Group Schemes and Local Densities

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§1. Introduction

The subject matter of this paper is an old one with a rich history, beginning with the work of Gauss and Eisenstein, maturing at the hands of Smith and Minkowski, and culminating in the fundamental results of Siegel. More precisely, if L is a lattice over $\mathbb{Z}$ (for simplicity), equipped with an integral quadratic form Q, the celebrated Smith-Minkowski-Siegel mass formula expresses the total mass of (L, Q), which is a weighted class number of the genus of (L, Q), as a product of local factors. These local factors are known as the local densities of (L, Q). Subsequent work of Kneser, Tamagawa and Weil resulted in an elegant formulation of the subject in terms of Tamagawa measures. In particular, the local density at a non-archimedean place p can be expressed as the integral of a certain volume form ω^{ld} over $\mathrm{Aut}_{\mathbb{Z}_p}(L, Q)$, which is an open compact subgroup of $\mathrm{Aut}_{\mathbb{Q}_p}(L, Q)$.

The question that remains is whether one can find an explicit formula for the local density. Through the work of Pall (for $p \neq 2$) and Watson (for $p = 2$), such an explicit formula for the local density is in fact known for an arbitrary lattice over $\mathbb{Z}_p$ (see [P] and [Wa]). The formula is obviously structured, though [CS] seems to be the first to comment on this. Unfortunately, the known proof (as given in [P] and [K]) does not explain this structure and involves complicated recursions. On the other hand, Conway and Sloane [CS, §13] have given a heuristic explanation of the formula.

In this paper, we will give a simple and conceptual proof of the local density formula, for $p \neq 2$. The view point taken here is similar to that of our earlier work [GHY], and the proof is based on the observation that there exists a *smooth* affine group scheme $\underline{G}$ over $\mathbb{Z}_p$ with generic fiber $\mathrm{Aut}_{\mathbb{Q}_p}(L, Q)$, which satisfies $\underline{G}(\mathbb{Z}_p) = \mathrm{Aut}_{\mathbb{Z}_p}(L, Q)$. This follows from general results of smoothening [BLR], as we explain in Section 3. For the purpose of obtaining an explicit formula, it is necessary to have an explicit construction of $\underline{G}$. The main contribution of this paper is to give such an explicit construction of $\underline{G}$ (in Section 5), and to determine its special fiber (in Section 6). Finally, by comparing ω^{ld} and the canonical volume form ω^{can} of $\underline{G}$, we obtain the explicit formula for the local density in Section 7. The smooth group schemes constructed in this paper should also be of independent interest.

Our method works over any non-archimedean local field of residue characteristic $p \neq 2$, and also works for any types of classical groups. Therefore, we obtain new explicit formulas for local densities of lattices in symplectic spaces, hermitian spaces, and quaternionic hermitian and anti-hermitian spaces. For lattices in a symplectic space, or a hermitian space over an *unramified* quadratic extension, we note that such explicit formulas were obtained earlier in [HS1], [HS2] and [Hi] by very different techniques.

The restriction that $p \neq 2$ is actually not required for symplectic spaces, hermitian spaces over an unramified quadratic extension and quaternionic hermitian spaces, as we explain in Section 9.

For the remaining types of spaces, the case $p = 2$ is much more involved and will not be pursued here.

Finally, we have included an appendix on the Smith-Minkowski-Siegel mass formula. One reason for including this is that most treatments of this topic in the literature either worked over a number field of class number 1 (usually $\mathbb{Q}$), or worked explicitly with integral matrices. As such, the lattice involved is implicitly assumed to be *free*. For lattices which are not free, there is a minor subtlety which we will clarify in the appendix. Furthermore, we have not been able to find a reference which treats the mass formula for general types of classical groups in sufficient detail. Even in the case of orthogonal groups, these issues are only completely worked out in the recent Ph.D. thesis of Hanke [Ha].

As a consequence of the mass formula and our results on the local densities, we obtain an explicit formula for the mass of an *arbitrary* lattice in a quaternionic hermitian space. This extends a recent result of Shimura, who obtained an exact formula for the mass of a particular lattice called the maximal lattice.

§2. Notations

(2.1) Let F be a non-archimedean local field of residue characteristic $p \neq 2$ and let A be its ring of integers. Fix a uniformizing element π of F and let $\kappa = A/\pi A$. Let q be the cardinality of κ.

(2.2) Let (K, σ) be one of the following F-algebras with involution:

- $K = F$, $\sigma =$ identity;
- $K = E$, a quadratic extension, $\sigma =$ the unique non-trivial automorphism of E/F;
- $K = F \oplus F$, $\sigma(x, y) = (y, x)$;
- $K = D$, the quaternion algebra over F, $\sigma =$ the standard involution;
- $K = M_2(F)$, the algebra of 2×2 matrices, $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(2.3) Let B be a maximal A-order in K. Then B is uniquely determined except in the case $K = M_2(F)$, in which we may and do assume that $B = M_2(A)$. If $K = E$ is a ramified quadratic extension of F, or $K = D$, we let π_K be a uniformizer of K and put $e = 2$; in all other cases, we let $\pi_K = \pi$ and put $e = 1$.

(2.4) Let ϵ be either 1 or -1. The triple (K, σ, ϵ) will be fixed throughout this paper (at least until the Appendix), and by a hermitian form we will always mean a (σ, ϵ)-hermitian form. We consider a B-lattice L (i.e. a free *right* B-module of finite rank) with a hermitian form $\langle -, - \rangle : L \times L \to B$. Our convention is

$$\begin{aligned} \langle v.a, w.b \rangle &= \sigma(a) \langle v, w \rangle b, \\ \langle w, v \rangle &= \epsilon \sigma(\langle v, w \rangle). \end{aligned}$$

We assume that $V = L \otimes_A F$ is non-degenerate with respect to $\langle -, - \rangle$, in the sense that $\langle x, V \rangle = 0$ implies that $x = 0$.

The right B-module L is also regarded as a left B-module by the rule $a.v = v.\sigma(a)$.

(2.5) Definition The dual lattice of L, denoted by $L^\perp$, is

$$L^\perp = \{v \in L \otimes_A F : \langle x, L \rangle \subset B\}.$$

The pair $(L, \langle -, - \rangle)$ will be fixed throughout this paper.

(2.6) Let G be the reductive algebraic group over F such that

$$G(E) = \mathrm{Aut}_{K \otimes_F E}(V \otimes_F E, \langle -, - \rangle)$$

for any F-algebra E. Then G is a classical group, not necessarily connected. The group $G_L = \mathrm{Aut}_B(L, \langle -, - \rangle)$ is an open compact subgroup of $G(F)$.

We denote by $\underline{\mathrm{GL}_B(L)}_{/A}$ the A-group scheme whose group of R-valued points is

$$\mathrm{GL}_{B \otimes_A R}(L \otimes_A R)$$

for any commutative A-algebra R. If B is commutative, this is just the Weil restriction of scalars $\mathrm{Res}_{B/A} \mathrm{GL}_B(L)$. We define $\underline{\mathrm{GL}_K(V)}_{/F}$ in the same way.

§3. The Local Density

In this section, we explain our view point and strategy for proving the local density formula. We first recall how the local density arises, and our exposition here follows that of [S].

(3.1) Let H be the F-vector space of hermitian forms on $V = L \otimes_A F$, and let $h_0 \in H$ be our fixed hermitian form $\langle -, - \rangle$. Define a map

$$\tilde{f} : \mathrm{End}_K(V) \to H$$

by $\tilde{f}(t) = h_0 \circ t$. Here, $h_0 \circ t$ is the hermitian form $(v, w) \mapsto \langle t.v, t.w \rangle$. Then the inverse image of h_0 under $\tilde{f}$ is simply G.

