Representations of Metaplectic Groups

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Abstract. We describe some recent developments in the representation theory of the metaplectic group Mp_{2n} over a p-adic field, such as the proof of a local Langlands correspondence, and certain restriction problems in the style of the Gross-Prasad conjecture.

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1. Introduction

1.1. The local Langlands correspondence. Let k be a nonarchimedean local field of characteristic 0 and residual characteristic p. If G is a connected reductive group over k, then the goal of the local Langlands program is to give a classification of the set $\mathrm{Irr}(G)$ of isomorphism classes of irreducible smooth representations of $G(k)$ in terms of objects of arithmetic interest. More precisely, if W_k denotes the Weil group of k (which is a dense subgroup of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$) and $WD_k = W_k \times \mathrm{SL}_2(\mathbb{C})$ is the Weil-Delighne group, then the objects of arithmetic interest in question are the L-parameters, which are homomorphisms

$$\phi : WD_k \longrightarrow {}^L G$$

where ${}^L G$ denotes the L-group of G. This classification, called the local Langlands correspondence (LLC), is a non-abelian generalization of local class field theory and should be thought of as a sophisticated version of the Cartan-Weyl theory of highest weights, which gives a classification of the irreducible representations of connected compact Lie groups.

The LLC is supposed to satisfy a number of properties which determine it uniquely. Indeed, there are two possible ways of characterizing this classification. One is to require that this classification respects certain fundamental invariants that one can associate to both sides, namely the local L-factors and ϵ-factors. Another is through the theory of endoscopy, which relates the characters of representations of $G(k)$ to stable distributions of its endoscopic groups in terms of the combinatorics of their L-parameters. It is in fact not a priori clear that these two requirements characterize the same parametrization, though one expects this to be the case.

The LLC has been established for the group $G = \mathrm{GL}_n$ by Harris-Taylor [HT] and Henniart [He], for all inner forms of SL_n by Gelbart-Knapp [GK] and Hiraga-Saito [HS], and for a number of low rank

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groups (see [Ro], [GT1,2] and [GTW]). The next giant step is obtained in the pending work of Arthur [A] which establishes the LLC for all (quasi-split) classical groups. Assuming this work of Arthur, Moeglin and Waldspurger have gone on to establish many key properties of this parametrization, such as reconciling the two characterizations mentioned above. For the purpose of this paper, I will take the view that the LLC is now essentially known, at least for the classical groups.

1.2. Restriction problems. Just as the Cartan-Weyl theorem did not signify the end of interest in the representation theory of compact connected Lie groups, the LLC is not an end in itself. Rather, the LLC provides one with the language and tools to address certain natural problems which arise in representation theory. An important class of such problems has been singled out for the special orthogonal groups by Gross and Prasad in two influential papers [GP1,2], and recently extended to all classical groups in [GGP].

The basic question is the following. Suppose that $\mathrm{SO}_{n-1} \subset \mathrm{SO}_n$ and π and σ are irreducible representations of SO_n and SO_{n-1} respectively. Then a recent result of Aizenbud-Gourevitch-Rallis-Schiffmann [AGRS] , supplemented by Waldspurger [W8], says that

$$\dim \mathrm{Hom}_{\mathrm{SO}_{n-1}}(\pi, \sigma) \leq 1.$$

The Gross-Prasad (GP) conjecture then describes precisely when this Hom space is nonzero, and this conjectural answer is expressed in terms of the LLC. In a series of remarkable papers [W4-7] and [MW], Waldspurger and Moeglin-Waldspurger have established the GP conjecture for special orthogonal groups, and there is every reason to expect that the case of unitary groups can be similarly handled.

1.3. Covering Groups. There have been many attempts in the literature to extend the Langlands philosophy to the case of covering groups, at least partially. Amongst these, we mention the work of Waldspurger [W1,2], Flicker [F], Flicker-Kazhdan [FK], Adams [Ad], Renard [R1,2], Savin [S1-2] and Weissman [We]. Given the recent spectacular progress in the LLC and its applications, the time seems ripe to consider a systematic extension of the Langlands program to the case of covering groups of $G(k)$. Though there has been quite a number of significant work in the archimedean case, notably by Adams, Vogan and their collaborators, the recent papers [Li1,2, 3] by Wenwei Li constitute the first steps towards such a systematic extension for all local fields.

This desire to extend the Langlands program to the covering groups is not simply an exercise in generalization for its own sake. Indeed, the simplest example of such covering groups which arises in nature is the metaplectic group Mp_{2n}, which is the unique 2-fold cover of the symplectic group Sp_{2n}. This group already features prominently in the theory of automorphic forms, as it is the setting for the study of half-integral weight modular forms, most notably the theta functions. Indeed, the group Mp_{2n} carries a distinguished family of representations known as the Weil representations, and is the setting for the theory of theta correspondence, which was initiated by Howe.

1.4. Content of this paper. The goal of this paper is not to discuss the case of general covering groups, but to describe some recent developments in the representation theory of the specific group Mp_{2n}. When $n = 1$, a complete classification of the irreducible (genuine) representations and the automorphic discrete spectrum of Mp_2 was obtained in the foundational work of Waldspurger [W1,2] and the main results of this paper constitute an extension of Waldspurger's local results to the case of general n.

Here is a summary of the content of the paper. After reviewing some basic structural results about Mp_{2n} in Section 2, and introducing the theory of theta correspondences in Section 3, the main results are discussed in Sections 4,5 and 6. These results are mostly joint with G. Savin [GS]; they establish an LLC for Mp_{2n} and prove many of its expected properties. In Section 7, we examine a Gross-Prasad type restriction problem formulated for symplectic/metaplectic groups in [GGP] and show how this problem can be resolved. In Section 8, we discuss some other developments, specifically the development of a Langlands-Shahidi method for Mp_{2n} by D. Szpruch, and the development of a theory of endoscopy for Mp_{2n} by W. W. Li. As we mentioned above, the work of W. W. Li is especially exciting, and leads to a number of interesting questions in the context of Mp_{2n} which we list as problems in Section 9. In the final Section 10, we formulate some global problems concerning the discrete automorphic spectrum of Mp_{2n}.

