ON AN EXACT MASS FORMULA OF SHIMURA

WEE TECK GAN, JONATHAN P. HANKE, and JIU-KANG YU

Abstract

In a series of recent papers, G. Shimura obtained an exact formula for the mass of a maximal lattice in a quadratic or hermitian space over a totally real number field. Using Bruhat-Tits theory, we obtain a quick and more conceptual proof of his formula when the form is totally definite.

1. Introduction

In a series of recent papers (see [S1], [S2], [S3]), G. Shimura obtained an explicit formula for the mass of a maximal lattice in a quadratic or hermitian space over a totally real number field k. (In [S3], k is actually arbitrary.) Recall that a lattice Λ is said to be maximal for a quadratic or hermitian form $\langle -, - \rangle$ if, for all $x \in \Lambda$, $\langle x, x \rangle$ lies in the ring of integers A of k and Λ is maximal with respect to this property. For such a lattice, the mass formula was obtained by Shimura as a consequence of his theory of Euler products and Eisenstein series for the corresponding orthogonal or unitary group.

On the other hand, if G is a connected reductive group over k, which is anisotropic at all archimedean places and quasi-split at all finite places, an explicit mass formula was obtained in [GrG] for an open compact subgroup

$$K = G(k \otimes \mathbb{R}) \times \prod_{v<\infty} K_v$$

of the adele group $G(\mathbb{A})$, with K_v a certain special open compact subgroup at each finite place v. This is an extension of a fundamental result of G. Prasad [P, Theorem 1.6]. As remarked at the end of [GrG], the restriction that G be quasi-split at all finite places is not necessary; if one replaces K_v by an Iwahori subgroup J_v at each finite place where G is not quasi-split, one can still obtain an explicit formula for the mass of the corresponding open compact subgroup. This is carried out in the following section, and the aim of the present paper is to rederive Shimura's formula from the above general mass formula, at least in the cases where the quadratic or hermitian form is totally definite.

DUKE MATHEMATICAL JOURNAL
Vol. 107, No. 1, © 2001
Received 10 April 2000.
2000 *Mathematics Subject Classification*. Primary 11E12, 11E41, 20G25.

Our derivation of Shimura's formula is based on the observation that the stabilizer K_v of a maximal lattice at a finite place is always a maximal parahoric subgroup of the orthogonal or unitary group G_v. In fact, in all but two cases, it is a special maximal compact subgroup, even at the places where G_v is not unramified, and in all cases, it corresponds to a vertex at an end of the relative local Dynkin diagram of G_v. We enumerate the various possibilities in Sections 3, 6, and 8, for the cases of a hermitian, quadratic, and quaternionic hermitian form, respectively. In particular, we describe the maximal reductive quotient of the special fiber of the smooth integral group scheme associated to the relevant maximal parahoric subgroup by Bruhat-Tits theory. These results are then used in Sections 4, 7, and 9 to compare the mass formula obtained in Section 2 with that of Shimura.

We emphasize here that the formula obtained in Section 2 expresses the mass in terms of the Tamagawa number of G, and hence we need to assume that the Tamagawa number is known. This is of course the case, by the work of many people. Shimura's proof of the mass formula, on the other hand, does not require the knowledge of the Tamagawa number. Our proof provides a more conceptual explanation of the form of the local factors at the bad places. We are also able to generalize Shimura's results slightly, in that we do not need to assume that the discriminant of the form is a unit when the form has odd rank.

Finally, we note that in the case of an even-rank hermitian form, our formula appears to differ from that of Shimura. Namely, at each finite place where the unitary group G_v is not quasi-split, the corresponding local factor in Shimura's formula contains a λ-invariant, which is absent in our formula. However, in the appendices, we resolve this discrepency by showing, using the theories of Bruhat and Tits and of Moy and Prasad, that Shimura's λ-invariant is in fact always equal to 1.

2. A general mass formula

Throughout this paper, k is a totally real number field of degree d over $\mathbb{Q}$, with ring of integers A and adele ring $\mathbb{A}$. Moreover, v denotes a finite place of k, and k_v denotes the corresponding completion of k, with ring of integers A_v and uniformizer π_v. The order of the residue field A_v/π_v is denoted by q_v.

Let G be a connected reductive group over k of absolute rank l, with $G(k \otimes \mathbb{R})$ compact. We let H be the quasi-split inner form of G. Let S_G be the set of finite places v such that $G_v = G \times_k k_v$ is not quasi-split. Since G is unramified almost everywhere, S_G is a finite set, and for $v \notin S_G$, $G_v \cong H_v$. Recall that a canonical integral model $\underline{H}_v^0$ of H_v was defined in [Gr, Section 4] using the theory of Bruhat and Tits. Hence $\underline{H}_v^0$ is a smooth affine group scheme over A_v with generic fiber H_v and connected special fiber, and it is associated to a special vertex in the reduced building of H_v. Indeed, if H_v is simply connected, then $\underline{H}_v^0(A_v)$ is a special maximal compact subgroup of $H(k_v)$, and if H_v is unramified, then $\underline{H}_v^0(A_v)$ is hyperspecial. Let $\overline{H}_v^0$ be the reductive quotient of the special fiber of $\underline{H}_v^0$. Then $\overline{H}_v^0$ is a connected reductive group over A_v/π_v, which is necessarily quasi-split.

Recall that a canonical Haar measure $|\omega_{G_v}|$ on $G(k_v)$ was defined in [Gr, Section 4]. Indeed, if ω_{H_v} is an invariant differential of top degree on $\underline{H}_v^0$ with nonzero reduction on the special fiber and if $|\omega_{H_v}|$ is the associated Haar measure on $H(k_v)$, then $|\omega_{G_v}|$ is the pullback of $|\omega_{H_v}|$ under an inner twisting $\varphi : G_v \to H_v$. An alternative definition of $|\omega_{G_v}|$ was given in [GrG]. Finally, recall that in [Gr] a motive $M = M_G$ of Artin-Tate type was attached to G. Let $L(M)$ be the L-function of M evaluated at $s = 0$. Then the restriction of M to a decomposition subgroup at v is isomorphic to the motive M_v of G_v, and the local L-factor $L_v(M)$ of $L(M)$ is equal to the L-function of M_v evaluated at $s = 0$. Further, M_H is isomorphic to M.

Now let

$$K = G(k \otimes \mathbb{R}) \times \prod_v K_v \tag{2.1}$$

be an open compact subgroup of $G(\mathbb{A})$ such that $K_v = \underline{G}_v(A_v)$ is the group of integral points of the smooth integral model $\underline{G}_v$ of G_v associated to a parahoric subgroup by Bruhat-Tits theory. Let $S \supset S_G$ be the finite set of finite places v of k such that $\underline{G}_v \not\cong \underline{H}_v^0$. Note that, for $v \in S$, $\underline{G}_v$ may not be connected. This differs from the usage of the term "parahoric" in [BT1]. Let μ_K be the Haar measure of $G(\mathbb{A})$ which gives K volume 1. Then the mass of K is, by definition,

$$\mathrm{Mass}(K) = \int_{G(k)\backslash G(\mathbb{A})} \mu_K. \tag{2.2}$$

Our aim in this section is to give an explicit formula for $\mathrm{Mass}(K)$.

Let $\mu = \mu_{G(k\otimes\mathbb{R})} \times \prod_v \mu_v$ be the Haar measure on $G(\mathbb{A})$, where $\mu_{G(k\otimes\mathbb{R})}$ gives $G(k \otimes \mathbb{R})$ volume 1, and where

$$\mu_v = L_v\big(M^\vee(1)\big) \cdot |\omega_{G_v}|. \tag{2.3}$$

By [Gr, Proposition 4.7], for $v \notin S$, μ_v gives $K_v \cong \underline{H}_v^0(A_v)$ volume 1. Now the proof of [GrG, Proposition 10.7] gives

$$\int_{G(k)\backslash G(\mathbb{A})} \mu = \frac{1}{2^{ld}} \cdot L(M) \cdot \tau(G), \tag{2.4}$$

where $\tau(G)$ is the Tamagawa number of G. In particular, if $S = S_G$ is empty, then $\mu = \mu_K$ and (2.4) is just [GrG, Proposition 10.7]. In general, we have

$$\mu_K = \left(\prod_{v\in S} \lambda_v\right) \cdot \mu, \tag{2.5}$$

where

$$\lambda_v = \left(L_v\big(M^\vee(1)\big) \cdot \int_{K_v} |\omega_{G_v}| \right)^{-1}. \tag{2.6}$$

It remains to obtain a more explicit expression for λ_v.

Let $\overline{G}_v$ be the maximal reductive quotient of the special fiber of $\underline{G}_v$. Then $\overline{G}_v$ is a possibly disconnected reductive group over A_v/π_v, and the natural projection

$$\underline{G}_v(A_v) \longrightarrow \overline{G}_v\big(A_v/\pi_v\big)$$

is surjective. Let J_v be the inverse image, under the natural projection, of a Borel subgroup of the connected component of $\overline{G}_v$. Then J_v is an Iwahori subgroup of $G(k_v)$ and is the group of integral points of a smooth integral model $\underline{J}_v$ of G_v. The maximal reductive quotient $\overline{J}_v$ of the special fiber of $\underline{J}_v$ is isomorphic to the (connected) maximally split maximal torus of $\overline{G}_v$. Hence, if $N(\overline{G}_v)$ denotes the number of positive roots of $\overline{G}_v$ over an algebraic closure of A_v/π_v, then

$$\#\big(K_v/J_v\big) = \frac{\#\overline{G}_v\big(A_v/\pi_v\big)}{\#\overline{J}_v\big(A_v/\pi_v\big) \cdot q_v^{N(\overline{G}_v)}}. \tag{2.7}$$

Hence,

$$\int_{K_v} |\omega_{G_v}| = \frac{q_v^{-N(\overline{G}_v)} \cdot \#\overline{G}_v\big(A_v/\pi_v\big)}{\#\overline{J}_v\big(A_v/\pi_v\big)} \cdot \int_{J_v} |\omega_{G_v}|. \tag{2.8}$$

Now let ν_{G_v} be an invariant differential of top degree on $\underline{J}_v$ with nonzero reduction on the special fiber, and let $|\nu_{G_v}|$ be the associated Haar measure on $G(k_v)$. Then we have (see [Gr, p. 295])

$$|\omega_{G_v}| = q_v^{-N(\overline{H}_v^0)} \cdot |\nu_{G_v}|, \tag{2.9}$$

where we recall that H_v is the quasi-split inner form of G_v. Since

$$\int_{J_v} |\nu_{G_v}| = q_v^{-\dim(\overline{J}_v)} \cdot \#\overline{J}_v\big(A_v/\pi_v\big) \tag{2.10}$$

and

$$L_v\big(M^\vee(1)\big)^{-1} = q_v^{-\dim(\overline{H}_v^0)} \cdot \#\overline{H}_v^0\big(A_v/\pi_v\big), \tag{2.11}$$

we have, after a short computation,

$$\lambda_v = \frac{q_v^{-N(\overline{H}_v^0)} \cdot \#\overline{H}_v^0\big(A_v/\pi_v\big)}{q_v^{-N(\overline{G}_v)} \cdot \#\overline{G}_v\big(A_v/\pi_v\big)}. \tag{2.12}$$

We summarize the above considerations in the following proposition.