(3.2) Regarding H and $M = \mathrm{End}_K(V)$ as varieties over F, let ω_H and ω_M be non-zero, translation invariant volume forms on H and M respectively. Then one can define a volume form ω on G in the following way. Let $M^* = \underline{\mathrm{GL}_K(V)}_{/F}$. It is easy to see that $f = \tilde{f}|_{M^*} : M^* \to H$ is smooth of relative dimension $\dim G$. Therefore, there is an exact sequence of locally free sheaves on M^*:

$$0 \to f^*\Omega_{H/F} \to \Omega_{M^*/F} \to \Omega_{M^*/H} \to 0.$$

This gives rise to an isomorphism

$$f^*\Big(\bigwedge^{\mathrm{top}} \Omega_{H/F}\Big) \otimes \bigwedge^{\mathrm{top}} \Omega_{M^*/H} \simeq \bigwedge^{\mathrm{top}} \Omega_{M^*/F}.$$

Let $\omega' \in \bigwedge^{\mathrm{top}} \Omega_{M^*/H}(M^*)$ be such that $f^*\omega_H \otimes \omega' = \omega_M|_{M^*}$. We then put $\omega = \omega'|_G$. It is easy to see that ω is a non-zero quasi-invariant differential on G, by which we mean that for any $g \in G(F)$, $g^*\omega = \chi(g) \cdot \omega$ for some rational character χ of G which is trivial on G°, the connected component of identity of G. Hence ω defines a Haar measure $|\omega|$ on $G(F)$. We shall sometimes denote ω by $\omega_M/f^*\omega_H$.

(3.3) Let H_L be the set of hermitian forms on V which take values in B when restricted to L. Then H_L is an A-lattice in H, which gives H an integral structure. Similarly, $\mathrm{End}_B(L)$ is an A-lattice in M, giving M an integral structure. The hermitian form h_0 is an element of H_L, and so we obtain a naive integral model $\underline{G}'$ of G. More precisely, for any commutative A-algebra R,

$$\underline{G}'(R) = \mathrm{Aut}_{B\otimes_A R}(L \otimes_A R, h_0 \otimes_A R).$$

(3.4) Lemma *Assume that* $\int_{H_L} |\omega_H| = 1$ *and* $\int_{\mathrm{End}_B(L)} |\omega_M| = 1$. *Put* $\omega^{\mathrm{ld}} = \omega_M/f^*(\omega_H)$. *Then*

$$\int_{G_L} |\omega^{\mathrm{ld}}| = \lim_{N\to\infty} q^{-N\dim G} \# \underline{G}'(A/\pi^N A),$$

where the limit stabilizes for N sufficiently large.

This lemma is well-known, at least when $(K, \epsilon) = (F, 1)$. See, for example, [S] or [Ha]. The general case is proved in the same way.

(3.5) Definition The local density of $(L, \langle -, - \rangle)$ is the quantity

$$\beta_L = \frac{1}{[G : G^\circ]} \cdot \lim_{N\to\infty} q^{-N\dim G} \# \underline{G}'(A/\pi^N A).$$

(3.6) The proof of the local density formula that one finds in the literature involves a computation of the stabilizing value of the above limit. This limit does not always stabilize at the first term, because the group scheme $\underline{G}'$ is not always smooth over A. The starting point of our work is the observation of the following Proposition.

(3.7) Proposition *There exists a unique* smooth *affine group scheme $\underline{G}$ over A such that $\underline{G}$ has generic fiber G and $\underline{G}(R) = \underline{G}'(R)$ for any étale A-algebra R.*

Proof. Applying the general theorem of group smoothening [BLR, Theorem 7.1.5] to $\underline{G}'$, we see that there exists a smooth group scheme $\underline{G}$ over A of finite type with generic fiber G satisfying $\underline{G}(R) = \underline{G}'(R)$ for any étale A-algebra R. By a result of Raynaud [SGA3, Exp. XVII, Appendice III, Prop. 2.1(iii)], $\underline{G}$ is affine.

The uniqueness follows from [BT2, 1.7]. ∎

(3.8) Remark The above proof also applies when $p = 2$, though when F is of equal characteristic 2 and G is a form of odd orthogonal group, one should replace G by its reduced subgroup to make sure that G is smooth.

(3.9) Let $\omega^{\rm can}$ be a differential of top degree on $\underline{G}_{/A}$, which is quasi-invariant under G and which has non-zero reduction on the special fiber. Note that $\omega^{\rm can}$ is well-defined up to a unit of A on each connected component of $\underline{G}$, and hence determines a Haar measure $|\omega^{\rm can}|$ on $G(F)$. Now the volume of G_L with respect to this Haar measure is given by:

$$\int_{G_L} |\omega^{\rm can}| = q^{-\dim G} \cdot \#\underline{G}(A/\pi A).$$

Moreover, $\#\underline{G}(A/\pi A)$ can be easily computed once we know its maximal reductive quotient. Hence, by Lemma (3.4), in order to obtain an explicit formula for the local density, it suffices to:

- determine the special fiber of $\underline{G}$, especially its maximal reductive quotient;
- relate the Haar measures $|\omega^{\rm ld}|$ and $|\omega^{\rm can}|$.

The abstract existence result in Proposition (3.7) does not help in the solution of the above two problems. In Section 5, we will give an explicit construction of $\underline{G}$, which makes all subsequent computations possible.

§4. Jordan Decomposition of Hermitian Modules

From now on till the end of §7, we assume that $K \neq M_2(F)$, because the case $K = M_2(F)$ is best handled by Morita context, and will be done in §8. Though we have restricted ourselves to the case where A is a complete discrete valuation ring, the results of this and the following two sections hold more generally for A a Henselian discrete valuation ring, with perfect residue field.

(4.1) For $a \in K^*$ such that $\sigma(a) = \epsilon a$, we denote by $\langle a \rangle$ the rank 1 lattice B equipped with the hermitian form $(x, y) \mapsto \sigma(x)ay$. For any $a \in K^*$, we denote by $\mathbf{H}_a$ the rank 2 lattice $B.e_1 + B.e_2$ with the hermitian form such that $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$, and $\langle e_1, e_2\rangle = a$.

(4.2) Proposition *Let L be as in (2.4).*

- *If $(K, \epsilon) = (F, 1)$, or $K = F \oplus F$, or $K = E$ is unramified, or $(K, \epsilon) = (D, -1)$, L is isomorphic to an orthogonal direct sum of lattices of the form $\langle a \rangle$.*
- *If $(K, \epsilon) = (F, -1)$, L is isomorphic to an orthogonal direct sum of lattices of the form $\mathbf{H}_a$.*
- *If $K = E$ is ramified or $(K, \epsilon) = (D, 1)$, L is isomorphic to an orthogonal direct sum of lattices of the form $\langle a \rangle$ or $\mathbf{H}_a$.*

Proof. The case $K = F$ is well-known, and the other cases are probably known too. Since the proofs for the various cases are similar, we will only sketch the argument for $(K, \epsilon) = (D, 1)$.

Let $r = \min\{\mathrm{ord}\langle v, v\rangle : v \in L\}$ and $s = \min\{\mathrm{ord}\langle v, w\rangle : v, w \in L\}$. Then r is an integer, s is half an integer, and obviously $s \leq r$. Since $I = \{\langle v, w\rangle : v, w \in L\}$ is a two-sided ideal in B, we must have $I = \pi_K^{2s} B$. Since the trace of $\langle x, y\rangle$ is $\langle x+y, x+y\rangle - \langle x, x\rangle - \langle y, y\rangle$, we have $\mathrm{tr}(I) \subset \pi_K^{2r} A$. This implies that $s = r$ or $s = r - 1/2$.

Suppose that $s = r$. Let $e_1 \in L$ be such that $\mathrm{ord}\langle e_1, e_1\rangle = r$ and choose $e_2, \ldots, e_n$ be such that $e_1, \ldots, e_n$ form a B-basis of L. We may replace e_i by $e_i - \langle e_1, e_1\rangle^{-1}\langle e_i, e_1\rangle e_1$ and assume that e_1

is perpendicular to $e_2, \dots, e_n$. Thus L is the orthogonal direct sum of $B.e_1$ and $B.e_2 + \cdots + B.e_n$. The result follows by induction.

Suppose that $s = r - 1/2$. Let $e_1, e_2 \in L$ be such that $s = \operatorname{ord}\langle e_1, e_2\rangle$. We claim that we may choose e_1 and e_2 so that $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$. Assuming this claim, an argument similar to the above shows that $B.e_1 + B.e_2$ is an orthogonal direct summand of L, and hence the result follows by induction.

To verify the claim, we use the following well-known presentation of D: D is generated by i, j such that $i^2 \in A^* \setminus (A^*)^2$, $j^2 = \pi$, and $ij = -ji$. By scaling $\langle -, -\rangle$ suitably, we can assume that $r = 0$ and $\langle e_1, e_2\rangle = \pi^{-1}j$. It suffices to show that for any $a, c \in A$, we can find $x \in B$ such that $a - \operatorname{tr}(\pi^{-1}xj) + cx\sigma(x) = 0$. We will actually show that there is a solution x in $A[j]$ (this fact is also needed in the case where $K = E$ is ramified).