2. Metaplectic Groups

Let $(W_n, \langle -,-\rangle)$ be a symplectic vector space of dimension $2n$ over k, with associated symplectic group $\mathrm{Sp}(W_n)$. The group $\mathrm{Sp}(W_n)$ has a unique two-fold central extension $\mathrm{Mp}(W_n)$ which is called the metaplectic group:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W_n) \longrightarrow \mathrm{Sp}(W_n) \longrightarrow 1.$$

As a set, we may write

$$\mathrm{Mp}(W_n) = \mathrm{Sp}(W_n) \times \{\pm 1\}$$

with group law given by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot c(g_1, g_2))$$

for some 2-cocycle c on $\mathrm{Sp}(W_n)$ valued in $\{\pm 1\}$. Without describing c explicitly, let us describe the restriction of this double cover over a maximal parabolic subgroup of $\mathrm{Sp}(W_n)$. For more details, the reader can consult [K2].

2.1. Parabolic subgroups.

Let X_k be a k-dimensional isotropic subspace of W_n, so that

$$W_n = X_k \oplus W_{n-k} \oplus X_k^*.$$

The stabilizer of X_k in $\mathrm{Sp}(W_n)$ is a maximal parabolic subgroup $P(X_k) = M(X_k) \cdot N(X_k)$. Its unipotent radical is $N(X_k)$ and is Levi factor is

$$M(X_k) \cong \mathrm{GL}(X_k) \times \mathrm{Sp}(W_{n-k}).$$

The metaplecitc covering splits uniquely over the unipotent radical $N(X_k)$ of $P(X_k)$. Thus, we may regard $N(X_k)$ canonically as a subgroup of $\mathrm{Mp}(W_n)$ and one has a Levi decomposition

$$\widetilde{P}(X_k) = \widetilde{M}(X_k) \cdot N(X_k)$$

We want to describe the covering over $M(X_k) \cong \mathrm{GL}(X_k) \times \mathrm{Sp}(W_{n-k})$.

Not surprisingly, the restriction of the covering to $\mathrm{Sp}(W_{n-k})$ is nothing but the unique two-fold cover $\mathrm{Mp}(W_{n-k})$ of $\mathrm{Sp}(W_{n-k})$. The covering over $\mathrm{GL}(X_k)$ can be described as follows. Consider the set

$$\mathrm{GL}(X_k) \times \{\pm 1\}$$

with multiplication law

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot (\det g_1, \det g_2))$$

where $(\det g_1, \det g_2)$ denotes the Hilbert symbol. Then $\widetilde{\mathrm{GL}}(X_k)$ is precisely this double cover of $\mathrm{GL}(X_k)$. Hence, we have

$$\widetilde{M}(X_k) = \left(\widetilde{\mathrm{GL}}(X_k) \times \mathrm{Mp}(W_{n-k}) \right) / \Delta \mu_2.$$

More generally, for any parabolic subgroup P, one has the Levi decomposition $\widetilde{P} = \widetilde{M} \cdot N$ with

$$\widetilde{M} \cong \widetilde{\mathrm{GL}}(k_1) \times_{\mu_2} ... \times_{\mu_2} \widetilde{\mathrm{GL}}(k_r) \times_{\mu_2} \mathrm{Mp}(W_{n-k_1-...-k_r}).$$

2.2. Representations of $\widetilde{\mathrm{GL}}(X_k)$. The (genuine) representation theory of $\widetilde{\mathrm{GL}}(X_k)$ can be easily related to the representation theory of $\mathrm{GL}(X_k)$. Indeed, if we fix an additive character ψ of k, then there is a natural genuine character χ_ψ of $\widetilde{\mathrm{GL}}(X_k)$. Using χ_ψ, one obtains a bijection between $\mathrm{Irr}(\mathrm{GL}(X_k))$ and the set $\mathrm{Irr}(\widetilde{\mathrm{GL}}(X_k))$ of genuine irreducible representations of $\widetilde{\mathrm{GL}}(X_k)$, via:

$$\tau \mapsto \tilde{\tau}_\psi = \tau \otimes \chi_\psi.$$

We stress that this bijection depends on the choice of the additive character ψ.

Note that we could restrict the genuine character χ_ψ to the center $\widetilde{Z}$ of $\mathrm{Mp}(W)$. We denote this character of $\widetilde{Z}$ by χ_ψ as well. This character allows one to define a central sign for irreducible representations σ of $\mathrm{Mp}(W)$, as explained in §5.

2.3. Parabolic Induction. After the above discussion, one sees that given an irreducible representation τ of $\mathrm{GL}(X_k)$ and an irreducible representation π of $\mathrm{Mp}(W_{n-k})$, one has an irreducible representation $\tilde{\tau}_\psi \boxtimes \pi$ of $\widetilde{M}(X_k)$. Thus, one may consider the parabolically induced representation

$$I_{P(X_k),\psi}(\tau, \pi) = \mathrm{Ind}_{\widetilde{P}(X_k)}^{\mathrm{Mp}(W_n)} \tilde{\tau}_\psi \boxtimes \pi \quad \text{(normalized induction)}.$$

More generally, for any parabolic subgroup $P = M \cdot N$ and irreducible representation τ_i of $\mathrm{GL}(k_i)$ and π of $\mathrm{Mp}(W_{n-k_1-...-k_r})$, one has the induced representation

$$I_{P,\psi}(\tau_1, ..., \tau_r, \pi).$$

Though $\mathrm{Mp}(W_n)$ is not a linear group, many basic results regarding the induction and Jacquet functors remain valid. For a justification of this, the reader can consult [HM].

