorem

Suppose π of $\mathrm{U}(n)$ corresponds to $(a_1, ..., a_n)$ and σ of $\mathrm{U}(n-1)$ corresponds to $(b_1, ..., b_{n-1})$. Then

$$\dim \mathrm{Hom}_{U(n-1)}(\pi, \sigma) \leq 1,$$

and equality holds if and only if

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq ......b_{n-1} \geq a_n.$$

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Assume that $W \subset V$ is quadratic or hermitian, so that $\dim_k V - \dim_k W = 1$. If π is a irreducible representation of $G(V)$ and σ one of $G(W)$, then

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This is due to:

- Aizenbud-Gourevitch-Rallis-Schiffmann (with a little input from Waldspurger) when k is p-adic;
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Question: when is it nonzero?

Example: $GL(n)$

Suppose that π and σ are generic representations of $\mathrm{GL}(n)$ and $\mathrm{GL}(n-1)$ respectively. Then one has

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In this case, we do not need a classification of the irreducible representations of $\mathrm{GL}(n)$ to state the answer to the restriction problem.

Here, genericity is a technical condition satisfied by most representations of $\mathrm{GL}(n)$: it means that the representations have the "largest dimension" amongst all representations.

Example: $\mathrm{U}(2) \times \mathrm{U}(1)$ revisted

Let's examine the example of $U(2) \times U(1)$ in greater detail.
We have looked at the positive definite case:

$$W \subset V,$$

where V has signature $(2, 0)$ and W has signature $(1, 0)$.

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If we take an irrep π_a of $\mathrm{U}(2)$ with Cartan-Weyl parameter $(a, -a)$, then π has trivial central character, and $\pi_a|_{\mathrm{U}(1)}$ consists of those χ_b's such that

$$-a \leq b \leq a.$$

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The group $\mathrm{U}(V') = \mathrm{U}(1,1)$ is closely related to the group $\mathrm{SL}_2(\mathbb{R})$. Indeed,

$$\mathrm{U}(1,1)/Z \cong \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}.$$

The map

$$\mathrm{U}(1,0) \to \mathrm{U}(1,1) \to \mathrm{U}(1,1)/Z$$

is injective, with image the maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$.

Fact:

There is a pair of irreps π_a^+ and π_a^- of $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$, such that

$$\pi_a^+|_{\mathrm{U}(1,0)} = \bigoplus_{b>a} \chi_b$$

and

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Conclusion:

If one consider $\pi \oplus \pi^+ \oplus \pi^-$ as a representation of $\mathrm{U}(1,0)$, then one gets each irrep of $\mathrm{U}(1,0)$ exactly once.

The example of $\mathrm{U}(2) \times \mathrm{U}(1)$ suggests:

If we group together certain irreps of $G(V)$ and $G(W)$, and their pure inner forms, then the restriction problem will have a nice solution like the case of $\mathrm{GL}(n)$.

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Such a grouping is precisely what is proposed by the local Langlands conjecture (as extended by Vogan), which gives a classification of the irreps of $G(V)$ and its pure inner forms.

In particular, it is a highly nontrivial extension of the Cartan-Weyl theory of highest weight.

Local Langlands-Vogan Conjecture (Preliminary Form)

Let $\mathrm{Irr}(G)$ denote the set of irreps of G. Then given a space V, one has

$$\bigsqcup_{V'} \mathrm{Irr}(G(V')) = \bigsqcup_{\phi} \Pi_{\phi}$$

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- the union on the left runs over all pure inner forms $G(V')$ of $G(V)$;
- the indexing set on the right runs over the set of Langlands parameters for $G(V)$;
- the sets Π_ϕ are finite sets of irreps of $G(V)$ and its pure inner forms: they are called Vogan packets.

Langlands Parameters

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$$W_k = \mathrm{Gal}(\overline{k}/k).$$

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Then

$$\phi : WD_k \longrightarrow G^\vee(\mathbb{C}).$$

L-parameters for quadratic case

- when V is quadratic, with odd dimension $2n+1$, then

$$\phi : WD_k \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C}),$$

is a $2n$-dimensional semisimple symplectic representation of WD_k.

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Thus, an L-parameter is a self-dual Galois representation with appropriate sign ϵ: there is an isomorphism

$$b : \phi \rightarrow \phi^{\vee}$$

with $b^{\vee} = \epsilon \cdot b$.