Write $x = u + jv$, $u, v \in A$, the equation that we want to solve is $c(u^2 - \pi v^2) + a - 2v = 0$. This equation defines a closed subscheme of $\operatorname{Spec} A[u, v]$, which is smooth over A. Therefore the solvability follows from Hensel's lemma. ∎

(4.3) Corollary *There exists an orthogonal decomposition $L = \bigoplus_{i \geq 0} L_i$ such that the dual lattice of L_i is $\pi_K^{-i} L_i$.*

§5. The Smooth Model $\underline{G}$

In this section, we give an explicit construction of the smooth integral model $\underline{G}$, which was shown to exist in Proposition (3.7).

(5.1) Let L be as in (2.4) and let $L = \bigoplus L_i$ be a decomposition as in Corollary (4.3). Then $L^\perp = \bigoplus \pi_K^{-i} L_i$. Obviously, if $g \in G_L$, then g stablizes $L^\perp$. We can interpret this fact in terms of matrices as follows.

Let $n_i = \operatorname{rank}_B L_i$ and let $n = \operatorname{rank}_B L = \sum n_i$. Assume that $n_i = 0$ unless $0 \leq i < N$. In the following, we will always divide an $n \times n$ matrix into $N \times N$ blocks such that the (i, j)-block is of size $n_i \times n_j$. By choosing a B-basis for L_i for each i, we can represent an element g of $\operatorname{End}_B(L)$ by such an $N \times N$ matrix $m = m(g)$. If $g \in G_L$, the fact that g stabilizes $L^\perp$ means that

$$(*) \qquad \text{the } (i,j)\text{-block } m_{ij} \text{ has entries in } \begin{cases} \pi_K^{j-i} B \text{ if } j \geq i, \\ B \text{ if } j \leq i. \end{cases}$$

(5.2) We will first construct an affine scheme of rings $\underline{M}$ over A, representing matrices formally of this type. Define a functor from the category of commutative *flat* A-algebras to the category of rings as follows. For any commutative *flat* A-algebra R, set

$$\underline{M}(R) = \{t \in \operatorname{End}_{B \otimes_A R}(L \otimes_A R) : t(L^\perp \otimes_A R) \subset L^\perp \otimes_A R\}.$$

Then the functor $\underline{M}$ is representable by a unique *flat* A-algebra $A(\underline{M})$. We denote the spectrum of this algebra again by $\underline{M}$ (so now $\underline{M}(R)$ is defined for *any* A-algebra R). The representability is easily verified by thinking of elements of $\underline{M}(R)$ as matrices satisfying $(*)$ and then directly writing down the affine ring of $\underline{M}$, which is simply a polynomial ring over A of $n^2 \cdot [K : F]$ variables. Moreover, it is easy to see that $\underline{M}$ has the structure of a scheme of rings, since the set of matrices of the form $(*)$ is closed under addition and multiplication.

(5.3) We stress that the above description of $\underline{M}(R)$ is only valid if R is a flat A-algebra. Suppose now that R is a κ-algebra. Then, by choosing a B-basis for each L_i, we can describe each element of $\underline{M}(R)$ formally as a matrix $(\pi_K^{\max(0,j-i)} u_{ij})$, where u_{ij} is an $n_i \times n_j$ matrix with entries in $R \otimes_A B$. Moreover, the addition law on the ring $\underline{M}(R)$ is defined in the obvious way. The multiplication law, on the other hand, is given as follows. To multiply (u_{ij}) and (u'_{ij}), we form the matrices $m = (\pi_K^{\max(0,j-i)} u_{ij})$ and $m' = (\pi_K^{\max(0,j-i)} u'_{ij})$, and write $m \cdot m' = (\pi_K^{\max(0,j-i)} u''_{ij})$. Then (u''_{ij}) is the product of (u_{ij}) and (u'_{ij}).

Since $\underline{M}$ is a scheme of rings, the functor $R \mapsto \underline{M}(R)^*$ is represented by a group scheme $\underline{M^*}$. It is easy to see that $\underline{M^*}$ is an open subscheme of $\underline{M}$, with generic fiber $M^* = \underline{\mathrm{GL}_K(V)}_{/F}$, and $\underline{M}^*$ is smooth over A.

(5.4) For any *flat* A-algebra R, let $\underline{H}(R)$ be the set of hermitian forms h on $L \otimes_A R$ (with values in $B \otimes_A R$) such that $h(L, L^\perp) \subset R \otimes_A B$ (recall that $\langle -, - \rangle$ is fixed throughout the paper and $L^\perp$ is the dual lattice of L defined in (2.5)). It is easy to see that $\underline{H}$ is represented by a *flat* A-scheme, again denoted by $\underline{H}$, which is isomorphic to an affine space of dimension $n^2[K : F] - \dim G$.

The group $\underline{M^*}(R)$ acts on the right of $\underline{H}(R)$ by $h.t = h \circ t$. It is easy to see that this action is represented by an action morphism

$$\underline{H} \times \underline{M^*} \to \underline{H}.$$

Note that our fixed hermitian form h_0 is an element of $\underline{H}(A)$. Now we have the following crucial result.

(5.5) Theorem *Let f be the morphism $\underline{M^*} \to \underline{H}$ defined by $f(t) = h_0 \circ t$. Then f is smooth of relative dimension* $\dim G$.

The proof of Theorem (5.5) depends on the following two lemmas.

(5.5.1) Lemma *Let S be a noetherian scheme and $f : X \to Y$ be a morphism of S-schemes. Assume that both X, Y are of finite type over S. Suppose*

(i) X is flat over S;

(ii) $f_s = f \times_S \kappa(s) : X_s \to Y_s$ is smooth for all $s \in S$.

Then f is smooth.

Proof. By (ii), f_s is flat for all $s \in S$. By [BLR, Prop. 2.4.2], f is flat. Let $x \in X$. Set $y = f(x)$ and let $s \in S$ be such that both x and y are lying above s. Since f_s is smooth at x, $X_y = (X_s)_y$ is smooth over $\kappa(y)$. By [BLR, Prop. 2.4.7], this implies that f is smooth at x. ∎

(5.5.2) Lemma *The morphism $f \otimes \kappa : \underline{M^*} \otimes \kappa \to \underline{H} \otimes \kappa$, is smooth of relative dimension* $\dim G$.

Proof. It is enough to check the statement over the algebraic closure $\bar{\kappa}$ of κ. By [H, III.10.4], it suffices to show that for any $m \in \underline{M^*}(\bar{\kappa})$, the induced map on the Zariski tangent space $f_* : T_m \to T_{f(m)}$ is surjective.

To facilitate computations, we think of elements of $\underline{M}(R)$ as as in (5.3). Similarly, we can think of elements of $\underline{H}(R)$ formally as $n \times n$ hermitian matrices h whose (i,j)-block is of the form $\pi_K^{\max(i,j)} h_{ij}$, where h_{ij} has entries in $R \otimes_A B$. The action morphism $\underline{H} \times \underline{M^*} \to \underline{H}$ is simply

$$(h, m) \mapsto \sigma({}^{\mathrm{t}}m).h.m,$$

where the multiplication is to be interpreted as in (5.3).

We introduce still another functor on flat A-algebras: define $\underline{M'}(R)$ to be the set of all $n \times n$ matrix h over $R \otimes_A B$ such that the (i,j)-block h_{ij} of h has entries in $\pi_K^{\max(i,j)} R \otimes_A B$. It is easy to see that $\underline{M'}$ is represented by a *flat* A-scheme, again denoted by $\underline{M'}$. The matrix products $(m, m') \to \sigma({}^{\mathrm{t}}m).m'$ and $(m, m') \to m'.m$ induce two morphisms $\underline{M} \times_A \underline{M'} \to \underline{M'}$ of schemes over A.

Then we can identify T_m with $\underline{M}(\bar{\kappa})$ and $T_{f(m)}$ with $\underline{H}(\bar{\kappa}) \subset \underline{M'}(\bar{\kappa})$. The map $f_* : T_m \to T_{f(m)}$ is then $X \mapsto \sigma({}^{\mathrm{t}}m).h_0.X + \sigma({}^{\mathrm{t}}X).h_0.m$.