PROPOSITION 2.13
Let $K = G(k \otimes \mathbb{R}) \times \prod_v K_v$ be an open compact subgroup of $G(\mathbb{A})$, with $K_v = \underline{G}_v(A_v)$ a parahoric subgroup for all v, and let S be the finite set of finite places where $\underline{G}_v \ncong \underline{H}_v^0$. Then

$$\mathrm{Mass}(K) = \left(\frac{1}{2^{ld}} \cdot L(M) \cdot \tau(G)\right) \cdot \prod_{v \in S} \lambda_v,$$

with λ_v given by (2.12).

This is the explicit mass formula that we seek. The point is that the local factor λ_v is effectively computable by Bruhat-Tits theory.

3. Hermitian spaces

The purpose of this and the following section is to prove [S1, Theorem 24.4] using Proposition 2.13. In this section, we consider hermitian spaces over a nonarchimedean local field. Since the discussion is purely local, we suppress v from the notation. Hence let F be a nonarchimedean local field of characteristic zero, and let E be a quadratic extension of F. The nontrivial automorphism of E over F is given by $x \mapsto \overline{x}$. Further, the ring of integers of F (resp., E) is denoted by A (resp., A_E), with uniformizer π (resp., π_E). Let $q = \#A/\pi$, and let $\mathfrak{D}$ be the different of E/F.

Now let V denote an m-dimensional vector space over E, equipped with a nondegenerate hermitian form $\langle -, - \rangle$, and let G be the corresponding unitary group. There are exactly two isomorphism classes of rank m hermitian spaces over E, classified by their discriminant, which is an invariant taking value in $F^\times/\mathbb{N}E^\times$. We choose a representative for each class of $F^\times/\mathbb{N}E^\times$ in A, with minimum possible valuation. Since $F^{\times 2} \subset \mathbb{N}E^\times$, each representative is either a unit or a uniformizer. Below, we enumerate the possible hermitian spaces.

Consider first the case when $m = 2n + 1$ is odd. Let $\mathbb{H}$ be the split rank-two hermitian space. Hence, $\mathbb{H}$ has basis $\{e, f\}$ over E and is equipped with the hermitian form $\langle -, - \rangle$ given by

$$\begin{pmatrix} \langle e, e\rangle & \langle e, f\rangle \\ \langle f, e\rangle & \langle f, f\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, for each $\alpha \in F^\times/\mathbb{N}E^\times$, let

$$V_\alpha = \mathbb{H}^n \oplus E_\alpha, \tag{3.1}$$

where E_α is the rank-one hermitian space over E with $\langle x, y\rangle = \alpha x \overline{y}$, for $x, y \in E$. This gives the two hermitian spaces of rank m over E. Moreover, the corresponding unitary groups G of the two spaces are isomorphic and quasi-split.

Table 1
Odd Unitary Groups

E	α	$\overline{G}$	$\overline{H}^0$	λ
unramified	$\alpha \in A^\times$	U_{2n+1}	U_{2n+1}	1
unramified	$\alpha \notin A^\times$	$U_{2n} \times U_1$	U_{2n+1}	$(q^{2n+1}+1)/(q+1)$
ramified	any α	$\mathbb{Z}/2\mathbb{Z} \times \mathrm{Sp}_{2n}$	SO_{2n+1}	$1/2$

Now let Λ be a maximal lattice in V. By definition, Λ is an A_E-module such that $\Lambda \otimes_{A_E} E \cong V$, and it is maximal with respect to the property that $\langle x, x\rangle \in A$ for all $x \in \Lambda$. Any two maximal lattices in V are conjugate under G (see [S1]). A maximal lattice Λ_α in V_α can be described as follows. Let

$$\Delta = A_E e \oplus \mathfrak{D}^{-1} f \tag{3.2}$$

be an A_E-lattice in $\mathbb{H}$. Then we have

$$\Lambda_\alpha = \Delta^n + A_E. \tag{3.3}$$

As shown in [T, Section 3.11], the stabilizer of Λ_α is a maximal parahoric subgroup $\underline{G}(A)$. In Table 1, we give the type of the maximal reductive quotient $\overline{G}$, as well as the local factor λ of (2.12). Here, and in later tables, we use U_m to denote a unitary group in m variables over A/π, Sp_{2n} to denote a symplectic group in $2n$ variables, SO_{2n+1} to denote the special orthogonal group in $2n+1$ variables, and $\mathbb{Z}/2\mathbb{Z}$ to denote the constant group scheme of order 2. Moreover, we let O_{2n} denote the split orthogonal group in $2n$ variables, and we let SO_{2n} denote its connected component. Note that there is a morphism $d_n : O_{2n} \to \mathbb{Z}/2\mathbb{Z}$, for which SO_{2n} is the kernel. Indeed, when the residue characteristic of F is different from 2, d_n is simply the determinant map, whereas if the residue characteristic is 2, d_n is the Dickson invariant. More uniformly, SO_{2n} is the subgroup of O_{2n} which acts trivially on the center of the even Clifford algebra of the underlying quadratic space.

Now we come to the case when $m = 2n$ is even. For $\alpha \in \mathbb{N}E^\times$, let

$$V_\alpha = \mathbb{H}^n. \tag{3.4}$$

The corresponding unitary group G is quasi-split. Now suppose that $\alpha \notin \mathbb{N}E^\times$. Let D be the division algebra over F with underlying vector space $E \oplus E \cdot z$, and with multiplication rules

$$z^2 = \alpha, \tag{3.5}$$
$$x \cdot z = z \cdot \overline{x}.$$

Table 2
Even Unitary Groups

E	α	$\overline{G}$	$\overline{H}^0$	λ
unramified	$\alpha \in \mathbb{N}E^\times$	U_{2n}	U_{2n}	1
unramified	$\alpha \notin \mathbb{N}E^\times$	$U_{2n-1} \times U_1$	U_{2n}	$\big(q^{2n}-1\big)/(q+1)$
ramified	$\alpha \in \mathbb{N}E^\times$	Sp_{2n}	Sp_{2n}	1
ramified	$\alpha \notin \mathbb{N}E^\times$	$\mathrm{Sp}_{2n-2} \times {}^2O_2$	Sp_{2n}	$1/2 \cdot \big(q^{2n}-1\big)/(q+1)$

Then D is the unique quaternion algebra over F; and let A_D denote its ring of integers, with uniformizer π_D. Equip D with the hermitian form

$$\langle x_1 + x_2 \cdot z, y_1 + y_2 \cdot z \rangle = x_1 \overline{y}_1 - \alpha x_2 \overline{y}_2. \tag{3.6}$$

Then we have

$$V_\alpha = \mathbb{H}^{n-1} \oplus D, \tag{3.7}$$

and the corresponding unitary group G is not quasi-split. This describes the two hermitian spaces of rank $2n$ over E.

Moreover, a maximal lattice in V_α is given by

$$\Lambda_\alpha = \begin{cases} \Delta^n & \text{if } \alpha \in \mathbb{N}E^\times, \\ \Delta^{n-1} \oplus A_D & \text{if } \alpha \notin \mathbb{N}E^\times. \end{cases} \tag{3.8}$$

Using results of [BT1] and [BT2], one can check that the stabilizer of Λ_α is a maximal parahoric subgroup $\underline{G}(A)$. In Table 2, we give the type of the maximal reductive quotient $\overline{G}$ of the special fiber of $\underline{G}$, as well as the local factor λ. Here, and in later tables, we use ${}^2O_{2n}$ to denote the quasi-split but nonsplit orthogonal group in $2n$ variables. As in the case of O_{2n}, there is a morphism ${}^2d_n : {}^2O_{2n} \to \mathbb{Z}/2\mathbb{Z}$ whose kernel is the connected component ${}^2\,\mathrm{SO}_{2n}$ of ${}^2O_{2n}$.

It is natural to ask if the smooth integral model $\underline{G}$ can be defined using the maximal lattice Λ. For this, let

$$\rho : G \longrightarrow \mathrm{Res}_{E/F}\big(\mathrm{GL}(V) \times \mathrm{GL}(V)\big)$$

be the direct sum of two copies of the standard representation of G. Here, $\mathrm{GL}(V)$ denotes the general linear group of V, and $\mathrm{Res}_{E/F}$ denotes the Weil restriction of scalars from E to F. Moreover, let

$$\tilde{\Lambda} = \big\{x \in V : \langle x, \Lambda \rangle \subset \mathfrak{D}^{-1}\big\}.$$

Now we have the following proposition.

PROPOSITION 3.9

Suppose that $m = 2n$ is even or E/F is tamely ramified. Then the integral model $\underline{G}$ is the scheme-theoretic closure of $\rho(G)$ in $\mathrm{Res}_{A_E/A}(\mathrm{GL}_{A_E}(\Lambda) \times \mathrm{GL}_{A_E}(\tilde{\Lambda}))$, where $\mathrm{GL}_{A_E}(\Lambda)$ is the integral model of $\mathrm{GL}(V)$ determined by the A_E-lattice Λ.

Proof

This follows from the final theorem of [BT2, Section 5]. To be honest, Bruhat and Tits proved the statement in this proposition for special unitary groups. However, it is easy to check that their argument works for the unitary groups that are considered here. □

Now we consider the case where $m = 2n + 1$ is odd and E/F is ramified. Though the case where E/F is tamely ramified has been covered in Proposition 3.9, it is more natural for us to include it in the following discussion. We first define a sublattice Λ' of Λ as follows. The quotient $\Lambda/\pi_E\Lambda$ is a vector space over $A_E/\pi_E \cong A/\pi$, and the hermitian form $\langle -, - \rangle$ gives rise to a degenerate symplectic form on $\Lambda/\pi_E\Lambda$ (see [T, Section 3.11]). Let Λ' be the inverse image in Λ of the kernel of this symplectic form. More explicitly,

$$\Lambda' = (\pi_E \Delta)^n \oplus A_E \subset \Lambda. \tag{3.10}$$

Now if we consider the same construction as above, but with $\tilde{\Lambda}$ replaced by Λ', then Bruhat and Tits showed that this gives the desired smooth integral model for the special unitary group. Unfortunately, it does not yield a smooth model for the unitary group. Instead, let T be the one-dimensional anisotropic torus over F which is split by E, and let $\underline{T}$ be its Neron-Raynaud model (see Appendix B). Consider the rational representation

$$\rho' : G \longrightarrow \mathrm{Res}_{E/F}\big(\mathrm{GL}(V) \times \mathrm{GL}(V)\big) \times T$$

defined by

$$\rho'(g) = \rho(g) \times \det(g).$$

Then we have the following proposition.