L-parameters for hermitian case

When V is hermitian of dimension n, then

$$\phi : WD_k \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

is an n-dimensional semisimple representation of WD_k which is

$$\begin{cases} \text{conjugate symplectic, if } n \text{ is even} \\ \text{conjugate orthogonal, if } n \text{ is odd.} \end{cases}$$

Thus, ϕ is a conjugate-self-dual representation with appropriate sign ϵ, i.e. there is an isomorphism

$$b : \phi \to (\phi^{\vee})^{\sigma}$$

with $(b^{\vee})^{\sigma} = \epsilon \cdot b$.

Relevant Pure Inner Forms

Let $W \subset V$ be given, so that $G(W) \subset G(V)$.

Definition

A pair $G(W') \times G(V')$ is a relevant pure inner form of $G(W) \times G(V)$ if

- $W' \subset V'$,
- $V/W \cong V'/W'$.

For example, when $W \subset V$ is hermitian, the group $G(V) \times G(W)$ has 4 pure inner forms, of which two are relevant.

Multiplicity One in Vogan packets

Let $W \subset V$ be given, and consider a *generic* L-parameter $\phi \times \phi'$ of $G(V) \times G(W)$, with Vogan packet $\Pi_{\phi} \times \Pi_{\phi'}$ of $G(V) \times G(W)$ and its pure inner forms. Then

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We may now make:

Conjecture

$$\sum_{\textit{relevant } (\pi,\sigma) \in \Pi_\phi \times \Pi_{\phi'}} \dim \mathrm{Hom}_{G(W)}(\pi \otimes \sigma, \mathbb{C}) = 1.$$

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Question:
Which relevant (π, σ) has nonzero contribution?

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Given $\phi : WD_k \to G^\vee(\mathbb{C})$, one defines a finite group

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i.e. the group of connected components of the centralizer of ϕ in $G^\vee$.

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$$A_\phi = \pi_0(Z_{G^\vee}(\phi)),$$

i.e. the group of connected components of the centralizer of ϕ in $G^\vee$.

Refined Langlands-Vogan Conjecture

Conjecture

Relative to a non-trivial character ψ of k or k/k_0, There is a natural bijection

$$J_\psi : \Pi_\phi \leftrightarrow Irr(A_\phi).$$

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Thus, an irrep of $G(V)$ (and its pure inner form) is given by

$$\pi = \pi(\phi, \eta)$$

with ϕ is an L-parameter and $\eta \in \mathrm{Irr}(A_\phi)$.

This is the generalization of the Cartan-Wely theory of highest weight. It has been announced by Arthur (at least for the quadratic case) about the details have not appeared.

Quadratic and Hermitian Cases

For the case at hand, when $G(V)$ is an orthogonal or unitary group, the component group A_ϕ is easily described.

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Then

$$A_\phi \cong \bigoplus_{i \in S} \mathbb{Z}/2\mathbb{Z} \cdot a_i$$

where S is the set of i's such that ϕ_i is (conjugate)-self-dual of the same sign ϵ as ϕ.

Thus, A_ϕ is an $\mathbb{F}_2$-vector space equipped with a canonical basis.

Example: Trilinear Forms for GL_2

Consider $SO(4) \times \mathrm{SO}(3)$.

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So the restriction problem is:

Given irreps π_1, π_2 and π_3 of $\mathrm{PGL}(2)$ with L-parameters ϕ_i, determine

$$\dim \mathrm{Hom}_{\mathrm{GL}(2)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}).$$

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The other pure inner form is the group $PD^\times \times PD^\times \times PD^\times$, where D is the quaternion division algebra. The Vogan packet containing $\pi_1 \otimes \pi_2 \otimes \pi_3$ is either a singleton, or contains another irrep

$$\pi_1^D \otimes \pi_2^D \otimes \pi_3^D \quad \text{(Jacquet-Langlands transfer)}$$

on this pure inner form.

Prasad's Theorem

Theorem

- $$\dim \mathrm{Hom}_{\mathrm{PGL}(2)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) + \dim \mathrm{Hom}_{PD^\times}(\pi_1^D \otimes \pi_2^D \otimes \pi_3^D, \mathbb{C})$$
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  $$= 1$$

- $$\mathrm{Hom}_{\mathrm{PGL}(2)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) \neq 0$$
  if and only if
  $$\epsilon(1/2, \phi_1 \otimes \phi_2 \otimes \phi_3) = 1.$$

Here the RHS is the local epsilon factor attached to the 8-dimensional representation $\phi_1 \otimes \phi_2 \otimes \phi_3$ of WD_k by Deligne and Langlands. It takes value ± 1.