The desired surjectivity now follows from the following three easy statements:

1. $X \mapsto h_0.X$ is a bijection $\underline{M}(\bar{\kappa}) \to \underline{M'}(\bar{\kappa})$.

2. $m' \mapsto \sigma({}^{\mathrm{t}}m') + \epsilon m'$ is a surjection $\underline{M'}(\bar{\kappa}) \to \underline{H}(\bar{\kappa})$.

3. For any $m \in \underline{M^*}(\bar{\kappa})$, $m' \mapsto \sigma({}^{\mathrm{t}}m).m'$ is a bijection from $\underline{M'}(\bar{\kappa})$ to itself. ∎

We now give the proof of Theorem (5.5). It is clear that $\underline{M^*}$ is flat over A and $f \otimes_A F$ is smooth. By Lemma (5.5.2), $f \otimes_A \kappa$ is smooth. Applying Lemma (5.5.1) with $(X, Y, S) = (\underline{M^*}, \underline{H}, \operatorname{Spec} A)$, the theorem follows. ∎

(5.6) Let $\underline{G}$ be the stabilizer of h_0 in $\underline{M}^*$. It is an affine group subscheme of $\underline{M}^*$, defined over A.

(5.7) Theorem *The group scheme $\underline{G}$ is smooth, and $\underline{G}(R) = \mathrm{Aut}_{B \otimes_A R}(L \otimes_A R, \langle -, - \rangle)$ for any étale A-algebra R.*

Proof. Regard h_0 as a morphism $\operatorname{Spec} A \to \underline{H}$. Then $\underline{G} \to \operatorname{Spec} A$ is simply the base change of $\underline{M}^* \to \underline{H}$ by this morphism. Therefore, the first statement follows from Theorem (5.5). The second assertion follows from the definition of $\underline{G}$. ∎

(5.8) We can now give another description of $\underline{G}$, which is more concise, but not as informative as the construction above. When $L \subset L^\perp \subset \pi_K^{-1} L$, G_L is a maximal parahoric subgroup of G and the following result specializes to the construction in [BT].

Proposition *Let*

$$\rho : G \to \underline{\mathrm{GL}_K(V)}_{/F} \times_F \underline{\mathrm{GL}_K(V)}_{/F}$$

be the direct sum of two copies of the standard representations. Then the schematical closure of $\rho(G)$ in $\underline{\mathrm{GL}_B(L)}_{/A} \times_A \underline{\mathrm{GL}_B(L^\perp)}_{/A}$ is isomorphic to $\underline{G}$.

Proof. Let $\underline{G'}$ be this schematical closure. Clearly $\underline{G'}(R) = \underline{G}(R)$ for any étale algebra R over A. Let

$$\tilde{\rho} : \underline{\mathrm{GL}_K(V)}_{/F} \to \underline{\mathrm{GL}_K(V)}_{/F} \times \underline{\mathrm{GL}_K(V)}_{/F}$$

be the direct sum of two copies of the identity homomorphism. Then it is easy to see that the schematical closure of $\tilde{\rho}(\underline{\mathrm{GL}_K(V)}_{/F})$ in $\underline{\mathrm{GL}_B(L)}_{/A} \times_A \underline{\mathrm{GL}_B(L^\perp)}_{/A}$ is nothing but $\underline{M^*}$. By the definition of schematical closure, $\underline{G'}$ is flat and there is a surjection pr' from the affine ring $A[\underline{M^*}]$ onto $A[\underline{G'}]$. By the construction in (5.4) and (5.6), $A[\underline{M^*}]$ also maps onto $A[\underline{G}]$, via a surjection pr, and there is a surjection $\phi : A[\underline{G}] \to A[\underline{G'}]$, such that $\mathrm{pr}' = \phi \circ \mathrm{pr}$. It is clear that $\phi \otimes F$ is the identity $F[G] \to F[G]$. Since both $\underline{G}$ and $\underline{G'}$ are flat over A, the maps $A[\underline{G}] \to F[G]$ and $A[\underline{G'}] \to F[G]$ are injective. Therefore, ϕ is injective, hence bijective. The proposition is proved. ∎

(5.9) Remark The results in this section cover the case $K = F \oplus F$. However, this case can also be dealt with separately: a free K-module V is of the form $W \oplus W$ for some F-vector space W, and then G is isomorphic to $\mathrm{GL}_F(W)$. It is easy to show that G_L is the intersection of the stablizers (in $\mathrm{GL}_F(W)$) of two lattices $M', M'' \subset W$. By the theory of elementary divisors, we may assume that there is a decomposition $M' = \bigoplus M_i$ and $M'' = \bigoplus \pi^{-i} M_i$. It follows that $\underline{G}$ is simply the group scheme $\underline{M^*}$ constructed in (5.2), with $(B, V, L, L^\perp)$ replaced by (A, W, M', M'').

§6. The Special Fiber of $\underline{G}$

The purpose of this section is to determine the structure of the special fiber $\tilde{G}$ of $\underline{G}$. We keep the notations from the previous section. For each i, put

$$L^{(i)} = \bigoplus_{j<i} \pi_K^{i-j} L_j + \bigoplus_{j \geq i} L_j = \{x \in L : \langle x, L \rangle \subset \pi_K^i B\}.$$

(6.1) Denote by $\tilde{M}$ the special fiber of $\underline{M^*}$. Let

$$\tilde{M}_i = \underline{\mathrm{GL}_{B/\pi_K B}(L^{(i)}/\pi_K L^{(i)})}_\kappa,$$

regarded as a κ-algebraic group, as in (2.6). For any κ-algebra R, let $m = (\pi_K^{\max(0,j-i)} u_{ij}) \in \tilde{M}(R)$. Then $u_{ii} \in \tilde{M}_i(R)$ for all i. Therefore, we have a morphism of algebraic varieties

$$r : \tilde{M} \to \prod \tilde{M}_i,$$

given by $m \mapsto (u_{ii})$. It is easy to see that r is a homomorphism of algebraic groups, and we have the following lemma.

Lemma *The kernel of r is the unipotent radical $\tilde{M}^+$ of $\tilde{M}$ and $\prod \tilde{M}_i$ is the maximal reductive quotient of $\tilde{M}$.*

(6.2) Suppose that $e = 1$. Consider the (σ, ϵ)-hermitian form $(x, y) \mapsto \pi^{-i}\langle x, y\rangle \bmod \pi$ on the $B/\pi B$-vector space $V_i' = L^{(i)}/\pi L^{(i)}$. Let V_i'' be the kernel of this hermitian form and $V_i = V_i'/V_i''$. We define G_i to be the isometry group of V_i with the induced hermitian form.

We can represent $\langle -, -\rangle$ by a hermitian matrix $\mathrm{diag}(\delta_0, \pi_K\delta_1, \cdots, \pi_K^{N-1}\delta_{N-1})$. Then $\delta_i \bmod \pi$ is a (σ, ϵ)-hermitian matrix over $B/\pi_K B$ representing the hermitian form on V_i.

For any κ-algebra R and any $m = (\pi_K^{\max(0,j-i)}u_{ij}) \in \tilde{G}(R)$, we have $u_{ii} \in G_i(R)$. Therefore, we have a homomorphism of algebraic groups $r : \tilde{G} \to \prod G_i$, $m \mapsto (u_{ii})$.

(6.2.1) Proposition *The maximal reductive quotient of $\tilde{G}$ is $\prod G_i$.*

Proof. The map $\tilde{G}(\bar{\kappa}) \to \prod G_i(\bar{\kappa})$ is surjective: in fact, if $G'(\bar{\kappa})$ is the subgroup consisting of those $m = (\pi_K^{\max(0,j-i)}u_{ij})$ such that $u_{ij} = 0$ for $i \neq j$, then $G'(\bar{\kappa}) \to \prod G_i(\bar{\kappa})$ is already surjective.

Since both $\tilde{G}$ and $\prod G_i$ are smooth, r is a quotient map, by [W, 15.2]. Since $\prod G_i$ is clearly reductive, it remains to show that the kernel U of r is unipotent and connected. Being a subgroup of $\tilde{M}^+$, U is clearly unipotent.