2.4. Orthogonal Groups. Now we come to the orthogonal groups. Let V_n be a vector space of dimension $2n + 1$ over k equipped with a nondegenerate quadratic form q_{V_n} of discriminant 1. Up to isomorphism, there are precisely two such quadratic spaces V. One of them, denoted by V_n^+, has maximal isotropic subspaces of dimension n, whereas the other has maximal isotropic subspaces of dimension $n-1$ and is denoted by V_n^-. As such, we call the former the split quadratic space and the latter the non-split one. We shall write

$$\epsilon(V_n) = \begin{cases} +1 \text{ if } V_n \text{ is split;} \\ -1, \text{ if } V_n \text{ is non-split.} \end{cases}$$

Let $\mathrm{O}(V_n)$ be the associated orthogonal group. Then observe that

$$\mathrm{O}(V_n) = \mathrm{SO}(V_n) \times \{\pm 1\}$$

where $\mathrm{SO}(V_n)$ is the special orthogonal group. The group $\mathrm{SO}(V_n)$ is split precisely when V_n is the split quadratic space.

Given any irreducible representation π of $\mathrm{SO}(V_n)$, there are two extensions of π to $\mathrm{O}(V_n)$, depending on whether the element $-1 \in \mathrm{O}(V_n)$ acts as $+1$ or -1. We denote these two extensions by π^+ and π^- respectively.

2.5. Parabolic subgroups. The parabolic subgroups of $\mathrm{SO}(V_n)$ are stabilizers of flags of isotropic subspaces in V_n. More precisely, if Y_k is a k-dimensional isotropic subspace of V_n, so that

$$V_n = Y_k + V_{n-k} + Y_k^*,$$

then the stabilizer $Q(Y_k) = L(Y_k) \cdot U(Y_k)$ is a maximal parabolic subgroup with unipotent radical $U(Y_k)$ and Levi factor $L(Y_k) \cong \mathrm{GL}(Y_k) \times \mathrm{SO}(V_{n-k})$. More generally, a general parabolic subgroup $Q = L \cdot U$ has Levi subgroup

$$L = \mathrm{GL}(k_1) \times \mathrm{GL}(k_2) \times ... \times \mathrm{GL}(k_r) \times \mathrm{SO}(V_{n-k_1-...-k_r}),$$

3. Weil Representations and Theta Correspondences

In this section, we introduce the Weil representations for $\mathrm{Mp}(W_n) \times \mathrm{O}(V_r)$ and recall the notion of theta correspondence.

3.1. Weil Representation. Fix an additive character ψ of k. Then the group $\mathrm{Mp}(W_n) \times \mathrm{O}(V_r)$ has a natural representation $\Omega_{V_r,W_n,\psi}$ depending on ψ. This representation can be realized on the space $S(X_n^* \otimes V_r)$ of Schwarz-Bruhat functions on $X_n^* \otimes V_r = \mathrm{Hom}(X_n, V_r)$. The action of $\mathrm{Mp}(W_n) \times \mathrm{O}(V_r)$ on $S(X_n^* \otimes V_r)$ via $\Omega_{V_r,W_n,\psi}$ is described as follows.

$$\begin{cases} (\Omega_\psi(h)\phi)(A) = \phi(h^{-1}A), \text{ if } h \in \mathrm{O}(V_r); \\ (\Omega_\psi(n)\phi)(A) = \psi(\frac{1}{2} \cdot \langle n(A), A\rangle) \cdot \phi(A), \text{ if } n \in N(X_n) = \mathrm{Sym}^2 X_n \subset \mathrm{Hom}(X_n^*, X_n); \\ (\Omega_\psi(m,\epsilon)\phi)(A) = \chi_\psi(m,\epsilon) \cdot |\det(m)|^{\frac{1}{2}\dim V_r} \cdot \phi(m^{-1} \cdot A) \text{ if } (m,\epsilon) \in \widetilde{M}(X_n) = \widetilde{\mathrm{GL}}(X_n); \\ \Omega_\psi(w)\phi)(A) = \gamma(\psi \circ q_{V_r})^n \cdot \int_{X_n^* \otimes V_r} \phi(B) \cdot \psi(\langle A, B\rangle)\, dB. \end{cases}$$

Here, in the last equation, w is a certain Weyl group element and $\gamma(\psi \circ q_{V_r})$ is the Weil index associated to the pair (ψ, q_{V_r}). Moreover, in the second equation, with $A \in X_n^* \otimes V_r$, the element $n(A)$ lies in $X_n \otimes V_r$, and the pairing between $X_n \otimes V_r$ and $X_n^* \otimes V_r$ is the tensor product of the natural pairing between X_n and X_n^* and the symmetric bilinear form b_{V_r} associated to the quadratic form q_{V_r} on V_r:

$$b_{V_r}(v_1, v_2) = q_{V_r}(v_1 + v_2) - q_{V_r}(v_1) - q_{V_r}(v_2).$$

3.2. Theta Correspondence. Given an irreducible representation π of $\mathrm{O}(V_r)$, the maximal π-isotypic quotient of $\Omega_{V_r,W_n,\psi}$ has the form $\pi \boxtimes \Theta_{V_r,W_n,\psi}(\pi)$ for some smooth representation $\Theta_{V_r,W_n,\psi}(\pi)$ of $\mathrm{Mp}(W_n)$ (called the big theta lift of π). The maximal semisimple quotient of $\Theta_{V_r,W_n,\psi}(\pi)$ is denoted by $\theta_{V_r,W_n,\psi}(\pi)$ and is called the small theta lift of π.

Similarly, if σ is an irreducible genuine representation of $\mathrm{Mp}(W_n)$, then one has its big theta lift $\Theta_{W_n,V_r,\psi}(\sigma)$ and its small theta lift $\theta_{W_n,V_r,\psi}(\sigma)$, which are smooth representations of $\mathrm{O}(V_r)$.