PROPOSITION 3.11

Suppose that $m = 2n + 1$ is odd and E/F is ramified. Then the integral model $\underline{G}$ is the scheme-theoretic closure of $\rho'(G)$ in $\mathrm{Res}_{A_E/A}(\mathrm{GL}_{A_E}(\Lambda) \times \mathrm{GL}_{A_E}(\Lambda')) \times \underline{T}$.

Proof

The main point is to show that the above scheme-theoretic closure $\underline{G}'$ is smooth. Let V and Λ be as given in (3.1) and (3.3), respectively. This determines a maximal torus Z of G as well as various root subgroups U_a (see [T, Section 1.15]). By [BT2, Section 3.11], the scheme-theoretic closure of $\rho'(U_a)$ is smooth. Moreover, one can check that the closure of $\rho'(Z)$ is also smooth and is equal to the Neron-Raynaud model of Z. By Lemma B.3, we thus have a closed immersion of $\mathrm{Res}_{A_E/A}(\mathrm{GL}(\Lambda) \times \mathrm{GL}(\Lambda')) \times \underline{T}$ into $\mathrm{GL}_N(A)$ for some integer N. Now the smoothness of $\underline{G}'$ follows from [BT1, Section 2.2.3].

Finally, for any unramified extension F' of F, it is easy to see that the groups $\underline{G}(A_{F'})$ and $\underline{G}'(A_{F'})$ are equal. Since $\underline{G}'$ is smooth and affine, this completely determines its group scheme structure, and thus $\underline{G}' = \underline{G}$, as required. □

This completes our discussion of hermitian spaces and their unitary groups over a local field.

4. Mass formula for unitary groups

In this section, we return to the notation of Section 2, so that k is a totally real number field of degree d over $\mathbb{Q}$. Let L be a totally imaginary quadratic extension of k, and let χ be the nontrivial character of the Galois group of L over k. Let V be an m-dimensional vector space over L equipped with a totally definite hermitian form $\langle -, - \rangle$. Let G be the corresponding unitary group. Then $G(k \otimes \mathbb{R})$ is compact, so that G is of the type considered in Section 2.

For each finite place v, let $L_v = L \otimes_k k_v$. Then $V \otimes_k k_v$ is an L_v-module, equipped with a hermitian form by extension of scalars. If v splits in L, then $G_v \cong \mathrm{GL}_m \cong H_v$ as algebraic groups, whereas if v does not split, G_v is the unitary group for the hermitian space $V \otimes_k k_v$ over the quadratic extension L_v of k_v.

Let Λ be a maximal lattice in V. Then $\Lambda_v = \Lambda \otimes_A A_v$ is a maximal lattice in $V \otimes_k k_v$ (see [S1, Lemma 8.10]). Let K_v be the stabilizer of Λ_v in G_v. Then the mass of Λ is, by definition,

$$\mathrm{Mass}(\Lambda) = \mathrm{Mass}(K), \tag{4.1}$$

where K is the open compact subgroup $G(k \otimes \mathbb{R}) \times \prod_v K_v$ of $G(\mathbb{A})$. Now if v splits in L, then it is easy to see the following (see [S1, p. 29]):

$$K_v \cong \mathrm{GL}_m(A_v) \cong \underline{H}_v^0(A_v).$$

Moreover, by results of the last section, for all v that do not split in L, K_v is a maximal parahoric subgroup. Hence, K is of the type considered in Section 2. In particular, the mass of Λ is given by Proposition 2.13, and the local factors λ_v are given in Tables 1 and 2 in Section 3. Moreover, we have, by definition,

$$L(M) = \prod_{r=1}^{m} L\big(1 - r, \chi^r\big), \tag{4.2}$$

and by results of R. Kottwitz [K] and T. Ono [O, p. 128],

$$\tau(G) = 2. \tag{4.3}$$

Putting these results together, we have the following two propositions.

PROPOSITION 4.4
Let $m = 2n+1$ be odd. Then $v \in S$ if and only if either L_v is a ramified quadratic extension of k_v, or L_v is an unramified quadratic extension of k_v and $V \otimes_k k_v \cong V_\alpha$, with $\alpha \notin \mathbb{N}L_v^\times$. Moreover,

$$\mathrm{Mass}(\Lambda) = \left(\frac{1}{2^{md}} \cdot L(M) \cdot \tau(G)\right) \cdot \prod_{v \in S} \lambda_v,$$

where $L(M)$ and $\tau(G)$ are given by (4.2) and (4.3), respectively, and for $v \in S$,

$$\lambda_v = \begin{cases} \dfrac{1}{2} & \text{if } L_v \text{ is ramified,} \\ \dfrac{q_v^m + 1}{q_v + 1} & \text{if } L_v \text{ is unramified and } V \otimes_k k_v \cong V_\alpha, \text{ with } \alpha \notin \mathbb{N}(L_v^\times). \end{cases}$$

PROPOSITION 4.5
Let $m = 2n$ be even. Then $v \in S$ if and only if v does not split in L and $V \otimes_k k_v \cong V_\alpha$, with $\alpha \notin \mathbb{N}L_v^\times$. Moreover,

$$\mathrm{Mass}(\Lambda) = \left(\frac{1}{2^{md}} \cdot L(M) \cdot \tau(G)\right) \cdot \prod_{v \in S} \lambda_v,$$

where $L(M)$ and $\tau(G)$ are given by (4.2) and (4.3), respectively, and for $v \in S$,

$$\lambda_v = \begin{cases} \dfrac{q_v^m - 1}{q_v + 1} & \text{if } L_v \text{ is unramified,} \\ \dfrac{1}{2} \cdot \dfrac{q_v^m - 1}{q_v + 1} & \text{if } L_v \text{ is ramified.} \end{cases}$$

Remarks. (i) When $m = 2n+1$ is odd, it was assumed in [S1, Theorem 24.4] that the discriminant of V is a unit. We do not assume this in Proposition 4.4. It is easy to check that the formula in Proposition 4.4 agrees with that of Shimura, using the functional equation relating $L(M)$ and $L(M^\vee(1))$. We leave this as an exercise for the reader.

(ii) The formula in Proposition 4.5 seems to differ from that of Shimura, in that, when $v \in S$ is such that L_v is wildly ramified, the local factor in Shimura's formula contains a certain invariant $\lambda(\theta)$. Our result above certainly suggests that $\lambda(\theta)$ is equal to 1. In the appendices, we give an independent proof of this fact.

(iii) As in [S1], one can consider lattices in V which are maximal with respect to an integral ideal $\mathfrak{a}$ of A. If $\mathfrak{a}$ is a principal ideal, such lattices are among those that we have studied. However, if $\mathfrak{a}$ is not principal, then a lattice Λ that is $\mathfrak{a}$-maximal may not be maximal in the usual sense for any hermitian form. Nevertheless, the mass of Λ can still be obtained from Proposition 2.13, as the local stabilizers of Λ are still maximal parahoric subgroups that correspond to vertices at the end of the relative local Dynkin diagram.

5. Higher-level congruence subgroups

In [S1, Theorem 24.4], the mass formula was obtained not only for the open compact subgroup $K = G(k \otimes \mathbb{R}) \times \prod_v K_v$ of Section 4 but also for certain open subgroups of K with K_v replaced by certain congruence subgroups. In this section, we indicate how one obtains the mass of such congruence subgroups from Propositions 4.4 and 4.5.

Let us first recall some definitions from [S1]. We return to the notation of Section 3, and we let Λ be a maximal lattice in $V \cong V_\alpha$, a rank m hermitian space over F with discriminant α. Recall that

$$\tilde{\Lambda} = \{x \in V : \langle x, \Lambda\rangle \subset \mathfrak{D}^{-1}\}, \tag{5.1}$$

where $\mathfrak{D}$ is the different of E over F. For $r \geq 0$ an integer, we set, following Shimura,

$$\begin{aligned} C_r &= \{g \in G(F) : (g-1)\Lambda \subset \pi^r \Lambda\}, \\ D_r &= \{g \in G(F) : (g-1)\tilde{\Lambda} \subset \pi^r \Lambda\}. \end{aligned} \tag{5.2}$$

Moreover, let

$$K_r = \ker\left(\underline{G}(A) \longrightarrow \underline{G}(A/\pi^r)\right) \tag{5.3}$$

be the rth principal congruence subgroup associated to the integral model $\underline{G}$ for which $\underline{G}(A)$ is the stabilizer of Λ. Note that

$$\#\big(\underline{G}(A)/K_r\big) = q^{r\cdot\dim(G)} \cdot \big(q^{-\dim(\overline{G})} \cdot \#\overline{G}(A/\pi)\big). \tag{5.4}$$

We would like to relate the group K_r to the groups C_r and D_r. For this, recall that if T is the one-dimensional anisotropic torus over F which is split by E, then we have the determinant map $\det : G \to T$. Let T_r be the rth principal congruence subgroup of T corresponding to the Neron-Raynaud model $\underline{T}$. Hence, by Lemma B.1,

$$T_r = \{t \in T(F) : \mathrm{ord}_E(t-1) \geq 2r + d\}, \tag{5.5}$$

where ord_E is the valuation on E giving π_E valuation 1. We then have the following lemma.

LEMMA 5.6
Assume that $m = 2n + 1$ is odd and E/F is ramified. Then

$$D_r = \{g \in C_r : \det(g) \in T_r\}.$$

Proof
For $r = 0$, this is [S1, Lemma 17.13(3)]. The proof there extends easily to the general case, and hence we omit the details. □

Now we have the following proposition.