The equations defining $\tilde{G}$ are the following:

$$\delta_i = \sigma({}^{\mathrm{t}}u_{ii})\delta_i u_{ii}, \qquad 0 \le i < N$$
$$0 = \sum_{i \le k \le j} \sigma({}^{\mathrm{t}}u_{ki})\delta_k u_{kj}, \qquad 0 \le i < j < N.$$

The equations defining U is obtained by setting $u_{ii} = 1$ in the above equations. It follows easily that as an algebraic variety over κ, U is isomorphic to an affine space of dimension $\sum_{i<j} n_i \cdot n_j \cdot [K : F]$. This shows that U is connected, and also gives a second proof of the unipotency of U. ∎

(6.2.2) Remark The above equations also show that $\tilde{G}$, as an algebraic variety, is isomorphic to the direct product of $\prod G_i$ and U. This gives a second proof of the smoothness of $\tilde{G}$. We can also remark that the surjectivity of $\tilde{G}(\bar{\kappa}) \to \prod G_i(\bar{\kappa})$ and the smoothness of $\prod G_i$ already implies that r is a quotient map, and therefore the smoothness of $\prod G_i$ and U gives a third proof of the smoothness of $\tilde{G}$.

(6.2.3) Proposition *The type of G_i is as follows:*

- *If $(K, \epsilon) = (F, 1)$, then*

$$G_i = \begin{cases} \mathrm{O}(n_i), & \text{if } \det(\delta_i)(\text{mod } \pi) \in \kappa^{\times 2}; \\ {}^2\,\mathrm{O}(n_i), & \text{otherwise.} \end{cases}$$

- *If $(K, \epsilon) = (F, -1)$, $G_i = \mathrm{Sp}(n_i)$.*
- *If K/F is an unramified quadratic extension, $G_i = \mathrm{U}(n_i)$.*
- *If $K = F \oplus F$, $G_i = \mathrm{GL}(n_i)$.*

Here, $\mathrm{O}(n_i)$ (respectively ${}^2\,\mathrm{O}(n_i)$) denotes the split (respectively non-split) orthogonal group over κ in n_i variables, $\mathrm{Sp}(n_i)$ the symplectic group in n_i variables and $\mathrm{U}(n_i)$ the unitary group in n_i variables.

(6.3) The remainder of this section is devoted to the case $e = 2$, which is somewhat more complicated than the case $e = 1$ treated above. Let σ_i be the reduction modulo π_K of the automorphism

$$x \mapsto \pi_K^{-i} \sigma(x) \pi_K^i$$

of B. Consider $V_i' = L^{(i)}/\pi_K L^{(i)}$. Then

$$(x, y) \mapsto \pi_K^{-i} \langle x, y \rangle \bmod \pi_K$$

is a $(\sigma_i, (-1)^i \epsilon)$-hermitian form on V_i'. Let V_i be the maximal non-degenerate quotient of V_i' with respect to this hermitian form. We define G_i to be the isometry group of V_i.

If we represent $\langle -, - \rangle$ by a block diagonal hermitian matrix $\mathrm{diag}(\pi_K^i \delta_i)$, then $\delta_i \bmod \pi_K$ is a $(\sigma_i, (-1)^i \epsilon)$-hermitian matrix over $B/\pi_K B$ representing the hermitian form on V_i.

Again, the map

$$m = (\pi_K^{\max(0, j-i)} u_{ij}) \mapsto (u_{ii} \bmod \pi_K)$$

is a homomorphism of algebraic groups $r : \tilde{G} \to \prod G_i$.

(6.3.1) Proposition *The group $\prod G_i$ is the maximal reductive quotient of $\tilde{G}$.*

The proof of this Proposition depends on a series of lemmas.

(6.3.2) Lemma *If $L = L_0$ (resp. $L = L_1$), the map $\tilde{G}(\bar{\kappa}) \to G_0(\bar{\kappa})$ (resp. $\tilde{G}(\bar{\kappa}) \to G_1(\bar{\kappa})$) is surjective.*

Proof. If $L = L_0$ (resp. $L = L_1$), then L is a self-dual lattice (resp. $\pi^{-1} L$ is a maximal lattice). Since $\underline{G}$ is smooth, it is the Bruhat-Tits scheme associated to a maximal parahoric subgroup (cf. [T], [BT], and [GHY]). Using Bruhat-Tits theory (see [BT2, 4.6.10] and [T, 3.5.1 and 3.5.2]), one can check that G_0 is actually the maximal reductive quotient of $\tilde{G}$. Then the surjectivity statement follows. ∎

(6.3.3) Lemma *Let $1 \to X \to Y \to Z \to 1$ be an exact sequence of group schemes which are locally of finite type over κ. Suppose that X is smooth, connected, and unipotent. Then $1 \to X(R) \to Y(R) \to Z(R) \to 1$ is exact for any κ-algebra R.*

Proof. Since the group schemes are locally of finite type over κ, $1 \to X \to Y \to Z \to 1$ is an exact sequence of sheaves on the (big) fppf site of $\mathrm{Spec}\,\kappa$. Hence it suffices to show that the pointed set $H^1_{\mathrm{fppf}}(\mathrm{Spec}\, R, X)$ is trivial for all κ-algebra R.

Since κ is perfect, X is a split unipotent group [Sp, Theorem 14.3.8(iii)]; that is, X is a successive extension of additive groups $\mathbb{G}_a$'s. Therefore, it suffices to show that $H^1_{\mathrm{fppf}}(\mathrm{Spec}\, R, \mathbb{G}_a) = 0$. This follows from [SGA4, Exp. VII, Proposition 4.3 and Remark 4.5]. ∎

(6.3.4) Recall that $\tilde{M}$ is the special fiber of $\underline{M}^*$, and $\tilde{M}^+$ the unipotent radical of $\tilde{M}$. Notice that $\underline{M}(R)$ is a $B/\pi B$-algebra for any κ-algebra R. Therefore, we can consider the subfunctor $\underline{\pi_K M} : R \mapsto \pi_K \underline{M}(R)$ of $\underline{M} \otimes \kappa$, and the subfunctor $\tilde{M}^1 : R \mapsto 1 + \pi_K \underline{M}(R)$ of $\tilde{M}$. Then we have the following easy lemma.

Lemma *(i) The functor $\tilde{M}^1$ is representable by a smooth, connected, unipotent group scheme over κ. Moreover, $\tilde{M}^1$ is a closed normal subgroup of $\tilde{M}^+$.*

(ii) The quotient group scheme $\tilde{M}^+/\tilde{M}^1$ represents the functor $R \mapsto \tilde{M}^+(R)/\tilde{M}^1(R)$, and is smooth, connected and unipotent.

(6.3.5) From the lemma, we can describe the functors of points of the schemes $\tilde{M}, \tilde{M}^+$ and $\tilde{M}^+/\tilde{M}^1$ as follows:

$$\begin{aligned}
\tilde{M}(R) &= \{(u_{ij}) : u_{ij} \in M_{n_i \times n_j}(R \otimes B/\pi B), u_{ii} \text{ is invertible for all } i\},\\
\tilde{M}^+(R) &= \{(u_{ij}) : u_{ij} \in M_{n_i \times n_j}(R \otimes B/\pi B), u_{ii} = 1 \pmod{\pi_K} \text{ for all } i\},\\
(\tilde{M}^+/\tilde{M}^1)(R) &= \{(u_{ij}) : u_{ij} \in M_{n_i \times n_j}(R \otimes B/\pi_K B), u_{ii} = 1 \text{ for all } i\}.
\end{aligned}$$

We remark that the above only describes the underlying schemes. The group law is to be interpreted as in (5.3): to multiply (u_{ij}) and (u'_{ij}), let $m = (\pi_K^{\max(0,j-i)} u_{ij})$, $m' = (\pi_K^{\max(0,j-i)} u'_{ij})$ and write $m.m' = (\pi_K^{\max(0,j-i)} u''_{ij})$; then (u''_{ij}) is the product of (u_{ij}) and (u'_{ij}).

(6.3.6) Recall that there is a closed immersion $\tilde{G} \to \tilde{M}$. We define $\tilde{G}^+$ and $\tilde{G}^1$ to be the kernels of the compositions

$$\tilde{G} \to \tilde{M} \to \tilde{M}/\tilde{M}^+$$

and

$$\tilde{G} \to \tilde{M} \to \tilde{M}/\tilde{M}^1$$

respectively. Then $\tilde{G}^1$ is the kernel of the morphism $\tilde{G}^+ \to \tilde{M}^+/\tilde{M}^1$, and hence is a closed normal subgroup of $\tilde{G}^+$. The induced morphism $\tilde{G}^+/\tilde{G}^1 \to \tilde{M}^+/\tilde{M}^1$ is a monomorphism, and thus $\tilde{G}^+/\tilde{G}^1$ is a closed subgroup scheme of $\tilde{M}^+/\tilde{M}^1$ by [SGA3, Exp. $\mathrm{VI_B}$, Cor. 1.4.2].