The following theorem summarizes some basic results of Howe, Kudla [K1], Moeglin-Vigneras-Waldspurger [MVW] and Waldspurger [W3] about the theta correspondence.

Theorem 3.1. *(i) The representation $\Theta_{V_r,W_n,\psi}(\pi)$ is either zero or has finite length.*

(ii) If π is supercuspidal, then $\Theta_{V_r,W_n,\psi}(\pi)$ is either zero or irreducible (and thus is equal to $\theta_{V_r,W_n,\psi}(\pi)$). Moreover, if π and π' are supercuspidal representations such that $\Theta_{V_r,W_n,\psi}(\pi) \cong \Theta_{V_r,W_n,\psi}(\pi')$, then $\pi \cong \pi'$.

(iii) If $p \neq 2$, then $\Theta_{V_r,W_n,\psi}(\pi)$ is either zero or has a unique irreducible quotient, so that $\theta_{V_r,W_n,\psi}(\pi)$ is irreducible. Moreover, for any irreducible representations π and π' of $\mathrm{O}(V_r)$,

$$\theta_{V_r,W_n,\psi}(\pi) \cong \theta_{V_r,W_n,\psi}(\pi') \Longrightarrow \pi \cong \pi'.$$

(iv) The analogous statements hold for $\Theta_{W_n,V_r,\psi}(\sigma)$ and $\theta_{W_n,V_r,\psi}(\sigma)$ if σ is an irreducible genuine representation of $\mathrm{Mp}(W_n)$.

4. Theta Dichotomy

Let us study the theta correspondence for $\mathrm{Mp}(W_n) \times \mathrm{O}(V_n)$ in greater depth. The following theorem was shown in [GGP] and [GS], but the most difficult step is a fundamental result of Kudla-Rallis [KR]; part (b) was also shown independently by C. Zorn [Z].

Theorem 4.1. *One has:*

(a) *given an irreducible representation π of $\mathrm{SO}(V_n)$, exactly one extension of π to $\mathrm{O}(V_n) = \mathrm{SO}(V_n) \times \{\pm 1\}$ has nonzero theta lift to $\mathrm{Mp}(W_n)$;*

(b) *given an irreducible representation σ of $\mathrm{Mp}(W_n)$, σ has nonzero theta lift to $\mathrm{O}(V_n^\epsilon)$ for exactly one ϵ.*

Now one may ask if it is possible to specify, in the context of (a), which extension $\pi^\pm$ of π participates in the theta correspondence with $\mathrm{Mp}(W_n)$. Analogously, given a representation σ in the context of (b), one may ask to which $\mathrm{O}(V_n)$ is the theta lift of σ nonzero. To describe the answers, we need to introduce some more notations.

Recall that we have set: $\epsilon(V_n^\epsilon) = \epsilon$. Further, observe that the sign ϵ in π^ϵ simply encodes the central character of π^ϵ: $\epsilon = \pi^\epsilon(-1)$. On the other hand, for an irreducible genuine representation σ of $\mathrm{Mp}(W_n)$, one may consider its central character ω_σ , which is a genuine character of $\tilde{Z}$ (the preimage in $\mathrm{Mp}(W_n)$ of the center Z of $\mathrm{Sp}(W_n)$). Now using the additive character ψ, one can define a genuine character χ_ψ of $\tilde{Z}$ (see 2.2). We define the central sign $z_\psi(\sigma)$ of σ by

$$z_\psi(\sigma) = \omega_\sigma(-1)/\chi_\psi(-1) \in \{\pm 1\},$$

where we note that the quotient above is independent of the choice of the preimage in $\tilde{Z}$ of $-1 \in Z$.

Now we have:

Theorem 4.2. *(i) Let π be an irreducible representation of $\mathrm{SO}(V_n)$. Then π^ϵ participates in theta correspondence (with respect to ψ) with $\mathrm{Mp}(W_n)$ if and only if*

$$\epsilon = \epsilon(V_n) \cdot \epsilon(1/2, \pi).$$

Here $\epsilon(s, \pi, \psi)$ is the standard epsilon factor defined by Lapid-Rallis [LR] using the doubling method; its value at $s = 1/2$ is independent of ψ.

(ii) Let σ be an irreducible representation of $\mathrm{Mp}(W_n)$. Then σ has nonzero theta lift (with respect to ψ) to $\mathrm{O}(V_n)$ if and only if the central character of σ satisfies:

$$z_\psi(\sigma) = \epsilon(V_n) \cdot \epsilon(1/2, \sigma, \psi) = \epsilon(V_n) \cdot \epsilon(1/2, \Theta_\psi(\sigma)).$$

Here $\epsilon(s, \sigma, \psi)$ is the standard epsilon factor defined in [G1] using the doubling method, following the approach of [LR].

In addition, we have [GS]:

THEOREM 4.3. *Assume that $p \neq 2$. For any irreducible tempered representation π of $\mathrm{O}(V_n)$, one has*

$$\Theta_{V_n,W_n,\psi}(\pi) = \theta_{V_n,W_n,\psi}(\pi).$$

Similarly, for any irreducible tempered representation σ of $\mathrm{Mp}(W_n)$, one has

$$\Theta_{W_n,V_n,\psi}(\sigma) = \theta_{W_n,V_n,\psi}(\sigma).$$

(Note that the representations in these identities may both be zero).

5. Local Langlands Correspondence

We are now ready to announce the LLC for $\mathrm{Mp}(W_n)$, which was first obtained with Gross-Prasad in [GGP]. For a detailed proof, see [GS].