PROPOSITION 5.7
Suppose that $r > 0$. If $m = 2n$ is even, then $K_r = C_r$. If $m = 2n + 1$ is odd, then

$$K_r = \begin{cases} C_r & \text{if } E/F \text{ is unramified;} \\ D_r & \text{if } E/F \text{ is ramified.} \end{cases}$$

Proof
If $m = 2n$ is even or E/F is unramified, the assertion follows immediately from Proposition 3.9. Now assume that $m = 2n + 1$ is odd and E/F is ramified. Then Proposition 3.11 implies that

$$K_r = \{g \in C_r : \det(g) \in T_r\},$$

so that the result follows by Lemma 5.6. Note that when E/F is tamely ramified, Proposition 3.9 implies that $K_r = C_r$. However, in this case, $C_r = D_r$. Indeed, if $g \in C_r$, so that $\det(g) \equiv 1 \pmod{\pi_E^{2r}}$, then in fact $\det(g) \equiv 1 \pmod{\pi_E^{2r+1}}$. □

COROLLARY 5.8
Suppose that $r > 0$. If $m = 2n$ is even, then

$$\frac{\#\underline{G}(A)}{D_r} = \begin{cases} \#\big(\underline{G}(A)/K_r\big) & \text{if } \alpha \in \mathbb{N}E^\times, \\ q \cdot \#\big(\underline{G}(A)/K_r\big) & \text{if } \alpha \notin \mathbb{N}E^\times. \end{cases}$$

If $m = 2n + 1$ is odd, assume that $\alpha \in A^\times$. Then

$$\#\big(\underline{G}(A)/D_r\big) = \#\big(\underline{G}(A)/K_r\big).$$

Proof
First assume that $m = 2n$ is even. If $\alpha \in \mathbb{N}E^\times$, then $\tilde{\Lambda} = \Lambda$, so that $D_r = C_r$. Since

$C_r = K_r$ by Proposition 5.7, the result follows in this case. If $\alpha \notin \mathbb{N}E^{\times}$, then we need to show that $\#C_r/D_r = q$. This follows from [S1, Lemma 17.3] and Corollary D.4. The case for odd m is clear. □

Now we return to the global situation and adopt the notation of Section 4. As in [S1, Theorem 24.4], we further assume that the discriminant of the hermitian space V is a unit if $m = 2n+1$ is odd. Suppose that $\mathfrak{c}$ is an integral ideal of A. For each place v of k, let $r_v \geq 0$ be defined by

$$\mathfrak{c}A_v = \pi_v^{r_v} A_v. \tag{5.9}$$

Hence, for almost all v, $r_v = 0$. Now let

$$K(\mathfrak{c}) = G(k \otimes \mathbb{R}) \times \prod_v K(\mathfrak{c})_v \tag{5.10}$$

be the open compact subgroup of $G(\mathbb{A})$ such that

$$K(\mathfrak{c})_v = \begin{cases} K_v & \text{if } r_v = 0, \\ D_{v,r_v} & \text{if } r_v \neq 0, \end{cases} \tag{5.11}$$

where, for v split in L, the group D_{v,r_v} is defined as in (5.2). Then $\mathrm{Mass}(K(\mathfrak{c}))$ is given in [S1, Theorem 24.4]. To derive the result from Propositions 4.4 and 4.5, note that

$$\mathrm{Mass}(K(\mathfrak{c})) = \#\big(K/K(\mathfrak{c})\big) \cdot \mathrm{Mass}(K), \tag{5.12}$$

so that we need to compute the local indices $\#K_v/D_{v,r_v}$ for those v such that $r_v \neq 0$. If v splits in L, then it is easy to see that D_{v,r_v} is the r_vth principal congruence subgroup of $\mathrm{GL}_m(A_v)$, so that its index in K_v is given by the general formula in (5.4). If v does not split in L, then the index is given by Corollary 5.8 and (5.4).

It is interesting to note that in [S1] Shimura first computed the mass of $K(\mathfrak{c})$ for some suitable ideal $\mathfrak{c}$, before obtaining the mass of Λ. Moreover, for this last step, it suffices to know the index $\#K_v/D_{v,1}$, which was given in [S1, Proposition 17.11].

6. Quadratic spaces

In this section, we return to the local setting of Section 3, and we consider quadratic spaces over the nonarchimedean local field F. Let V be an m-dimensional vector space over F, equipped with a quadratic form $Q : V \to F$. Let $\langle -, - \rangle$ be the associated symmetric bilinear form. Hence, for $x, y \in V$,

$$\langle x, y \rangle = \frac{1}{2}\big(Q(x+y) - Q(x) - Q(y)\big) \tag{6.1}$$

and $Q(x) = \langle x, x\rangle$. We also let G be the corresponding special orthogonal group of V.

As is well known, the quadratic spaces over F of rank m are classified by their discriminant $\delta \in F^\times/F^{\times 2}$ and their Hasse-Witt invariant $w = \pm 1$. We can and do assume that δ is chosen to be either a unit or a uniformizer. Moreover, the Hasse-Witt invariant we are using here is normalized so that, for example, the split quadratic space always has Hasse-Witt invariant 1. Given δ and w, we let $V_{\delta,w}$ be the corresponding quadratic space. We enumerate the spaces $V_{\delta,w}$ and the stabilizers of their maximal lattices. As in Section 3, we consider the even- and odd-rank cases separately.

Let $\mathbb{H}$ be the split rank-two quadratic space over F. Hence, $\mathbb{H}$ has basis $\{e, f\}$ over F, and

$$\begin{pmatrix} \langle e,e\rangle & \langle e,f\rangle \\ \langle f,e\rangle & \langle f,f\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}.$$

Also, let D be the quaternion algebra over F, regarded as a quadratic space using the reduced norm. Now assume that $m = 2n$ is even, with $n \geq 2$. If $\delta \in F^{\times 2}$, then let

$$\begin{aligned} V_{\delta,1} &= \mathbb{H}^n, \\ V_{\delta,-1} &= \mathbb{H}^{n-2} \oplus D. \end{aligned} \tag{6.2}$$

The special orthogonal group G corresponding to $V_{\delta,1}$ is split, and it is the quasi-split inner form of that for $V_{\delta,-1}$.

If $\delta \notin F^{\times 2}$, let $E_\delta = F(\sqrt{\delta})$. For $\alpha \in F^\times/\mathbb{N}E_\delta^\times$, chosen to be either a unit or a uniformizer, we let $E_{\delta,\alpha}$ be the rank-two quadratic space with the quadratic form $Q_\alpha(x) = \alpha\mathbb{N}(x)$. Then we have

$$V_{\delta,w} = \mathbb{H}^{n-1} \oplus E_{\delta,\alpha}, \tag{6.3}$$

with $\alpha \in \mathbb{N}E_\delta^\times$ if and only if $w = 1$. The special orthogonal groups corresponding to these spaces are isomorphic and quasi-split.

Let

$$\Delta = Ae \oplus Af \tag{6.4}$$

be an A-lattice in $\mathbb{H}$, and let A_D be the ring of integers of D. A maximal lattice in $V_{\delta,w}$ is given by

$$\Lambda_{\delta,w} = \begin{cases} \Delta^n & \text{if } \delta \in F^{\times 2} \text{ and } w = 1, \\ \Delta^{n-2} \oplus A_D & \text{if } \delta \in F^{\times 2} \text{ and } w = -1, \\ \Delta^{n-1} \oplus A_{E_\delta} & \text{if } \delta \notin F^{\times 2}. \end{cases} \tag{6.5}$$

Using results of [BT1] and [BT2], one can check that the stabilizer of $\Lambda_{\delta,w}$ is a maximal parahoric subgroup $\underline{G}(A)$. In Table 3, we list the type of the maximal

Table 3
Even Special Orthogonal Groups

δ	w	$\overline{G}$	$\overline{H}^0$	λ
$\delta \in F^{\times 2}$	1	SO_{2n}	SO_{2n}	1
$\delta \in F^{\times 2}$	-1	$S({}^2O_{2n-2} \times {}^2O_2)$	SO_{2n}	$\frac{(q^{n-1}-1)(q^n-1)}{2(q+1)}$
$\delta \notin F^{\times 2}$ and E_δ unramified	1	${}^2 SO_{2n}$	${}^2 SO_{2n}$	1
$\delta \notin F^{\times 2}$ and E_δ unramified	-1	$S(O_{2n-2} \times {}^2O_2)$	${}^2 SO_{2n}$	$\frac{(q^{n-1}+1)(q^n+1)}{2(q+1)}$
$\delta \notin F^{\times 2}$ and E_δ ramified	± 1	$\mathbb{Z}/2\mathbb{Z} \times SO_{2n-1}$	SO_{2n-1}	$1/2$

reductive quotient $\overline{G}$ of the special fiber of $\underline{G}$, as well as the local factor λ given by (2.12). Moreover, we have written $S(O_{2n} \times {}^2O_{2n'})$ for the kernel of the morphism $d_n \cdot {}^2d_{n'} : O_{2n} \times {}^2O_{2n'} \to \mathbb{Z}/2\mathbb{Z}$ and similarly for $S({}^2O_{2n} \times {}^2O_{2n'})$.

Now we come to the case when $m = 2n+1$ is odd. For $\delta \in F^\times/F^{\times 2}$, we have

$$V_{\delta,1} = \mathbb{H}^n \oplus F_\delta, \tag{6.6}$$

where F_δ is the rank-one quadratic space with quadratic form $Q_\delta(x) = \delta x^2$. The special orthogonal groups of $V_{\delta,1}$ are all isomorphic to the split group.

To describe $V_{\delta,-1}$, let W_δ be the three-dimensional quadratic space whose underlying vector space is the space of trace zero elements in D, and whose quadratic form is given by

$$x \longmapsto -\delta^{-1} \cdot \mathbb{N}(x). \tag{6.7}$$

Then, for $n \geq 1$, we have

$$V_{\delta,-1} = \mathbb{H}^{n-1} \oplus W_\delta. \tag{6.8}$$

The special orthogonal group of $V_{\delta,-1}$ is not quasi-split, and its isomorphism class does not depend on δ.

Let

$$\Delta_\delta = \{x \in W_\delta : \mathbb{N}(x) \in \delta A\}. \tag{6.9}$$

Then Δ_δ is the unique maximal lattice in W_δ. More explicitly, let u be a unit in F such that $E = F(\sqrt{u})$ is the unramified quadratic extension over F. Then $D = E \oplus E \cdot z$, with z^2 a uniformizer in A. Hence, $W_\delta = F \cdot \sqrt{u} \oplus E \cdot z$, and

$$\Delta_\delta = A \cdot \delta\sqrt{u} \oplus A_E \cdot z. \tag{6.10}$$

A maximal lattice in $V_{\delta,w}$ is then given by

$$\Lambda_{\delta,w} = \begin{cases} \Delta^n + A & \text{if } w = 1, \\ \Delta^{n-1} \oplus \Delta_\delta & \text{if } w = -1. \end{cases} \tag{6.11}$$

Table 4
Odd Special Orthogonal Groups

δ	w	$\overline{G}$	$\overline{H}^0$	λ
$\delta \in A^\times$	1	SO_{2n+1}	SO_{2n+1}	1
$\delta \notin A^\times$	1	O_{2n}	SO_{2n+1}	$(q^n+1)/2$
$\delta \in A^\times$	-1	$\mathrm{SO}_{2n-1} \times {}^2O_2$	SO_{2n+1}	$\big(q^{2n}-1\big)/(2(q+1))$
$\delta \notin A^\times$	-1	${}^2O_{2n}$	SO_{2n+1}	$(q^n-1)/2$

As before, it follows from results of [BT1] and [BT2] that the stabilizer of $\Lambda_{\delta,w}$ is a maximal parahoric subgroup $\underline{G}(A)$. In Table 4, we list the type of the maximal reductive quotient $\overline{G}$ and the local factor λ.