Epsilon Factors of (Conjugate)-Symplectic Parameters

To specify which pair of relevant representations $(\pi, \sigma) \in \Pi_\phi \times \Pi_{\phi'}$ has nonzero contribution to Hom, we need to specify a character χ_ψ of the finite group $A_\phi \times A_{\phi'}$.

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A key observation, motivated by Prasad's theorem, is:

The representations ϕ and ϕ' are (conjugate)-self-dual with opposite signs, since $\dim V$ and $\dim W$ are of opposite parity.

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A key observation, motivated by Prasad's theorem, is:

The representations ϕ and ϕ' are (conjugate)-self-dual with opposite signs, since dim V and dim W are of opposite parity.

Proposition

If ϕ is a conjuagte symplectic representation of WD_k, and ψ is a character of k/k_0, then

$$\epsilon(1/2, \phi, \psi) = \pm 1.$$

If ϕ is conjugate orthogonal, then

$$\epsilon(1/2, \phi, \psi) = 1.$$

A Distinguished Character of the Component Group

We shall define such a character χ_ψ in the hermitian case. Suppose

$$A_\phi = \bigoplus_{i \in S} \mathbb{Z}/2\mathbb{Z} \cdot a_i$$

and

$$A_{\phi'} = \bigoplus_{j \in S'} \mathbb{Z}/2\mathbb{Z} \cdot b_j.$$

Definition

Set

$$\chi(a_i) = \epsilon(1/2, \phi_i \otimes \phi', \psi)$$

and

$$\chi(b_j) = \epsilon(1/2, \phi \otimes \phi'_j, \psi).$$

Note that the representation $\phi_i \otimes \phi'$ is conjugate symplectic, so these epsilon factors are equal to ± 1.

Local Gross-Prasad Conjecture

Conjecture

The unique relevant representation $(\pi, \sigma) \in \Pi_\phi \times \Pi_{\phi'}$ *such that*

$$\mathrm{Hom}_{G(W')}(\pi \otimes \sigma, \mathbb{C}) \neq 0$$

corresponds to the character χ_ψ *of* $A_\phi \times A_{\phi'}$ *under the bijection*

$$J_\psi : \Pi_\phi \times \Pi_{\phi'} \leftrightarrow Irr(A_\phi \times A_{\phi'})$$

Results: Quadratic Case

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- $SO(3) \times SO(2)$: Waldspurger, Tunnell and Saito.
- $SO(4) \times SO(3)$: D. Prasad
- In a series of 5 recent papers, Waldspurger and Moeglin-Waldspurger have proven:

Theorem

The local GP conjecture for $\mathrm{SO}(n) \times \mathrm{SO}(n-1)$ *is true, assuming the Langlands-Vogan correspondence is known and has some expected properties.*

Results: Hermitian Case

- $U(1) \times \mathrm{U}(2)$ and $\mathrm{U}(2) \times \mathrm{U}(3)$: checked in my second paper with Gross-Prasad.
- there is every reason to expect that the techniques of Waldspurger will give the general case, but it will take many pages to verify the details!

Global Setting

The local GP conjecture has a global counterpart.

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Let E be a number field with involution σ such that $F = F^{\sigma}$. Let $\mathbb{A}_F$ be the ring of adeles of F.

For example: if $F = \mathbb{Q}$, then

$$\mathbb{A}_{\mathbb{Q}} = \prod_p{}' \mathbb{Q}_p \times \mathbb{R}.$$

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$$\mathbb{A}_{\mathbb{Q}} = \prod_p{}' \mathbb{Q}_p \times \mathbb{R}.$$

One may consider quadratic or hermitian spaces

$$W \subset V$$

over E, and the associated groups

$$G(W) \subset G(V)$$

over F.

Global Period Integral

Let Π be a cuspidal automorphic representation of $G(V)(\mathbb{A}_F)$, so that

$$\Pi \subset L^2(G(V)(F)\backslash G(V)(\mathbb{A}_F))$$

Similarly, let Σ be a cuspidal representation of $G(W)(\mathbb{A}_F)$

Definition

The global period integral on $\Pi \otimes \Sigma$ is the linear functional

$$\mathcal{P} : \Pi \otimes \Sigma \longrightarrow \mathbb{C}$$

given by

$$\mathcal{P}(f \otimes \varphi) = \int_{G(W)(F)\backslash G(W)(\mathbb{A}_F)} f(h) \cdot \varphi(h)\, dh$$

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Note that

$$\mathcal{P} \in \mathrm{Hom}_{G(W)(\mathbb{A}_F)}(\Pi \otimes \Sigma, \mathbb{C}).$$

The Global Problem

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Since $\Pi = \otimes'_v \pi_v$ and $\Sigma = \otimes'_v \sigma_v$, if

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for some place v, then $\mathcal{P}$ is necessarily zero.