THEOREM 5.1. *Assume that p is odd. For each non-trivial additive character $\psi : k \to \mathbb{C}^\times$, there is a bijection*

$$\Theta_\psi : \mathit{Irr}(\mathrm{Mp}(W_n)) \longleftrightarrow \mathit{Irr}(\mathrm{SO}(V_n^+)) \sqcup \mathit{Irr}(\mathrm{SO}(V_n^-)).$$

This bijection is defined by:

$$\theta_{V_n,W_n,\psi}(\pi^\epsilon) \longleftrightarrow \pi \in \mathit{Irr}(\mathrm{SO}(V_n)),$$

where π^ϵ is the unique extension of π to $\mathrm{O}(V_n)$ which participates in the theta correspondence with $\mathrm{Mp}(W)$.

The archimedean analog of this theorem was due to Adams-Barbasch [AB]. For representations with Iwahori-fixed vectors, an approach to the LLC via isomorphisms of Iwahori-Hecke algebras was established in [GS2].

COROLLARY 5.2. *Assume the local Langlands correspondence for $\mathrm{SO}(V_n^\pm)$. Then one obtains a local Langlands correspondence for $\mathrm{Mp}(W_n)$, i.e. a bijection (depending on ψ)*

$$\mathcal{L}_\psi : \mathit{Irr}(\mathrm{Mp}(W_n)) \longleftrightarrow \Phi(\mathrm{Mp}(W_n))$$

where $\Phi(\mathrm{Mp}(W_n))$ is the set of pairs (ϕ, η) such that

- *$\phi : WD_k \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$ is a $2n$-dimensional symplectic representation of the Weil-Deligne group WD_k of k;*
- *η is an irreducible representation of the (finite) component group*

$$A_\phi = \pi_0(Z_{\mathrm{Sp}_{2n}(\mathbb{C})}(\phi)).$$

As we noted in the introduction, the local Langlands correspondence for $\mathrm{SO}(V_n^\pm)$ is known for $\dim V = 5$ (by [GT1] and [GTW]), and the general case will appear in the forthcoming book of Arthur [A]. We should also mention the work of Jiang-Soudry [JS] who established the LLC for generic representations of $\mathrm{SO}(V_n^+)$.

One may ask if the LLC given in Corollary 5.2 satisfies certain typical properties. For example, one would expect certain natural invariants, such as L-factors and ϵ-factors, to be preserved under the LLC. To a large extent, such questions amount to whether the bijection Θ_ψ satisfies the analogous properties. The following theorem addresses these issues and summarizes certain results obtained jointly with G. Savin [GS] and A. Ichino [GI].

Theorem 5.3. *Suppose that $\pi \in Irr(\mathrm{SO}(V_n))$ and $\sigma \in Irr(\mathrm{Mp}(W_n))$ correspond under Θ_ψ. Then we have:*

(i) π is a discrete series representation if and only if σ is a discrete series representation.

(ii) π is tempered if and only if σ is tempered. Moreover, suppose that

$$\pi \subset I_Q(\tau_1, ..., \tau_r, \pi_0),$$

where Q is a parabolic subgroup of $\mathrm{SO}(V_n)$ with Levi subgroup $\mathrm{GL}_{n_1} \times ... \times \mathrm{GL}_{n_r} \times \mathrm{SO}(V_{n_0})$, the τ_i's are unitary discrete series representations of GL_{n_i}, and π_0 is a discrete series representation of $\mathrm{SO}(V_{n_0})$. Then

$$\sigma \subset I_{\tilde{P}}(\tau_1, ..., \tau_r, \Theta_\psi(\pi_0)),$$

where $\tilde{P}$ is the parabolic subgroup of $\mathrm{Mp}(W_n)$ with Levi subgroup $\tilde{\mathrm{GL}}_{n_1} \times_{\mu_2} ... \times_{\mu_2} \tilde{\mathrm{GL}}_{n_r} \times \mathrm{Mp}(W_{n_0})$. In particular, Θ_ψ gives a bijection between the (isomorphism classes of) irreducible constituents of $I_Q(\tau_1, ..., \tau_r, \pi_0)$ and $I_{\tilde{P}}(\tau_1, ..., \tau_r, \Theta_\psi(\pi_0))$.

(iii) In general, suppose that

$$\pi = J_Q(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \pi_0), \quad s_1 > s_2 > .... > s_r > 0$$

is a Langlands quotient of $\mathrm{SO}(V)$, where Q is as in (ii), the τ_i's are unitary tempered representations of GL_{n_i}, and π_0 is a tempered representation of $\mathrm{SO}(V_{n_0})$. Then

$$\sigma = J_{\tilde{P}}(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \Theta_\psi(\pi_0))$$

where $\tilde{P}$ is as in (ii).

(iv) If π and σ are discrete series representations, then

$$deg(\pi) = deg(\sigma),$$

where deg denotes the formal degree with respect to appropriately chosen Haar measures.

(v) If π is a generic representation of $\mathrm{SO}(V_n^+)$, then σ is a ψ-generic representation of $\mathrm{Mp}(W_n)$. If σ is ψ-generic and tempered, then π is generic.

(vi) If π is a discrete series representation of $\mathrm{SO}(V_n)$ and ρ is an irreducible unitary supercuspidal representation of GL_r, then one has a Plancherel measure $\mu(s, \pi \times \rho, \psi)$ associated to the induced representation $I_P(s, \pi \boxtimes \rho)$. If $\sigma = \Theta_\psi(\pi)$, then one has

$$\mu(s, \pi \times \rho, \psi) = \mu(s, \sigma \times \rho, \psi).$$

(vii) If χ is a 1-dimensional character of GL_1, then one has

$$\begin{cases} L(s, \pi \times \chi) = L_\psi(s, \sigma \times \chi) \\ \epsilon(s, \pi \times \chi, \psi) = \epsilon(s, \sigma \times \chi, \psi) \end{cases}$$

where the local factors in question are those defined in [LR] and [G1] using the doubling method of Piatetski-Shapiro and Rallis [PSR].