As in the case of unitary groups, one can describe the smooth integral model $\underline{G}$ using Λ. Let $\rho : G \to \mathrm{GL}(V)$ be the standard representation of G. Then, following the proof of Proposition 3.11, we have the following proposition.

PROPOSITION 6.12
Unless the residue characteristic of F is 2 and G is residually split but nonsplit (see [T]), $\underline{G}$ is isomorphic to the scheme-theoretic closure of $\rho(G)$ in $\mathrm{GL}(\Lambda)$.

Remarks. (i) The only case that is excluded in Proposition 6.12 is when the residue characteristic of F is 2, $m = 2n$ is even, and $V \cong V_{\delta,w}$, with E_δ a ramified quadratic extension of F.

(ii) Note that the maximal lattice Λ defines a naive integral model $\underline{G}'$ of G; namely, for any A-algebra B,

$$\underline{G}'(B) = \big\{g \in \mathrm{SL}(\Lambda \otimes_A B) : Q_B(gv) = Q_B(v) \text{ for all } v \in \Lambda \otimes_A B\big\},$$

where Q_B is the quadratic form obtained from Q by extension of scalars. It is this integral model $\underline{G}'$ that intervenes in the Siegel mass formula. More precisely, the local representation density that appears in Siegel's formula is the limit as $r \to \infty$ of terms involving $\#\underline{G}'(A/\pi^r)$. Unfortunately, $\underline{G}'$ is in general not a smooth group scheme, and this is the reason why the above limit does not stabilize at the first term when $r = 1$. In particular, $\underline{G}'$ is not isomorphic to the smooth model $\underline{G}$ in general. However, to apply Proposition 2.13, it is only necessary for us to know that $\underline{G}'$ and $\underline{G}$ have the same group of integral points.

Now we consider the remaining case not covered by Proposition 6.12, that is, when the residue characteristic of F is 2, and G is residually split but nonsplit. Note that the integral model $\underline{G}$ is disconnected, with component group of order 2. Let $\underline{G}^0$ denote its connected component of identity. We describe $\underline{G}^0$ using the maximal lattice Λ.

Let $V = V_{\delta,w}$ and $\Lambda = \Lambda_{\delta,w}$ be as given in (6.3) and (6.5), so that

$$\begin{cases} V = \mathbb{H}^{n-1} \oplus E_{\delta,\alpha}, \\ \Lambda = \Delta^{n-1} \oplus A_{E_\delta}. \end{cases}$$

Note that the special orthogonal group of the rank-two quadratic space $E_{\delta,\alpha}$ is isomorphic to the one-dimensional anisotropic torus T_δ which is split by E_δ. Let d be the exponent of the different ideal $\mathfrak{D}$ of the wildly ramified quadratic extension E_δ of F, and set

$$\{h, h'\} = \begin{cases} \left\{\frac{d+1}{2}, \frac{d-1}{2}\right\} & \text{if } d \text{ is odd,} \\ \left\{\frac{d}{2}, \frac{d}{2}\right\} & \text{if } d \text{ is even.} \end{cases}$$

Further, let

$$\tilde{\Lambda} = \left\{x \in V : 2\langle x, \Lambda\rangle \subset A\right\}. \tag{6.13}$$

Let $\{e_1, f_1, \ldots, e_{n-1}, f_{n-1}\}$ be the standard basis of Δ^{n-1}, as described in (6.4). It is not difficult to see that one can choose a basis $\{e_n, f_n\}$ for the maximal lattice A_{E_δ} in $E_{\delta,\alpha}$ such that

$$\tilde{\Lambda} = \Delta^{n-1} \oplus \left(A\pi^{-h} e_n \oplus A\pi^{-h'} f_n\right). \tag{6.14}$$

For $g \in G(F)$, let $M(g)$ be the $2n \times 2n$ matrix representing g with respect to the basis $\{e_1, f_1, \ldots, e_n, f_n\}$, and write

$$M(g) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where M_{11} is a $(2n-2) \times (2n-2)$ matrix, and so on. Now define a rational representation

$$\rho' : G \longrightarrow \mathrm{GL}\left(F^2 \oplus V\right)$$

by

$$\rho'(g) = \begin{pmatrix} 1 & 0 & 0 \\ M_{12}N & M_{11} & M_{12} \\ (M_{22}-1)N & M_{21} & M_{22} \end{pmatrix},$$

where

$$N = \begin{pmatrix} \pi^{-h} & 0 \\ 0 & \pi^{-h'} \end{pmatrix}.$$

Then we have the following proposition, whose proof is similar to that of Proposition 3.11.

PROPOSITION 6.15
The connected smooth group scheme $\underline{G}^0$ *is isomorphic to the scheme-theoretic closure of* $\rho'(G)$ *in* $\mathrm{GL}(A^2 \oplus \Lambda)$.

Let K_r be the kernel of the reduction map $\underline{G}(A) \to \underline{G}(A/\pi^r)$. Further, for $r \geq 0$ an integer, define the groups C_r and D_r by

$$\begin{aligned} C_r &= \{g \in G(F) : (g-1)\Lambda \subset \pi^r \Lambda\}, \\ D_r &= \{g \in G(F) : (g-1)\tilde{\Lambda} \subset \pi^r \Lambda\}. \end{aligned} \tag{6.16}$$

Then the following proposition relates K_r to the group C_r or D_r.

PROPOSITION 6.17
Suppose that $r > 0$. *If the residue characteristic of* F *is* 2 *and* G *is residually split but nonsplit, then* $K_r = D_r$. *Otherwise,* $K_r = C_r$.

Proof
The second case is immediate from Proposition 6.12. As for the first case, if $M(g)$ is the matrix representing $g \in G(F)$ as above, then Proposition 6.15 implies that $g \in K_r$ if and only if

$$\begin{cases} M_{11} - 1 \equiv 0 \ (\mathrm{mod}\, \pi^r), \\ M_{21} \equiv 0 \ (\mathrm{mod}\, \pi^r), \\ M_{12} N \equiv 0 \ (\mathrm{mod}\, \pi^r), \\ (M_{22} - 1) N \equiv 0 \ (\mathrm{mod}\, \pi^r). \end{cases}$$

In view of (6.14), this is equivalent to saying that $g \in D_r$. □

This completes our discussion of quadratic spaces and special orthogonal groups over a local field.

7. Mass formula for special orthogonal groups

Now we return to the global situation of Section 2, so that k is a totally real number field. Let V be a totally definite quadratic space of rank m over k, and let G be the corresponding special orthogonal group. Let Λ be a maximal lattice in V. Then $\Lambda_v = \Lambda \otimes_A A_v$ is a maximal lattice in $V \otimes_k k_v$. Let K_v be the stabilizer of Λ_v in G_v. By results of Section 6, the open compact subgroup $K = G(k \otimes \mathbb{R}) \times \prod_v K_v$ is of the type considered in Section 2. Moreover, the mass of Λ is, by definition,

$$\mathrm{Mass}(\Lambda) = \mathrm{Mass}(K). \tag{7.1}$$

Hence, Mass(Λ) is given by Proposition 2.13. Furthermore, if $\delta \in k^\times/k^{\times 2}$ is the discriminant of V, let $L = k(\sqrt{\delta})$ be the corresponding étale quadratic algebra. If L is a field, let χ be the nontrivial quadratic character of the Galois group of L over k. Otherwise, let χ be trivial. Then

$$L(M) = \begin{cases} \prod_{r=1}^{n} \zeta_k(1-2r) & \text{if } m = 2n+1, \\ \prod_{r=1}^{n-1} \zeta_k(1-2r) \cdot L(1-n, \chi) & \text{if } m = 2n, \end{cases} \tag{7.2}$$

where ζ_k is the zeta function of k. On the other hand, it is well known that

$$\tau(G) = 2. \tag{7.3}$$

Combining these with the results of the last section on the local factors λ_v, we have the following two propositions.

PROPOSITION 7.4
Let $m = 2n+1$. Then $v \notin S$ if and only if $V \otimes_k k_v \cong V_{\delta,1}$, with $\delta \in A_v^\times$. Moreover,

$$\mathrm{Mass}(\Lambda) = \left(\frac{1}{2^{nd}} \cdot L(M) \cdot \tau(G)\right) \cdot \prod_{v \in S} \lambda_v,$$

where for $v \in S$,

$$\lambda_v = \begin{cases} \dfrac{q_v^{2n}-1}{2(q_v+1)} & \text{if } V \otimes_k k_v \cong V_{\delta,-1}, \text{ with } \delta \in A_v^\times; \\ \dfrac{q_v^n + w}{2} & \text{if } V \otimes_k k_v \cong V_{\delta,w}, \text{ with } \delta \notin A_v^\times. \end{cases}$$

PROPOSITION 7.5
Let $m = 2n$. Then $v \notin S$ if and only if $V \otimes_k k_v \cong V_{\delta,1}$ with $L \otimes_k k_v$ split or unramified. Moreover,

$$\mathrm{Mass}(\Lambda) = \left(\frac{1}{2^{nd}} \cdot L(M) \cdot \tau(G)\right) \cdot \prod_{v \in S} \lambda_v,$$

where for $v \in S$,

$$\lambda_v = \begin{cases} \dfrac{(q_v^{n-1}-1)(q_v^n-1)}{2(q_v+1)} & \text{if } V \otimes_k k_v \cong V_{\delta,-1}, \text{ with } \delta \in k_v^{\times 2}; \\ \dfrac{(q_v^{n-1}+1)(q_v^n+1)}{2(q_v+1)} & \text{if } V \otimes_k k_v \cong V_{\delta,-1}, \text{ with } \delta \notin k_v^{\times 2} \text{ and } E_\delta \text{ unramified}; \\ \dfrac{1}{2} & \text{if } V \otimes_k k_v \cong V_{\delta,w}, \text{ with } \delta \notin k_v^{\times 2} \text{ and } E_\delta \text{ ramified.} \end{cases}$$

We leave the comparison of the above formulas with those of Shimura [S2] to the reader. As in the case of hermitian spaces, one can consider higher-level congruence subgroups, using Proposition 6.17. We do not pursue this matter here.

8. Quaternionic hermitian spaces

In this section, we return to the local situation again, and we consider quaternionic hermitian forms over the nonarchimedean local field F. Let V be an m-dimensional vector space over the quaternion algebra D, and let $\langle -, - \rangle$ be a hermitian form on V. Let $\mathbb{H}$ denote the rank-two quaternionic hermitian space with basis $\{e, f\}$ over D, and such that

$$\begin{pmatrix} \langle e, e\rangle & \langle e, f\rangle \\ \langle f, e\rangle & \langle f, f\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.1}$$

Then we have

$$V = \begin{cases} \mathbb{H}^n & \text{if } m = 2n, \\ \mathbb{H}^n \oplus D & \text{if } m = 2n+1, \end{cases} \tag{8.2}$$

where D is regarded as a rank-one quaternionic hermitian space, equipped with the form $\langle x, y\rangle = x\bar{y}$. The corresponding unitary group G is an inner form of the symplectic group Sp_{2m}.