(6.3.7) Lemma *(i) $\tilde{G}^1$ is connected, smooth, and unipotent.*

(ii) $\tilde{G}^+/\tilde{G}^1$ is connected, smooth, and unipotent.

Proof. The equations defining $\tilde{G}$ are the following:

$$\begin{aligned}
\delta_i = \sigma_i({}^{\mathrm{t}}u_{ii})\delta_i u_{ii} &\\
&- \sigma_i({}^{\mathrm{t}}u_{i-1,i})\delta_{i-1}\pi_K u_{i-1,i} + \sigma_i({}^{\mathrm{t}}u_{i+1,i})\pi_K \delta_{i+1} u_{i+1,i},\\
&\qquad 0 \le i < N\\
0 = \sum_{i \le k \le j} \sigma_j({}^{\mathrm{t}}u_{ki})\big(\pi_K^{k-j}\delta_k \pi_K^{j-k}\big) u_{kj} &\\
&- \sigma_j({}^{\mathrm{t}}u_{i-1,i})\big(\pi_K^{i-j}\delta_{i-1}\pi_K^{j-i}\big)\pi_K u_{i-1,j} + \sigma_j({}^{\mathrm{t}}u_{j+1,i})\pi_K \delta_{j+1} u_{j+1,j},\\
&\qquad 0 \le i < j < N.
\end{aligned}$$

The equations defining $\tilde{G}^1$ are obtained by setting

$$\begin{aligned} u_{ii} &= 1 + \pi_K u'_{ii}, \\ u_{ij} &= \pi_K u'_{ij} \ (i \neq j) \end{aligned}$$

in the above equations. It is then easy to check that the underlying algebraic variety of $\tilde{G}^1$ is simply an affine space. This proves (i).

Now (i) and Lemma (6.3.3) imply that $\tilde{G}^+/\tilde{G}^1$ represents the functor $R \mapsto \tilde{G}^+(R)/\tilde{G}^1(R)$. If $m = (u_{ij}) \in (\tilde{M}^+/\tilde{M}^1)(R)$ is such that $m \in (\tilde{G}^+/\tilde{G}^1)(R)$, then obviously (u_{ij}) satisfies the following equations (which are given as equalities in $R \otimes_A B/\pi_K B$):

$$\begin{aligned} 1 &= u_{ii}, && 0 \leq i < N, \\ 0 &= \sum_{i \leq k \leq j} \sigma_j({}^{\mathrm{t}}u_{ki})(\pi_K^{k-j}\delta_k \pi_K^{j-k})u_{kj} && 0 \leq i < j < N. \end{aligned}$$

Let $G^{\ddagger}$ be the subfunctor of $\tilde{M}^+/\tilde{M}^1$ consisting of those (u_{ij}) satisfying the above equations. Then it is easy to check that $G^{\ddagger}$ is represented by a smooth, connected closed subscheme of $\tilde{M}^+/\tilde{M}^1$, and is isomorphic to an affine space over κ.

For ease of notation, let $G^{\dagger} = \tilde{G}^+/\tilde{G}^1$. Since $G^{\dagger}$ and $G^{\ddagger}$ are both closed subschemes of $\tilde{M}^+/\tilde{M}^1$ and $G^{\dagger}(\bar{\kappa}) \subset G^{\ddagger}(\bar{\kappa})$, $(G^{\dagger})^{\mathrm{red}}$ is a closed subscheme of $(G^{\ddagger})^{\mathrm{red}} = G^{\ddagger}$. It is easy to check that $\dim G^{\dagger} = \dim G^{\ddagger}$. Since $G^{\ddagger}$ is irreducible, we must have $(G^{\dagger})^{\mathrm{red}} \simeq G^{\ddagger}$, and hence $G^{\dagger} = G^{\ddagger}$ because $G^{\dagger}$ is a subfunctor of $G^{\ddagger}$. This proves (ii). ∎

We can now prove Proposition (6.3.1). Again, if we consider the subgroup scheme $\tilde{G}'$ of $\tilde{G}$ consisting of diagonal block matrices (i.e. those $m = (\pi_K^{\max(0,j-i)} u_{ij})$ with $u_{ij} = 0$ unless $i = j$), then the map $\tilde{G}'(\bar{\kappa}) \to \prod G_i(\bar{\kappa})$ is already surjective, by Lemma (6.3.2).

Therefore, the map $\tilde{G} \to \prod G_i$ is a quotient map, whose kernel is by definition $\tilde{G}^+$. By the preceding lemma, $\tilde{G}^+$ is connected, smooth and unipotent. Since $\prod G_i$ is obviously reductive, the proposition is proved completely. ∎

(6.3.8) Remark As in (6.2.2), the above analysis can be used to give another proof of the smoothness of $\tilde{G}$, and thus another proof of the smoothness of $\underline{G} \to \operatorname{Spec} A$. We omit the details.

(6.3.9) Proposition *The type of G_i is as follows:*

– *If K/F is a ramified quadratic extension and $\epsilon = (-1)^s$, then*

$$G_i = \begin{cases} \mathrm{O}(n_i), & \text{if } i+s \text{ is even and } \det(\delta_i)(\bmod\ \pi_K) \in \kappa^{\times 2}; \\ {}^2\mathrm{O}(n_i), & \text{if } i+s \text{ is even and } \det(\delta_i)(\bmod\ \pi_K) \notin \kappa^{\times 2}; \\ \mathrm{Sp}(n_i), & \text{if } i+s \text{ is odd.} \end{cases}$$

– *If $K = D$ and $\epsilon = 1$, then*

$$G_i = \begin{cases} \mathrm{U}(n_i), & \text{if } i \text{ is even;} \\ \mathrm{Res}_{\kappa_2/\kappa}(Sp(n_i)), & \text{if } i \text{ is odd.} \end{cases}$$

– *If $K = D$ and $\epsilon = -1$, then*

$$G_i = \begin{cases} \mathrm{U}(n_i), & \text{if } i \text{ is even;} \\ \mathrm{Res}_{\kappa_2/\kappa}(\mathrm{O}(n_i)), & \text{if } i \text{ is odd and } \det(\delta_i) (\mathrm{mod}\ \pi_K) \in \kappa_2^{\times 2}; \\ \mathrm{Res}_{\kappa_2/\kappa}({}^2\mathrm{O}(n_i)), & \text{if } i \text{ is odd and } \det(\delta_i) (\mathrm{mod}\ \pi_K) \notin \kappa_2^{\times 2}. \end{cases}$$

Here, κ_2 is the quadratic extension of κ, and $\mathrm{Res}_{\kappa_2/\kappa}$ denotes the Weil restriction of scalars.

§7. Comparison of Volume Forms and Final Formulas

(7.1) In the construction of (3.2), pick ω'_M and ω'_H to be such that

$$\int_{\underline{M}(A)} |\omega'_M| = 1, \qquad \text{and} \qquad \int_{\underline{H}(A)} |\omega'_H| = 1.$$

Put $\omega^{\mathrm{can}} = \omega'_M / f^*\omega'_H$. By Theorem (5.5), we have an exact sequence of locally free sheaves on $\underline{M}^*$.

$$0 \to f^*\Omega_{\underline{H}/A} \to \Omega_{\underline{M^*}/A} \to \Omega_{\underline{M^*}/\underline{H}} \to 0.$$

It follows that ω^{can} is of the type discussed in (3.9).