(viii) Assume that π is generic, so that σ is ψ-generic. Then for any irreducible representation ρ of GL_r, one has the equalities

$$\begin{cases} L(s, \pi \times \rho) = L(s, \sigma \times \rho, \psi) \\ \epsilon(s, \pi \times \rho, \psi) = \epsilon(s, \sigma \times \rho, \psi). \end{cases}$$

Here the factors on the LHS are those defined by Shahidi [Sh], and those on the RHS are defined by Szpruch [Sz].

It is not difficult to see the parametrization L_ψ) is determined by the properties of the above theorem, at least on the level of L-packets.

6. Variation of ψ

One remaining issue is the variation of the bijection Θ_ψ or $\mathcal{L}_\psi$ when ψ varies. In this section, we shall assume the LLC for $\mathrm{SO}(V_n^\pm)$.

For $c \in k^\times$, let ψ_c be the character $\psi_c(x) = \psi(cx)$. Then we would like to know the relation between $\mathcal{L}_\psi(\sigma)$ and $\mathcal{L}_{\psi_c}(\sigma)$. Recall that $\mathcal{L}_\psi(\sigma) = (\phi, \eta)$, where

$$\phi : WD_k \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$$

and η is an irreducible character of the component group $A_\phi = \pi_0(Z_{\mathrm{Sp}_{2n}}(\phi))$. The component group A_ϕ can be explicitly described as follows. If we decompose ϕ as a $2n$-dimensional representation:

$$\phi = \bigoplus_i n_i \phi_i,$$

then

$$A_\phi = \bigoplus_{i:\phi_i \text{ is symplectic}} \mathbb{Z}/2\mathbb{Z}a_i.$$

Now we have the following theorem which verifies [GGP, Conjecture 11.3]:

Theorem 6.1. *For $\sigma \in Irr(\mathrm{Mp}(W_n))$ and $c \in k^\times$, let $\mathcal{L}_\psi(\sigma) = (\phi, \eta)$ and $\mathcal{L}_{\psi_c}(\sigma) = (\phi_c, \eta_c)$. Then:*

(i) $\phi_c = \phi \otimes \chi_c$, where χ_c is the quadratic character defined by $\chi_c(x) = (c, x)$.

It follows by (i) that we have a canonical identification of component groups $A_\phi = A_{\phi_c}$, so that it makes sense to compare η and η_c.

(ii) the characters η and η_c are related by:

$$\eta_c(a_i)/\eta(a_i) = \epsilon(1/2, \phi_i) \cdot \epsilon(1/2, \phi_i \otimes \chi_c) \cdot \chi_c(-1)^{\frac{1}{2}\dim \phi_i}.$$

It is interesting to note that the proof of this theorem uses the Gross-Prasad conjecture for tempered representations of special orthogonal groups, which was recently shown by Waldspurger in a remarkable series of papers [W4-7]. A consequence of the theorem is:

Corollary 6.2. *Suppose that $\mathcal{L}_\psi(\sigma) = (\phi, \eta)$. Then σ has ψ_c-Whittaker model if and only if*

$$\eta(a_i) = \epsilon(1/2, \phi_i, \psi) \cdot \epsilon(1/2, \phi_i \otimes \chi_c, \psi) \cdot \chi_c(-1)^{\frac{1}{2}\dim \phi_i}.$$

7. The Gross-Prasad Conjecture

With a good understanding of the LLC for $\mathrm{Mp}(W_n)$, we can use it to address certain restriction problems commonly studied in representation theory. In [GGP], a class of restriction problems was formulated for symplectic/metaplectic groups. This problem concerns the Fourier-Jacobi models, but we shall treat only the basic case here. The general case is contained in [G2].

Let $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W_n))$ and $\tau \in \mathrm{Irr}(\mathrm{Sp}(W_n))$. Also, let ω_ψ be the Weil representation of $\mathrm{Mp}(W_n)$ associated to the additive charcater ψ. Then it was recently shown by B. Y. Sun [Sun] that

$$\dim \mathrm{Hom}_{\mathrm{Sp}(W_n)}(\sigma \otimes \tau \otimes \omega_\psi^\vee, \mathbb{C}) \leq 1.$$

The GP conjecture formulated in [GGP] gives a precise criterion for the non-vanishing of this Hom space.

Suppose that $\mathcal{L}_\psi(\sigma) = (\phi, \eta)$ with $\phi = \oplus_i n_i \phi_i$ and η is a character of

$$A_\phi = \prod_{i:\phi_i \text{ symplectic}} \mathbb{Z}/2\mathbb{Z}\, a_i.$$

Similarly, under the LLC for $\mathrm{Sp}(W_n)$ (which depends also on ψ [see GGP]), one has $\mathcal{L}_\psi(\tau) = (\phi_0, \eta_0)$, where

$$\phi_0 = \bigoplus_j m_j \phi_{0,j} : WD_k \longrightarrow \mathrm{SO}_{2n+1}(\mathbb{C})$$

and η_0 is a character of

$$A_{\phi_0} \subset \prod_{j:\phi_{0,j} \text{ orthogonal}} \mathbb{Z}/2\mathbb{Z}\, b_j \qquad \text{(of index 2)}.$$

Using the theta correspondence, one can transport the restriction problem in question to one on the orthogonal side, where it has been solved by Waldspurger. To effect this transfer, in addition to using the results of [GS] discussed in §5, one also needs a deep understanding of the theta correspondence for $\mathrm{Sp}_{2n} \times \mathrm{O}_{2n+2}$ and $\mathrm{O}_{2n} \times \mathrm{Sp}_{2n}$. For example, one would need to know that the big theta lift is equal to the small theta lift and how the theta correspondence is described in terms of the parameters (ϕ_0, η_0). When the representations involved are tempered, such questions have, to a large extent, been addressed by Muić [M1,2,3] and Moeglin [Mo3]. However, as Moeglin explained to me, though the above theta correspondences for tempered representations were completely described in [Mo3] in terms of L-parameters, it has not been verified that the internal parametrization in an L-packet by the characters η_0 used in [Mo3] agrees with the one furnished by Arthur's work [A]. With this caveat, one can deduce:

"Theorem" 7.1. *Suppose that $\sigma \in Irr(\mathrm{Mp}(W_n))$ and $\tau \in Irr(\mathrm{Sp}(W_n))$ are tempered representations, with $\mathcal{L}_\psi(\sigma) = (\phi, \eta)$ and $\mathcal{L}_\psi(\tau) = (\phi_0, \eta_0)$. Then*

$$\mathrm{Hom}_{\mathrm{Sp}(W_n)}(\sigma \otimes \tau \otimes \omega_\psi^\vee, \mathbb{C}) \neq 0$$

if and only if

$$\eta(a_i) = \epsilon(1/2, \phi_i \otimes (1 + \phi_0), \psi)$$

and η_0 is the restriction to $A^+_{\phi_0}$ of the character on A_{ϕ_0} defined by

$$\eta_0(b_i) = \epsilon(1/2, \phi_{0,i} \otimes \phi, \psi) \cdot \det \phi_{0,i}(-1)^n.$$

Indeed, the statement about η is established unconditionally and it is the statement about η_0 that requires the compatibility noted before the "Theorem".

8. Other Recent Developments

In this section, we discuss some other recent developments in the representation theory of $\mathrm{Mp}(W_n)$.

8.1. The Langlands-Shahidi method (D. Szpruch). In his thesis [Sz], Dani Szpruch has developed the Langlands-Shahidi theory to define certain local L-factors and ϵ-factors for ψ-generic representations of $\mathrm{Mp}(W_n)$. More precisely, if σ is a ψ-generic (genuine) representation of $\mathrm{Mp}(W_n)$ and ρ is an irreducible generic representation of GL_r, then following the work of Shahidi [Sh], Szpruch defined the local factors $L(s, \sigma \times \rho, \psi)$ and $\epsilon(s, \sigma \times \rho, \psi)$. Moreover, he verified that these local factors satisfy some expected properties, which characterized them uniquely. These are the local factors which intervene in Theorem 5.3(viii) above.

8.2. A theory of endoscopy (W. W. Li). In his thesis work [Li1,3], Wenwei Li has developed a definitive theory of endoscopy for $\mathrm{Mp}(W_n)$, extending earlier works of Adams [Ad] and Renard [Re1,2]. Here is a summary of [Li1,3]:

- For each pair (a,b) with $a+b=n$, Li defined transfer factors
$$\Delta_{a,b} : \mathrm{Mp}(W_n) \times (\mathrm{SO}(V_a^+) \times \mathrm{SO}(V_b^+)) \to \mathbb{C}.$$
- Li showed the existence of transfer mappings
$$C_c^\infty(\mathrm{Mp}(W_n)) \xrightarrow{t_{a,b}} C_c^\infty(\mathrm{SO}(V_a^+) \times \mathrm{SO}(V_b^+)),$$
relating orbital integrals on $\mathrm{Mp}(W_n)$ to stable orbital integrals on $\mathrm{SO}(V_a^+) \times \mathrm{SO}(V_b^+)$. This gives rise to a dual map
$$\mathcal{D}_{st}(\mathrm{SO}(V_a^+) \times \mathrm{SO}(V_b^+)) \xrightarrow{t_{a,b}^*} \mathcal{D}_{gen}(\mathrm{Mp}(W_n)),$$
from the space of stable distributions on $\mathrm{SO}(V_a^+) \times \mathrm{SO}(V_b^+)$ to the space of invariant (genuine) distributions on $\mathrm{Mp}(W_n)$.
- Finally, Li established the (ordinary and weighted) fundamental lemma for the transfer $t_{a,b}$, by using Harish-Chandra descent to reduce the desired identity to the nonstandard fundamental lemma for the Lie algebras $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$, which was formulated by Waldspurger and established by B.C. Ngo [N] and Chaudourard-Laumon [CL1,2] .

Thus, one sees that the elliptic endoscopic groups of $\mathrm{Mp}(W_n)$ are the groups $\mathrm{SO}(V_a^+) \times \mathrm{SO}(V_b^+)$ with $a+b=n$.

9. Some Problems

In this section, we formulate some problems which are suggested by the prior discussion.

9.1. Stability and Character identities for $\mathrm{Mp}(W_n)$. One hallmark of the LLC is its relation with character theory; in particular the notion of stability and the theory of endoscopy. The main results discussed in this paper do not address these issues. But, with the theory of W. W. Li mentioned in the previous section, it now makes sense to consider the following problems:

- If $\Pi_{\psi,\phi}$ is the tempered L-packet of $\mathrm{Mp}(W_n)$ associated to the L-parameter ϕ, then show that the sum
$$\sum_{\sigma \in \Pi_{\psi,\phi}} \mathrm{Trace}(\sigma)$$
is a stable distribution (as defined in [Li1]);
- Show that other linear combinations of $\mathrm{Trace}(\sigma)$'s are transfers of corresponding stable distribution from endoscopic groups.

9.2. Depth zero supercuspidal L-packets. In the paper [DR], Debacker and Reeder constructed certain L-packets of depth zero supercuspidal representations associated to certain tamely ramified L-parameters of arbitrary unramified groups G. Moreover, they showed that their L-packets are stable. In his thesis [K], Kaletha showed further that these L-packets satisfy the character identities required by the theory of endoscopy, thus showing that the L-packets of Debacker-Reeder are bona-fide L-packets.