As before, we say that a lattice Λ in V is maximal if $\langle x, x\rangle \in A$ for all $x \in \Lambda$, and Λ is maximal with respect to this property. All such lattices are conjugate under G (see [S3]). Moreover, if we let

$$\Delta = A_D e \oplus A_D \pi_D^{-1} f, \tag{8.3}$$

then a maximal lattice in V is given by

$$\Lambda = \begin{cases} \Delta^n & \text{if } m = 2n, \\ \Delta^n \oplus A_D & \text{if } m = 2n+1. \end{cases} \tag{8.4}$$

The stabilizer of Λ in G can be shown to be a maximal parahoric subgroup $\underline{G}(A)$. The maximal reductive quotient of the special fiber of $\underline{G}$ is

$$\overline{G} = \begin{cases} \mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\mathrm{Sp}_{2n}) & \text{if } m = 2n, \\ \mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\mathrm{Sp}_{2n}) \times U_1 & \text{if } m = 2n+1. \end{cases} \tag{8.5}$$

Here, we have written $\mathbb{F}_q$ for the residue field A/π and $\mathbb{F}_{q^2}$ for its quadratic extension A_D/π_D. Moreover, $\mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\mathrm{Sp}_{2n})$ denotes the symplectic group over $\mathbb{F}_{q^2}$, regarded as an algebraic group over $\mathbb{F}_q$ by the Weil restriction of scalars. Since $\overline{H}^0 \cong \mathrm{Sp}_{2m}$, we have

$$\lambda = \begin{cases} \prod_{r=1}^{n} \left(q^{4r-2} - 1\right) & \text{if } m = 2n, \\ \left(\prod_{r=1}^{n} \left(q^{4r-2} - 1\right) \right) \cdot \dfrac{q^{4n+2} - 1}{q + 1} & \text{if } m = 2n + 1. \end{cases} \tag{8.6}$$

It should be noted that in [S3] Shimura considered a different notion of maximality from the one above. We say that a lattice Λ_0 in V is maximal integral if $\langle x, y \rangle \in A_D$ for all $x, y \in \Lambda_0$, and Λ_0 is maximal with respect to this property. Equivalently, Λ_0 is self-dual with respect to $\langle -, - \rangle$. All such lattices are conjugate under G (see [S3]), and if we let

$$\Delta_0 = A_D e \oplus A_D f, \tag{8.7}$$

then a maximal integral lattice in V is given by

$$\Lambda_0 = \begin{cases} \Delta_0^n & \text{if } m = 2n, \\ \Delta_0^n \oplus A_D & \text{if } m = 2n + 1. \end{cases} \tag{8.8}$$

The stabilizer of Λ_0 in G can be shown to be a maximal parahoric subgroup $\underline{G}(A)$. The maximal reductive quotient of its special fiber is

$$\overline{G} = U_m. \tag{8.9}$$

In particular, we have

$$\lambda = \prod_{r=1}^{m} \left(q^r + (-1)^r\right). \tag{8.10}$$

9. Mass formula for quaternionic unitary groups

Now we return to the global situation, and we assume that D is a quaternion algebra over the totally real number field k, which is ramified at all real places. Let S_D be the set of finite places where D is ramified. Fix a maximal order A_D of D.

Let V be a totally definite quaternionic hermitian space of rank m over D, and let G be the corresponding unitary group. Then $G(k \otimes \mathbb{R})$ is compact, and if $v \notin S_D$, then $G_v \cong \mathrm{Sp}_{2m}$. Let Λ be a maximal A_D-lattice in V. Then it is easy to see that, for $v \notin S_D$, the stabilizer K_v of Λ_v is the hyperspecial maximal compact subgroup $\mathrm{Sp}_{2m}(A_v) \cong \underline{H}_v^0(A_v)$. For $v \in S_D$, the stabilizer K_v is given in Section 8. In any case, the open compact subgroup $K = G(k \otimes \mathbb{R}) \times \prod_v K_v$ is of the type considered in Section 2. As before, the mass of Λ is defined to be $\mathrm{Mass}(K)$. Since

$$L(M) = \prod_{r=1}^{m} \zeta_k(1 - 2r) \tag{9.1}$$

and

$$\tau(G) = 1, \tag{9.2}$$

we have the following proposition.

PROPOSITION 9.3
We have $v \in S$ if and only if $v \in S_D$. Moreover,

$$\mathrm{Mass}(\Lambda) = \left(\frac{1}{2^{md}} \cdot L(M)\right) \cdot \prod_{v \in S} \lambda_v,$$

where for $v \in S$,

$$\lambda_v = \begin{cases} \prod_{r=1}^{n} \left(q_v^{4r-2} - 1\right) & \text{if } m = 2n, \\ \left(\prod_{r=1}^{n} \left(q_v^{4r-2} - 1\right)\right) \cdot \dfrac{q_v^{4n+2} - 1}{q_v + 1} & \text{if } m = 2n + 1. \end{cases}$$

Now let Λ_0 be a maximal integral A_D-lattice in V. Then for $v \notin S_D$ the stabilizer K_v of $\Lambda_{0,v}$ is the hyperspecial maximal compact subgroup $\mathrm{Sp}_{2m}(A_v) = \underline{H}_v^0(A_v)$, whereas for $v \in S_D$, the stabilizer K_v is given in Section 8. Hence we have the following proposition.

PROPOSITION 9.4
We have $v \in S$ if and only if $v \in S_D$. Moreover,

$$\mathrm{Mass}(\Lambda_0) = \left(\frac{1}{2^{md}} \cdot L(M)\right) \cdot \prod_{v \in S} \lambda_v,$$

where for $v \in S$,

$$\lambda_v = \prod_{r=1}^{m} \left(q_v^r + (-1)^r\right).$$

Again, we leave the comparison of the above formula with [S3] to the reader.

Appendices: On the λ-invariant of Shimura

The purpose of these appendices is to prove certain technical results used in the paper and to show that the invariant $\lambda(\theta)$ defined in [S1, Lemma 17.9(2)] is always equal to 1.

A. Preliminaries

We return to the local situation of Sections 3, 6, and 8. Hence, F is a nonarchimedean local field of characteristic zero. Let E be a separable *ramified* quadratic extension of F, and let U (resp., U_E) be the group of units of F (resp., E). Recall that the groups U and U_E have natural filtrations $(U^i)_{i\geq 0}$ and $(U_E^i)_{i\geq 0}$ (see [Se]). Let $\mathfrak{D}$ be the different of E/F, and suppose that $\mathfrak{D} = (\pi_E^d)$. Also, let Tr and $\mathbb{N}$ denote the trace and norm map from E to F. Denote by Γ the Galois group of E/F , whose nontrivial element is given by $x \mapsto \bar{x}$. Recall that Γ has a filtration by ramification groups $(\Gamma_i)_{i\geq 0}$ (see [Se]).

In this section, we collect together some facts that we need later.

LEMMA A.1

(i) $\mathrm{Tr}(\pi_E^{-i} A_E) \subset A$ *if and only if* $i \leq d$.

(ii) *The ramification group* Γ_i *is trivial if and only if* $i \geq d$.

(iii) *An element* $x \in E^\times$ *satisifes* $\mathbb{N}x \in U^{d-1}$ *if and only if* $x \in U_E^{d-1}$.

(iv) *The group* U_E^i *is contained in* $\mathbb{N}E^\times$ *if and only if* $i \geq d$.

(v) *We have* $1 - d = \sup\{\mathrm{ord}_E(x) : x \in E^\times$ *and* $\mathrm{Tr}(x) = 1\}$, *where* ord_E *denotes the valuation on* E *giving a uniformizer valuation* 1.

(vi) *We have* $\{\mathrm{ord}_E(x) : x \in E^\times$ *and* $\mathrm{Tr}(x) = 0\} = d + 2\mathbb{Z}$.

Proof

(i) See [Se, Chapter III, Proposition 3.7].

(ii) See [Se, Chapter V, Lemma 3.4].

(iii) By [Se, Chapter V, Proposition 3.4], $\mathbb{N}\big(U_E^{d-1}\big) \subset U^{d-1}$. Conversely, suppose that $0 \leq i \leq d-2$, $x \in U_E^i$, and $\mathbb{N}x \in U^{i+1}$. Then by [Se, Chapter V, Corollary 3.1], $x \in U_E^{i+1}$. The desired statement now follows by induction.

(iv) By [Se, Chapter V, Corollary 3.3], $\mathbb{N}\big(U_E^d\big) = U^d$, and by [Se, Chapter V, Proposition 3.5(iii)], there exists $x \in U^{d-1} - U^d$ such that $x \notin \mathbb{N}\big(U_E^{d-1}\big)$. By (iii) above, this implies that $x \notin \mathbb{N}E^\times$.

(v) First note that $\mathrm{Tr}\left(\pi_E^{2-d} A_E\right) \subset \pi A$ by (i). Recall that $d = \mathrm{ord}(a^2 - 4\pi)$ for some $a \in F$, where ord is the valuation on F. If d is odd, then $d = \mathrm{ord}(4\pi)$, so that $\mathrm{Tr}(1/2) = 1$ and $\mathrm{ord}_E(1/2) = 1 - d$. If d is even, then $d = \mathrm{ord}(a^2)$, so that $\mathrm{Tr}(\pi_E/a) = 1$ and $\mathrm{ord}_E(\pi_E/a) = 1 - d$.

(vi) It suffices to find an element $x \in E^\times$ with $\mathrm{Tr}(x) = 0$ and to see that $\mathrm{ord}_E(x)$ has the same parity as d. It is easy to check that $x = 2\pi_E - a$, where a is as in the proof of (v) above, works. □

B. Unitary group in one variable

Fix an algebraic closure of F and extend to it the valuation ord of F. Let T be the one-dimensional anisotropic torus over F which is split by E. We can regard T as the unitary group of a rank-one hermitian space over E. In particular, we regard $T(F)$ as the group of norm-one elements in $E^\times$.

Let $\underline{T}$ be the Neron-Raynaud model of T, and let $\underline{T}^0$ be its connected component. If $X^2 - aX + \pi$ is the minimal polynomial of π_E over F, then the affine ring of T is

$$A(T) = F[X, Y]\big/\big(X^2 + aXY + \pi Y^2 - 1\big).$$

The affine rings $A(\underline{T})$ and $A(\underline{T}^0)$ of $\underline{T}$ and $\underline{T}^0$ are subrings of $A(T)$ and are given by the following lemma.