(7.2) Lemma *Let $d = \dim_\kappa(B/\pi_K B)$. Then*

$$\begin{aligned} \omega_M &= \pi^{N_E}\omega'_M, & N_E &= \sum_{j>i} d\cdot(j-i)\cdot n_i\cdot n_j, \\ \omega_H &= \pi^{N_H}\omega'_H, & N_H &= \sum_{j>i} d\cdot j\cdot n_i\cdot n_j + \sum_i d_i, \\ \omega^{\mathrm{ld}} &= \pi^{N_E - N_H}\omega^{\mathrm{can}}. \end{aligned}$$

Here

$$d_i = i\cdot(d\cdot n_i^2 - \dim G_i) \quad \text{if } e = 1.$$

If $K = E$ is a ramified quadratic extension,

$$d_i = \begin{cases} t\cdot n_i^2 & \text{if } i = 2t \text{ is even,} \\ t\cdot n_i^2 + \dim G_i & \text{if } i = 2t+1 \text{ is odd;} \end{cases}$$

and if $K = D$,

$$d_i = \begin{cases} t\cdot n_i(2n_i - \epsilon) & \text{if } i = 2t \text{ is even,} \\ t\cdot n_i(2n_i - \epsilon) + n_i^2 & \text{if } i = 2t+1 \text{ is odd.} \end{cases}$$

(7.3) Theorem *Let $q = \#\kappa$. The local density of $(L, \langle -, - \rangle)$ is*

$$\beta_L = \frac{1}{[G : G^\circ]} q^N \prod_i \Big(q^{-\dim G_i} \# G_i(\kappa) \Big),$$

where

$$N = N_H - N_E = \sum_{j>i} d\cdot i\cdot n_i\cdot n_j + \sum_i d_i.$$

(7.4) Remark Though we have assumed that $n_i = 0$ for $i < 0$, it is easy to check that the formula in the preceding theorem remains true without this assumption.

§8. Morita Context

In this section, we use the elementary aspects of the theory of Morita context (see [J]) to reduce the case $K = M_2(F)$ to the case $K = F$. We make a slight change of notations: $\mathbb{F} = M_2(F)$, $\mathbb{A} = M_2(A)$. Let

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 - e_1 = \sigma(e_1).$$

(8.1) Let V be an F-vector space. Then $\mathbb{V} = V \oplus V$ is an $\mathbb{F}$-module in a natural way. The functor $V \mapsto \mathbb{V}$ is an equivalence of categories. A quasi-inverse is the functor sending an $\mathbb{F}$-module $\mathbb{V}$ to $\mathbb{V}.e_1$. The module $\mathbb{V}$ is free over $\mathbb{F}$ if and only if $\dim V$ is even.

(8.2) Similarly, the category of A-modules is equivalent to the category of $\mathbb{A}$-modules via the functors $L \mapsto L \oplus L$ and $\mathbb{L} \mapsto \mathbb{L}.e_1$. The module $\mathbb{L}$ is free over $\mathbb{A}$ if and only if L is free of even rank over A.

(8.3) Suppose that $\mathbb{V}$ is a free $\mathbb{F}$-module and $V = \mathbb{V}.e_1$. The correspondence $\mathbb{L} \leftrightarrow L$ is a bijection between $\mathbb{A}$-lattices in $\mathbb{V}$ and A-lattices in V.

(8.4) Let $\mathbb{V}$ be an $\mathbb{F}$-module and $\langle\langle -, - \rangle\rangle$ be a sesquilinear form on $\mathbb{V}$. Let $V = \mathbb{V}.e_1$ and let $\langle -, - \rangle$ be the restriction of $\langle\langle -, - \rangle\rangle$ on V. Then $\langle -, - \rangle$ is an F-bilinear mapping with value in the 1-dimensional F-vector space $\sigma(e_1).\mathbb{F}.e_1$. Since $\mathbb{F} = \mathbb{F}.e_1.\mathbb{F}$ and $\mathbb{V} = \mathbb{V}.\mathbb{F}.e_1.\mathbb{F} = V.\mathbb{F}$, the mapping $\langle\langle -, - \rangle\rangle \mapsto \langle -, - \rangle$ from the space $\mathbb{S}$ of sesquilinear forms $\mathbb{V}$ to the space S of bilinear forms on V is injective. By comparing dimensions, we see that it is bijective if $\mathbb{V}$ is free over $\mathbb{F}$.

(8.5) Suppose that $\mathbb{V}$ is free over $\mathbb{F}$. Then $\langle\langle -, - \rangle\rangle$ is ϵ-hermitian if and only if $\langle -, - \rangle$ is $(-\epsilon)$-hermitian. Moreover, in either cases, $\langle\langle -, - \rangle\rangle$ is non-degenerate if and only if $\langle -, - \rangle$ is non-degenerate.

(8.6) From now on, assume that $\mathbb{V}$ is free over $\mathbb{F}$ and $\langle\langle -, - \rangle\rangle$ is ϵ-hermitian and non-degenerate. Let $\mathbb{L}$ be an $\mathbb{A}$-lattice in $\mathbb{V}$ corresponding to the A-lattice L in V. It is well-known that $g \mapsto g|_V$ is an isomorphism of algebraic groups from $\mathbb{G} = \mathrm{Aut}_{\mathbb{F}}(\mathbb{V}, \langle\langle -, - \rangle\rangle)$ to $G = \mathrm{Aut}_F(V, \langle , \rangle)$. This isomorphism induces a group isomorphism from $\mathbb{G}_L = \mathrm{Aut}_{\mathbb{A}}(\mathbb{L}, \langle\langle -, - \rangle\rangle)$ to $G_L = \mathrm{Aut}_A(L, \langle -, - \rangle)$.

(8.7) Let $\boldsymbol{\omega}^{\mathrm{ld}}$ (resp. ω^{ld}) be the volume form on $\mathbb{G}$ (resp G) defined in section (3.2) by using the lattice $\mathbb{L}$ (resp. L). Then the pull-back of ω^{ld} by the isomorphism $\mathbb{G} \to G$ is simply $\boldsymbol{\omega}^{\mathrm{ld}}$. By Lemma (3.4) and the last statement of (8.6), the local density of $\mathbb{L}$ is the same as that of L.

§9. The Case $p = 2$

(9.1) As mentioned in the introduction, the results of this paper hold for F of residue characteristic 2 in certain cases. One of the reasons for restricting to the case $p \neq 2$ is that the Jordan decomposition described in Section 4 is in general more complicated in the case $p = 2$. However, for lattices in symplectic spaces, hermitian and anti-hermitian spaces over $F \oplus F$ or an unramified quadratic extension E, and hermitian spaces over $M_2(F)$ or D, the results of Proposition (4.2) remain true, though we caution the reader that the proof for the quaternionic hermitian case given there is only valid if $p \neq 2$.

(9.2) Another reason for assuming that $p \neq 2$ is that Theorem (5.5) is not true in general when $p = 2$. Indeed, in the proof of Lemma (5.5.2), statement 2 is no longer true in general. However, for the types of spaces mentioned above, the proof of Theorems (5.5) and (5.7) remains valid. Furthermore, for these spaces, the isometry groups G_i defined in Section 6 do not involve orthogonal groups, and hence are smooth even in characteristic 2. Thus, the determination of the structure of the special fiber $\tilde{G}$ given in Section 6 carries through without change.

(9.3) In conclusion, the local density formula in Theorem (7.3) remains valid when $p = 2$ for symplectic spaces, hermitian and anti-hermitian spaces over $F \oplus F$ or an unramified quadratic extension E, and hermitian spaces over $M_2(F)$ or D.

§10. Appendix: The Mass Formula

In this appendix, we establish the Smith-Minkowski-Siegel mass formula, with the aim of clarifying the role played by the local densities defined in Section 3. As we will be working globally, we will need to differ from our previous notations at times.

(10.1) Let k be a number field of degree d over $\mathbb{Q}$, with ring of integers A, and adele ring $\mathbb{A}$. For a place v of k, we will let k_v be the corresponding completion of k, and for a finite place v, A_v will denote the ring of integers of k_v, and q_v will be the cardinality of the residue field of A_v. Let S_∞ denote the set of archimedean places, and let r_1 (respectively r_2) be the number of real (respectively complex) places. Hence, $r_1 + 2r_2 = d$.

(10.2) Let K one of the following k-algebras:

- $K = k$;
- $K = E$, a quadratic extension of k;
- $K = D$, a quaternion division algebra over k,

equipped with the obvious involution σ as in (2.2). Let $t = [K : k]$, the dimension of K as a k-vector space. Note that there is a trace map $\mathrm{Tr} : K \to k$, which gives the k-vector space K a natural symmetric bilinear trace form: $(x, y) \mapsto \mathrm{Tr}(x \cdot \sigma(y))$. Fix a maximal A-order B of K, and let $B^\perp$ be the A-lattice dual to B with respect to the trace form. Then set $d_{K/k} = [B^\perp : B] \in \mathbb{Z}_{>0}$. More precisely,

$$d_{K/k} = \begin{cases} 1, \text{ if } K = k; \\ |\mathbb{N}_{K/\mathbb{Q}}(\mathfrak{D})|, \text{ if } K = E; \\ \prod_{v \in S_D} q_v^2, \text{ if } K = D. \end{cases}$$

Here, $\mathfrak{D}$ is the different ideal of E/k, and S_D is the set of finite places of k where D_v is ramified.