So the question is: in the absence of local obstructions, is there further global obstruction to the non-vanishing of $\mathcal{P}$.

Global Gross-Prasad Conjecture

Conjecture

Suppose that Π and Σ are tempered. Then $\mathcal{P}$ is nonzero if and only if

- *for all v,*

$$\mathrm{Hom}_{G(W)(F_v)}(\pi_v \otimes \sigma_v, \mathbb{C}) \neq 0$$

- $$L(1/2, \Pi \times \Sigma) \neq 0.$$

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Ginzburg-Jiang-Rallis have shown the implication ($\Rightarrow$), under some assumptions on Π and Σ.

Refined GP Conjecture (Ichino-Ikeda)

Ichino and Ikeda have conjectured a precise formula relating $|\mathcal{P}|^2$ and $L(1/2, \Pi \times \Sigma)$, when V is quadratic.

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(When V is unitary, the analogous conjecture was formulated in the pending PhD thesis of Neal Harris at UCSD).
For each v, they defined a natural linear functional

$$\mathcal{P}_v : \pi_v \otimes \overline{\pi_v} \otimes \sigma_v \otimes \overline{\sigma_v} \longrightarrow \mathbb{C}$$

using "integration of matrix coefficients". It was shown by Waldspurger that (for tempered reps)

$$\mathcal{P}_v \neq 0 \Leftrightarrow \mathrm{Hom}_{G(W_v) \times G(W_v)}(\pi_v \otimes \sigma_v, \mathbb{C}) \neq 0.$$

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So

$$\prod_v \diamond_v \in \mathrm{Hom}_{G(W)(\mathbb{A}_F) \times G(W)(\mathbb{A}_F)}(\Pi \otimes \Sigma, \mathbb{C})$$

and is nonzero if and only if RHS is nonzero.

Ikeda-Ichino Conjecture

On the other hand, the global period integral $\mathcal{P}$ defines an element (possibly zero)

$$\mathcal{P} \otimes \overline{\mathcal{P}} \in \mathrm{Hom}_{G(W)(\mathbb{A}_F) \times G(W)(\mathbb{A}_F)}(\Pi \otimes \overline{\Pi} \otimes \Sigma \otimes \overline{\Sigma}, \mathbb{C}).$$

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Since the Hom space above is at most 1-dimensional, $\mathcal{P} \otimes \overline{\mathcal{P}}$ and $\prod_v \mathcal{P}_v$ must be scalar multiples of each other.

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Conjecture

One has

$$\mathcal{P} \otimes \overline{\mathcal{P}} = 2^{\beta} \cdot \Delta(G(V)) \cdot \frac{L(1/2, \Pi \times \Sigma)}{L(1, \Pi, Ad) \cdot L(1, \Sigma, Ad)} \cdot \prod_v \mathcal{P}_v,$$

where β is an explicit constant, and $\Delta(G(V))$ is a product of zeta values depending only on V.

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where β is an explicit constant, and $\Delta(G(V))$ is a product of zeta values depending only on V.

Observe that this implies the global GP conjecture.

Results

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For the refined global conjecture for $\mathrm{U}(n) \times \mathrm{U}(n-1)$, Jacquet and Rallis have pioneered a approach via a relative trace formula. But a lot of work remains to be done.

Derivatives

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Conjecture

Assume that $\epsilon(1/2, \Pi \times \Sigma) = -1$, so that $L(1/2, \Pi \times \Sigma) = 0$. Then the following are equivalent:

- $L'(1/2, \Pi \times \Sigma) \neq 0$;
- $\Pi_f \times \Sigma_f$ *occurs in a certain Chow group of a Shimura variety associated to* $V \times W$*, and the natural height pairing is nonzero when restricted to* $\Pi_f \otimes \Sigma_f$*.*

This is a generalization of the Gross-Zagier formula (as extended by Yuan-Zhang-Zhang).

THANK YOU FOR YOUR ATTENTION!