LEMMA B.1
If d is odd,

$$A(\underline{T}) = A\big[\pi^{-(d-1)/2}(X-1), \pi^{-(d-1)/2}Y\big],$$
$$A(\underline{T}^0) = A\big[\pi^{-(d+1)/2}(X-1), \pi^{-(d-1)/2}Y\big].$$

If d is even,

$$A(\underline{T}) = A\big[\pi^{-d/2}(X-1), \pi^{-(d-2)/2}Y\big],$$
$$A(\underline{T}^0) = A\big[\pi^{-d/2}(X-1), \pi^{-d/2}Y\big].$$

Proof
One checks that the above rings define smooth models of T over A. Then it is easy to show that the group of integral points of the model defined by $A(\underline{T})$ is equal to $T(F)$, so that $A(\underline{T})$ is indeed the affine ring of $\underline{T}$. Finally, one checks that the model defined by $A(\underline{T}^0)$ is connected and is a closed one-dimensional subscheme of $\underline{T}$. □

Remark. The above lemma corrects an error in [BT1, pp. 111–112, Section 4.4.13, Case B].

Set $T_0 = \underline{T}(A) = T(F)$, and set $T_0^0 = \underline{T}^0(A)$. Let us identify $T_0 = T(F)$ with $E^\times/F^\times$ via $x \mapsto x/\overline{x}$, for $x \in E^\times/F^\times$. Lemma B.1 implies the following lemma.

LEMMA B.2
We have

$$T_0^0 = \big\{x \in E^\times/F^\times : \mathrm{ord}(x) = 0 \textit{ in } \mathbb{Z}/2\mathbb{Z}\big\}.$$

In particular, $\#T_0/T_0^0 = 2$.

Indeed, Lemma B.1 allows us to write down explicit integral models for $\underline{T}$ and $\underline{T}^0$. Given integers n and n' satisfying $n \geq n' \geq n-1$, we consider the three-dimensional rational representation

$$\rho_{n,n'} : T(F) \longrightarrow \mathrm{GL}_3(F)$$

defined by

$$\rho_{n,n'}(x + y\pi) = \begin{pmatrix} 1 & 0 & 0 \\ \pi^{-n}(x-1) & x & -\pi^{n'-n+1}y \\ \pi^{-n'}y & \pi^{n-n'}y & ay + x \end{pmatrix}.$$

Then we have the following lemma.

LEMMA B.3
If d is odd, then $\underline{T}$ (resp., $\underline{T}^0$) is the scheme-theoretic closure in $\mathrm{GL}_3(A)$ *of the image of $\rho_{(d-1)/2,(d-1)/2}$ (resp.,$\rho_{(d+1)/2,(d-1)/2}$).*

If d is even, then $\underline{T}$ (resp., $\underline{T}^0$) is the scheme-theoretic closure in $\mathrm{GL}_3(A)$ *of the image of $\rho_{d/2,(d-2)/2}$ (resp., $\rho_{d/2,d/2}$).*

C. Quasi-split unitary group in two variables

Let $\mathbb{H}$ be the split rank-two hermitian space, as defined in Section 3, and let H be the corresponding unitary group. Note that H is quasi-split; we regard H as a subgroup of the general linear group of the underlying vector space of $\mathbb{H}$. We first recall some results from [MP1] and [MP2].

The maximal split torus of H is the one-dimensional torus

$$S = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in F^\times \right\},$$

and its centralizer in H is the torus

$$Z = \left\{ \begin{pmatrix} t & 0 \\ 0 & \bar{t}^{-1} \end{pmatrix} : t \in E^\times \right\}.$$

Note that Z has a natural filtration $(Z_r)_{r\geq 0}$ by open compact subgroups (see [MP2, Section 3.2, p. 103]). More explicitly, if we identify Z with $E^\times$ as above, then

$$Z_r = \left\{ t \in U_E : \mathrm{ord}(t-1) \geq r \right\}.$$

For $x \in E$, let $u_+(x)$ (resp., $u_-(x)$) be the upper (resp., lower) triangular unipotent 2×2 matrix, whose upper right (resp., lower left) entry is x. The root subgroups of H with respect to S are U^+ and U^-, where

$$U^\pm = \left\{ u_\pm(x) : \mathrm{Tr}(x) = 0 \right\}.$$

Note that the algebraic subgroup H_1 of H, which consists of elements of determinant one, is isomorphic to SL_2. An explicit isomorphism can be written down as follows. Pick $x_0 \in E$ whose trace is zero, and set $s = \mathrm{diag}(1, x_0)$. Then $g \mapsto sgs^{-1}$ is an isomorphism of H_1 with SL_2 over F. By Lemma A.1(vi), we may and do assume that $\mathrm{ord}(x_0) = \mathrm{ord}_E(x_0)/2 = d/2$.

The above isomorphism shows that there is a unique point y on the apartment $A(H_1, S) = A(H, S)$ (in the building of H) associated to S such that the filtration on $U^{\pm}$ determined by y, as defined by Bruhat and Tits [BT1], is given by

$$U^+_{y,r} = \left\{ u_+(x) : \mathrm{ord}(x) \ge r + \frac{d}{2} \right\},$$
$$U^-_{y,r} = \left\{ u_-(x) : \mathrm{ord}(x) \ge r - \frac{d}{2} \right\}.$$

By definition (see [MP2, Section 3]), for $r \ge 0$, the Moy-Prasad group $H_{y,r}$ is the open compact subgroup of $H(F)$ generated by Z_r, $U^+_{y,r}$, and $U^-_{y,r}$. Let $\underline{H}_y$ be the integral model of H associated to y by Bruhat-Tits theory. Then $H_y = \underline{H}_y(A)$ is the stabilizer of y under the action of $H(F)$ on the building and is equal to the Moy-Prasad group $H_{y,0}$. It is a special maximal compact subgroup of $H(F)$, and the maximal reductive quotient of the special fiber of $\underline{H}_y$ is isomorphic to SL_2.

The group $X_\bullet(S) \otimes \mathbb{R} \cong \mathbb{R}$ acts on $A(H_1, S)$ by translation. Let y' be the point obtained from y via translation by $-1/2$. Then y' is the midpoint of a fundamental chamber (which is a line segment) of $A(H_1, S)$, and y is a vertex. Let $\underline{H}_{y'}$ be the integral model associated to the stabilizer $H_{y'}$ in $H(F)$ of the chamber containing y'. Let $\underline{H}^0_{y'}$ be its connected component, which is the pointwise stabilizer of the chamber. Then $H^0_{y'} = \underline{H}^0_{y'}(A)$ is an Iwahori subgroup of $H(F)$. We have the Iwahori factorization, which is a direct product of sets:

$$H^0_{y'} = U^+_{y',0} \cdot Z_0 \cdot U^-_{y',0},$$

where $U^+_{y',r} = U^+_{y,r-1/2}$ and $U^-_{y',r} = U^-_{y,r+1/2}$. Moreover, $H_{y'}$ is generated by $H^0_{y'}$ and $N(Z)_{y'}$, the stabilizer of y' in the normalizer $N(Z)$ of Z. One checks by a direct computation that

$$N(Z)_{y'} = Z_0 \cup \left\{ g = \begin{pmatrix} 0 & a^{-1} \\ \overline{a} & 0 \end{pmatrix} : \mathrm{ord}_E(a) = d - 1 \right\}.$$

For g as above, with $\mathrm{ord}_E(a) = d - 1$, we have

$$\det(g) = -\overline{a}a^{-1} = \overline{(x_0 a)}(x_0 a)^{-1},$$

and $\mathrm{ord}_E(x_0 a) = 2d - 1$. Hence, $H^0_{y'}$ is precisely the subgroup of $H_{y'}$ consisting of those elements whose determinants lie in T^0_0 (as defined in Appendix B). In particular, by Lemma B.2, $H^0_{y'}$ has index 2 in $H_{y'}$. As before, we have the Moy-Prasad group $H_{y',r}$, generated by Z_r, $U^+_{y',r}$, and $U^-_{y',r}$. Moreover, $H_{y',0} = H^0_{y'}$.

Consider the lattice

$$L_i = A_E e \oplus \pi_E^i A_E f$$

in $\mathbb{H}$, and let H_{L_i} be its stabilizer in $H(F)$. It is easy to check that

$$H_{L_i} \cap U^{\pm} = U^{\pm}_{y,\pm(-i-d)/2}.$$

It follows that

$$H_y = H_{L_{-d}},$$
$$H_{y'} = H_{L_{1-d}}.$$

Note that L_{-d} is a maximal lattice in $\mathbb{H}$.

Henceforth, let $L = L_{1-d}$. For an integer $n > 0$, set

$$H_{L,n/2} = \left\{ g \in H(F) : (g-1)L \subset \pi_E^n L \right\}.$$

Then we have the following proposition.

PROPOSITION C.1
For any integer $n > 0$, $H_{L,n/2}$ is equal to the Moy-Prasad group $H_{y',n/2}$.

Proof
First, one checks by a direct computation that the groups $Z_{n/2}$ and $U^{\pm}_{y',n/2}$ are contained in $H_{L,n/2}$, so that $H_{y',n/2} \subset H_{L,n/2}$. Next, we claim that $H_{L,n/2} \subset H^0_{y'}$. Since $H_{L,n/2} \subset H_{y'} = H_L$, it suffices to check that $\det(H_{L,n/2}) \subset T^0_0$. Suppose that

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in H_{L,n/2}.$$

This implies that

$$\begin{cases} \mathrm{ord}_E(\alpha - 1) \geq n, \\ \mathrm{ord}_E(\delta - 1) \geq n, \\ \mathrm{ord}_E(\beta) \geq n + d - 1, \\ \mathrm{ord}_E(\gamma) \geq n + 1 - d. \end{cases}$$

If $\gamma = 0$, it is easy to show that $\det(g) \in T^0_0$. If $\gamma \neq 0$, then one shows that $\mathrm{Tr}(\overline{\alpha}\gamma) = 0$ and

$$\det(g) = \frac{-\gamma}{\overline{\gamma}} = \frac{(x_0\gamma)}{(\overline{x_0\gamma})}.$$

By Lemma A.1(vi), $\operatorname{ord}_E(\gamma) \equiv d \pmod 2$. Hence, $\operatorname{ord}_E(x_0\gamma)$ is even, and $\det(g) \in T_0^0$, as required.