It is natural to ask if a similar construction as [DR] can be carried out for Mp_{2n} (or more general covering groups). Depth zero supercuspidal representations for general covering groups have been constructed by Howard-Weissman [HW], and the issue is how to organize them into L-packets. One would then like to show that these L-packets agree with those defined by Theorem 5.1. In addition, with W. W. Li's theory of endoscopy in place, one would like to verify the stability properties and character identities of these depth zero supercuspidal L-packets. This is an ongoing project with M. Weissman.

10. Square-integrable automorphic forms

The main results of this paper, which are all local in nature, are but local preparations for the global problem of determining the automorphic discrete spectrum of $\mathrm{Mp}(W_n)$, when W_n is defined over a number field F with adele ring $\mathbb{A}$. Thus, we are interested in the decomposition of the space

$$L^2_{disc} = L^2_{gen,disc}(\mathrm{Sp}(W_n)(F)\backslash \mathrm{Mp}(W_n)(\mathbb{A}))$$

of genuine (square-integrable) automorphic forms on $\mathrm{Mp}(W_n)$. More precisely, one would like a description of L^2_{disc} in the same spirit as Arthur's results for the classical groups [A]. When $n = 1$, such a result was obtained by Waldspurger, who described the automorphic discrete spectrum of Mp_2 in terms of that of SO_3 by a deep study of the global theta correspondence for $\mathrm{Mp}_2 \times \mathrm{SO}_3$. In this final section, we shall consider this global problem for general W. In fact, we shall only consider the tempered part $L^2_{temp} \subset L^2_{disc}$.

10.1. The automorphic discrete spectrum. Since the local L-parameters of $\mathrm{Mp}(W_n)$ are $2n$-dimensional symplectic representations of WD_k, it is reasonable to expect that the discrete global L-parameters for $\mathrm{Mp}(W_n)$ are discrete global L-parameters for $\mathrm{SO}(V_n)$. Thus, a discrete global L-parameter of $\mathrm{Mp}(W_n)$ should be a multiplicity free $2n$-dimensional symplectic representation of the hypothetical Langlands group L_F:

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r$$

with each irreducible summand ϕ_i also symplectic. Following Arthur, one can reformulate this notion without referring to the hypothetical L_F. In this reformulation, such a discrete global parameter ϕ is given by the data of a collection of pairwise inequivalent cuspidal representations π_i of GL_{2n_i} with

(1) $\sum_i n_i = n$ and
(2) $L(\pi_i, \wedge^2, s)$ having a pole at $s = 1$ for each i.

Given ϕ as above, one inherits:

- local L-parameters ϕ_v for $\mathrm{Mp}(W_n \otimes F_v)$and thus local L-packets
$$\Pi_{\phi_v,\psi_v} = \{\sigma_{\eta_v} : \eta_v \in \mathrm{Irr}(A_{\phi_v})\}$$
- taking tensor products of these local packets, one gets a global packet
$$\Pi_{\phi,\psi} = \{\sigma_\eta = \bigotimes_v \sigma_{\eta_v} : \eta \in \mathrm{Irr}(\prod_v A_{\phi_v})\}$$
- a global component group
$$A_\phi = \mathbb{Z}/2\mathbb{Z}a_1 \times ... \times \mathbb{Z}/2\mathbb{Z}a_r$$

and a diagonal map

$$\Delta : A_\phi \to \prod_v A_{\phi_v}$$

10.2. The Langlands-Arthur multipicity formula. The following conjecture was made in [GGP]:

CONJECTURE 10.1. *One has*

$$L^2_{temp} = \bigoplus_{\phi \text{ discrete symplectic}} L^2_{\phi,\psi}$$

with

$$L^2_{\phi,\psi} = \bigoplus_{\eta \in \mathrm{Irr}(\prod_v A_{\phi_v})} m_\eta \cdot \sigma_\eta,$$

where

$$m_\eta = \langle \Delta^*(\eta), \epsilon_\psi \rangle,$$

and ϵ_ψ is the quadratic character of A_ϕ given by

$$\epsilon_\psi(a_i) = \epsilon(1/2, \pi_i).$$

In particular, the multiplicity formula given in this conjecture is the metaplectic analog of the Langlands-Arthur multiplicity formula for linear groups, and generalizes Waldspurger's results for $\dim W_n = 2$.

Here are some global problems one may wish to pursue.

10.3. Global theta correspondence. Study the global theta correspondence for $\mathrm{Mp}(W_n) \times \mathrm{SO}(V_n)$, as V_n ranges over all quadratic spaces over F of discriminant 1, and use this to verify the above conjecture. This will be the extension of Waldspurger's work to the higher dimension case. The case when $n = 2$ is an ongoing project with C. Zorn.

10.4. Stable trace formula. In a recent preprint [Li2], W. W. Li has begun the study of the trace formula for general covering groups. He has also begun the stabilization of the trace formula for $\mathrm{Mp}(W_n)$. Though a lot of work still remains to be done, one may expect the stabilization to give an identity of the form:

$$\mathrm{TF}_{\mathrm{Mp}(W_n)}(f) = \sum_{a+b=n} c_{a,b} \cdot \mathrm{STF}_{\mathrm{SO}(V_a^+) \times \mathrm{SO}(V_b^+)}(t_{a,b}(f)),$$

where TF stands for "trace formula" , STF for "stable trace formula" and $c_{a,b}$ is equal to $1/4$ if $ab \neq 0$ and $1/2$ otherwise. By stabilizing only the elliptic part, one may hope to obtain a simple stable trace formula for $\mathrm{Mp}(W)$ and use it to establish:

- the stability properties of tempered local L-packets and character identities for the L-packets of $\mathrm{Mp}(W_n)$;
- the multiplicity formula for cuspidal representations of $\mathrm{Mp}(W_n)$ with sufficiently many supercuspidal local components.

10.5. Non-tempered part. Finally, one would like to formulate a global conjecture for the non-tempered part of the discrete spectrum, and use the above techniques to verify it.

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