Finally, we show that $H_{L,n/2} \subset H_{y',n/2}$. Take $g \in H_{L,n/2}$ as above. Since $g \in H_{y'}^0$, we can write $g = u^+(x) \cdot z_0 \cdot u^-(y)$ by the Iwahori factorization. The above congruence conditions on the entries of g then imply that $u^+(x) \in U_{y',n/2}^+$, $u^-(y) \in U_{y',n/2}^-$, and $z \in Z_{n/2}$, as required. □

D. Anisotropic unitary group in two variables

Let D be the unique quaternion algebra over F. Regard D as a hermitian space over E using (3.5) and (3.6). For the convenience of the reader, we repeat the definition here. We write D as $E \oplus E \cdot z$, with multiplication rules $z^2 = \alpha$, where $\alpha \notin \mathbb{N}E^\times$, and $x \cdot z = z \cdot \overline{x}$. By Lemma A.1(iv), we may and do assume that $\alpha \in U^{d-1} - U^d$. Moreover, the hermitian form on D is given by

$$\langle x_1 + x_2 \cdot z, y_1 + y_2 \cdot z\rangle = x_1\overline{y}_1 - \alpha x_2 \overline{y}_2.$$

The unitary group G corresponding to this hermitian space is anisotropic, and it is an inner form of H.

The unique maximal lattice M in D is simply the ring of integers A_D. We have the following lemma.

LEMMA D.1
The A_E-lattice M has basis $\left\{1, \pi_E^{1-d}(1+z)\right\}$.

Proof
By the choice of α, it is easy to check that the two elements 1 and $\pi_E^{1-d}(1+z)$ lie in M. Now suppose that $a = x + y(1+z)$ lies in M, so that $\langle a, a\rangle \in A$. We claim that $\operatorname{ord}_E(y) \geq 1 - d$. If not, then we have $\operatorname{ord}(\mathbb{N}(1 + x/y)\alpha^{-1} - 1) \geq d$. So $\mathbb{N}(1+x/y)\alpha^{-1}$ lies in U^d, which by Lemma A.1(iv) is contained in the image of the norm from E. This contradicts the choice of α, and the claim is proved. In particular, it follows that $y(1+z) \in M$, and hence so does x, which implies that $x \in A$. □

Now let F' be the unramified quadratic extension of F. Then $E' = E \otimes_F F'$ is a ramified quadratic extension of F', and the action of Γ on E extends to E', with fixed field F'. In addition, E' is unramified over E, and we can identify the Galois group of E'/E with that of F'/F. Further, the different of E'/F' is $(\pi_{E'}^d)$. Now $D' = D \otimes_F F'$ is a split hermitian space over E'. To write down an explicit isomorphism, pick two distinct elements α_1 and α_2 in E' such that $\mathbb{N}_{E'/F'}(\alpha_i) = \alpha$. This is possible by local class field theory. By Lemma A.1(iii) and the fact that $\mathbb{N}_{E'/F'}U_{E'}^d = U_{F'}^d$ (see [Se, Chapter V, Corollary 3.3]), we have $\alpha_i \in U_{E'}^{d-1} - U_{E'}^{d}$. Further, since we can find an element $x \in U_{E'}^{d-1} - U_{E'}^{d}$ such that $\mathbb{N}_{E'/F'}(x) = 1$, we can assume that

$$\mathrm{ord}_{E'}(\alpha_1 - \alpha_2) = d - 1.$$

Now put $f_i = \alpha_i + z$, for $i = 1$ or 2, and let $c = \langle f_1, f_2 \rangle = \alpha_1 \overline{\alpha}_2 - \alpha \in E'$. Then $\mathrm{ord}_{E'}(c) = d - 1$. Put

$$e_1 = c^{-1} f_1 \qquad \text{and} \qquad e_2 = f_2.$$

Then $\{e_1, e_2\}$ is a basis of D' over E', and the map

$$a_1 e_1 + a_2 e_2 \longmapsto (a_1, a_2)$$

defines the required isomorphism of D' with the split hermitian space $\mathbb{H}' = \mathbb{H} \otimes_E E'$. Using this isomorphism, we identify $G(F')$ with $H' = H(F')$.

Let M' be the lattice $M \otimes_{A_E} A_{E'}$ in D'. It is spanned by the two elements

$$\begin{cases} 1 = (\alpha_1 - \alpha_2)^{-1} \cdot (f_1 - f_2) \text{ and} \\ \pi_E^{1-d}(1 + z) = \pi_E^{1-d}(\alpha_1 - \alpha_2)^{-1} \cdot \big((1 - \alpha_2) f_1 + (\alpha_1 - 1) f_2\big). \end{cases}$$

Using the fact that $\mathrm{ord}_{E'}(\alpha_1 - \alpha_2) = \mathrm{ord}_{E'}(1 - \alpha_1) = \mathrm{ord}_{E'}(1 - \alpha_2) = d - 1$, it is easy to check that M' is spanned by $\pi_E^{1-d} f_1$ and $\pi_E^{1-d} f_2$, that is, by e_1 and $\pi_E^{1-d} e_2$. In particular, the above isomorphism of D' with $\mathbb{H}'$ identifies M' with $L' = L \otimes_{A_E} A_{E'}$.

Since $M = A_D$, the stabilizer G_M of M is the whole group $G(F)$. Moreover, for any integer $n > 0$, let

$$G_{M,n/2} = \big\{ g \in G_M : (g - 1)M \subset \pi_E^n M \big\}.$$

The group $G_{M,(2m+1)/2}$ is the group D_m in [S1, (17.1.2), p. 135]. The above discussion proves the following proposition.

PROPOSITION D.2

The group G_M is the $\mathrm{Gal}(E'/E)$*-fixed points of $H'_{y'}$. Moreover, for any integer $n \geq 0$, $G_{M,n/2}$ is the* $\mathrm{Gal}(E'/E)$*-fixed points of $H'_{y',n/2}$; that is, $G_{M,n/2}$ is the Moy-Prasad group $G_{y',n/2}$, where y' is the unique point in the building of G.*

COROLLARY D.3

The invariant $\lambda(\theta)$ defined in [S1, Lemma 17.9(2), p. 139] is always equal to 1.

Proof

It suffices to show that, for any $g \in G_{M,3/2}$, $\det(g) \in U_E^{d+2}$. By the proposition, we can regard g as an element of $H'_{y',3/2}$. Since $H'_{y',3/2}$ is generated by $Z'_{3/2}$ and $U'^{\pm}_{y',3/2}$, it suffices to show that every element of $Z'_{3/2}$ has determinant in $U^{d+2}_{E'}$. But this follows from [S1, Lemma 17.5, p. 137]. □

The proposition also allows us to compute the index $\#G_M/G_{M,n/2}$.

COROLLARY D.4
We have

$$\#G_M/G_{M,n/2} = \begin{cases} N \cdot q^{2n-2} & \text{if } n \text{ is odd,} \\ N \cdot q^{2n-1} & \text{if } n \text{ is even,} \end{cases}$$

where $N = 2(q+1)$.

Proof
By Proposition D.2,

$$\#G_M/G_{M,n/2} = \big(\#G_M/G_{y',1/2}\big) \cdot \big(\#G_{y',1/2}/G_{y',n/2}\big).$$

By Bruhat-Tits theory, the first factor is the order of the reductive quotient of the special fiber of $\underline{G}_{y'}$, the integral model of G associated to y'. As we have seen, the identity component of the reductive quotient is a one-dimensional anisotropic torus, and its group of components is equal to $\mathbb{Z}/2\mathbb{Z}$. Hence, we have

$$\#G_M/G_{y',1/2} = N.$$

On the other hand, if we let $G_{y',r:r^+}$ denote the quotient $G_{y',r}/G_{y',r^+}$ for $r > 0$, then the proposition, together with [Yu, Proposition 2.2], implies that $G_{x,r:r^+}$ is the set of $\mathrm{Gal}(F'/F)$-fixed points of $H'_{y',r:r^+}$. Now, for $r > 0$, we have the Moy-Prasad isomorphism $H'_{y',r:r^+} \cong \mathfrak{h}'_{y',r:r^+}$ (see [MP2, Section 3.3]), where $\mathfrak{h}'$ is the Lie algebra of H' and $(\mathfrak{h}'_{y',r})$ is the analogous filtration of the Lie algebra. The Moy-Prasad isomorphism is valid here because Z' is the multiplicative group of E'; it is not true for the group of norm-one elements in E' when E'/F' is wildly ramified. Now, using [Yu, Proposition 2.2] again, we have

$$\#G_{y',r:r^+} = \#\mathfrak{g}_{y',r:r^+}.$$

From this, the result follows easily. □

Acknowledgments. We would like to thank Professors Benedict Gross, Gopal Prasad, and Peter Sarnak for their interest in this work and for a number of helpful conversations.

References

[BT1] F. BRUHAT and J. TITS, *Groupes réductifs sur un corps local, II: Schémas en groupes; existence d'une donnée radicielle valuée,* Inst. Hautes Études Sci. Publ. Math. **60** (1984), 197–376.

[BT2] ———, *Schémas en groupes et immeubles des groupes classiques sur un corps local, II: Groupes unitaires*, Bull. Soc. Math. France **115** (1987), 141–195.

[Gr] B. H. GROSS, *On the motive of a reductive group*, Invent. Math. **130** (1997), 287–313.

[GrG] B. H. GROSS and W. T. GAN, *Haar measure and the Artin conductor*, Trans. Amer. Math. Soc. **351** (1999), 1691–1704.

[K] R. KOTTWITZ, *Tamagawa numbers*, Ann. of Math. (2) **127** (1988), 629–646.

[MP1] A. MOY and G. PRASAD, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), 393–408.

[MP2] A. MOY and G. PRASAD, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), 98–121.

[O] T. ONO, "On Tamagawa numbers" in *Algebraic Groups and Discontinuous Subgroups (Boulder, Colo., 1965)*, Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence, 1966, 122–132.

[P] G. PRASAD, *Volumes of S-arithmetic quotients of semisimple groups*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 91–117.

[S1] G. SHIMURA, *Euler Products and Eisenstein Series*, CBMS Reg. Conf. Ser. Math. **93**, Amer. Math. Soc., Providence, 1997.

[S2] ———, *An exact mass formula for orthogonal groups*, Duke Math. J. **97** (1999), 1–66.

[S3] ———, *Some exact formulas on quaternion unitary groups*, J. Reine Angew. Math. **509** (1999), 67–102.

[Se] J.-P. SERRE, *Local Fields*, Grad. Texts in Math. **67**, Springer, New York, 1979.

[T] J. TITS, "Reductive groups over local fields" in *Automorphic Forms, Representations, and L-functions, I (Oregon State Univ., Corvallis, 1977)*, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, 1979, 29–69

[Yu] J.-K. YU, *Construction of tame supercuspidal representations*, to appear in J. Amer. Math. Soc.

Gan
Mathematics Department, Princeton University, Fine Hall, Princeton, New Jersey 08544, USA; wtgan@math.princeton.edu

Hanke
Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854, USA; jonhanke@math.rutgers.edu

Yu
Mathematics Department, Princeton University, Fine Hall, Princeton, New Jersey 08544, USA; jkyu@princeton.edu