(10.3) Consider a finite dimensional vector space V over K, equipped with a (σ, ϵ)-hermitian form $h_0 = \langle -, - \rangle$, where $\epsilon = \pm 1$. Let G be the corresponding isometry group, which is a (possibly disconnected) reductive algebraic group over k. Let L be a lattice in V, by which we mean a finitely generated projective B-submodule of V such that $L \otimes_A k = V$, and assume that $\langle -, - \rangle$ is B-valued on L. Note that L is not necessarily free.

(10.4) The genus of L is indexed by the finite double coset space:

$$\Sigma = G(k) \backslash G(\mathbb{A}) / U$$

where $U = G(k \otimes \mathbb{R}) \times \prod_{v \text{ finite}} U_v$, and U_v is the stabilizer of $L_v = L \otimes_A A_v$ in $G(k_v)$. For $\alpha \in \Sigma$, represented by $g_\alpha \in G(\mathbb{A})$, let

$$\Gamma_\alpha = G(k) \cap g_\alpha U g_\alpha^{-1}.$$

Then Γ_α is an arithmetic group, and since the connected center of G is an anisotropic torus over k, $\Gamma_\alpha \backslash G(k \otimes \mathbb{R})$ is of finite volume with respect to any Haar measure of $G(k \otimes \mathbb{R})$.

(10.5) If k is totally real, and $\langle -, - \rangle$ is totally definite, then Γ_α is finite, and following Eisenstein, the mass of L is defined to be:

$$\text{Mass}(L) = \sum_{\alpha \in \Sigma} \frac{1}{\# \Gamma_\alpha}.$$

In general, the definition of the mass of L (due to Siegel [Si]) depends on the choice of a Haar measure on the real Lie group $G(k \otimes \mathbb{R})$. There seems to be two natural choices for this. A reductive algebraic group over $\mathbb{R}$ has a unique compact form and a unique split form, and each of these possesses a natural Haar measure. For the compact form, the natural Haar measure is the one which gives the group volume 1. For the split form, it is the one determined by an invariant differential ω_0 of top degree on the Chevalley model over $\mathbb{Z}$. One can then transfer these Haar measures to any other form of the group, as in [GrG]. This gives two natural Haar measures on $G(k \otimes \mathbb{R})$, denoted by $|\omega_c|$ and $|\omega_0|$ respectively.

(10.6) The relation between these two measures can be found in [Gr, Section 7]. More precisely,

$$|\omega_0| = \lambda^d \cdot [G : G^\circ]^{r_1 + r_2} \cdot |\omega_c|,$$

where

$$\lambda = \prod_i \frac{(2\pi)^{d_i}}{(d_i - 1)!},$$

and the d_i's run over the degrees of G.

The Haar measure we use on $G(k \otimes \mathbb{R})$ will be the one coming from the compact form. Hence the mass of L is, by definition,

$$\text{Mass}(L) = \sum_{\alpha \in \Sigma} \int_{\Gamma_\alpha \backslash G(k \otimes \mathbb{R})} |\omega_c|.$$

Note that if k is totally real and $\langle -, - \rangle$ is totally definite, so that $G(k \otimes \mathbb{R})$ is compact, this agrees with the previous definition, which explains our choice. It is interesting to note that in [Si], the mass of L was defined using a Haar measure which is $|\omega_0|$ multiplied by a precise power of 2.

(10.7) Let ω be any invariant differential of top degree on G/k. Then ω gives rise to a Haar measure $|\omega|_v$ of $G(k_v)$ for each place v, and we will let $|\omega|_\infty$ denote the measure $\otimes_{v\in S_\infty}|\omega|_v$ on $G(k\otimes\mathbb{R})$. The Tamagawa measure of G is the measure

$$|\omega|_{\mathbb{A}} = d_k^{-\frac{\dim(G)}{2}} \cdot \bigotimes_v \frac{1}{[G:G^\circ]} \cdot |\omega|_v$$

on $G(\mathbb{A})$, where d_k is the discriminant of k over $\mathbb{Q}$. To be honest, this is only correct when G is semi-simple. When G is not semi-simple, which occurs when $K = E$ or when V is a two dimensional quadratic space, one should put in certain convergence factors. However, since the connected center of G is anisotropic over k, the product

$$\prod_{v \text{ finite}} \int_{U_v} |\omega|_v$$

is conditionally convergent for any open compact subgroup $U = \prod_{v \text{ finite}} U_v$, so that the measure defined above makes sense and agrees with the one defined using suitable convergence factors. For more discussion on these issues, we refer the reader to [S]. Let

$$\tau(G) = \int_{G(k)\backslash G(\mathbb{A})} |\omega|_{\mathbb{A}}$$

be the Tamagawa number of G.

(10.8) Recall from Section 3 that, for each finite place v, the lattice L_v determines a Haar measure $|\omega_{L_v}^{\text{ld}}|$, which gives rise to a local density β_{L_v} as defined in (3.5). The mass formula relates Mass(L) to the quantities β_{L_v}. Indeed, a standard and formal computation gives:

$$\text{Mass}(L) = c(L) \cdot \frac{\tau(G)\cdot d_k^{\dim(G)/2}}{\prod_{v \text{ finite}} \beta_{L_v}},$$

where

$$\begin{aligned} c(L) &= [G:G^\circ]^{r_1+r_2} \cdot \frac{|\omega_c|}{|\omega|_\infty} \cdot \prod_{v \text{ finite}} \frac{|\omega_{L_v}^{\text{ld}}|}{|\omega|_v} \\ &= \lambda^{-d} \cdot \frac{|\omega_0|}{|\omega|_\infty} \cdot \prod_{v \text{ finite}} \frac{|\omega_{L_v}^{\text{ld}}|}{|\omega|_v}. \end{aligned}$$

Note that all but finitely many terms of the above product is 1, and the product of the local densities β_{L_v} is only conditionally convergent when G is not semi-simple.

(10.9) Hence, the computation of Mass(L) is reduced to the computations of the local densities β_{L_v}, *provided* that $\tau(G)$ and $c(L)$ can be given explicitly, and this is the content of the mass formula. The determination of the Tamagawa number $\tau(G)$ is a deep problem which was solved in complete generality only fairly recently. For the groups we are dealing with, however, the value of $\tau(G)$ has been known for a while by the work of Weil, and is given by:

$$\tau(G) = \begin{cases} 1, & \text{if } K = k \text{ or } D; \\ 2, & \text{if } K = E. \end{cases}$$

The explicit determination of $c(L)$, on the other hand, is an easier problem, and is the main purpose of this Appendix.

(10.10) In the classical case of quadratic forms, if the lattice L is *free*, one can choose a global invariant differential ω such that $|\omega|_v = |\omega^{\mathrm{ld}}_{L_v}|$ for every finite place v. In this case, $c(L)$ can be interpreted as an archimedean local density. If L is not free, we do not have such an interpretation; nor do we see any reason for the existence of an ω as above. The computation of $c(L)$ in the case of quadratic forms is essentially the content of [Ha]. In the following, we shall determine $c(L)$ in general. One begins with the following elementary observation.

(10.11) Lemma *Let $n = \dim_K V$ and $d(L) = [L^\perp : L]$ be the discriminant of L. Then the quantity $\beta = c(L)/d(L)^{n(V)/2}$ is independent of L, where*

$$n(V) = \frac{2\dim_k H}{n\cdot t} = \begin{cases} n+\epsilon, & \text{if } K = k;\\ n, & \text{if } K = E;\\ n-\frac{\epsilon}{2}, & \text{if } K = D.\end{cases}$$

Here, H is the k-vector space of (σ,ϵ)-hermitian forms on V.

(10.12) Now we come to the computation of $c(L)$. We first introduce some notations. Let $K^\epsilon = \{u \in K : \sigma(u) = \epsilon u\}$ be the ϵ-eigenspace of σ, and let $t' = \dim_k K^\epsilon$. Henceforth, fix a basis

$$\begin{cases} \mathcal{E} = \{e_1, ..., e_n\} \text{ of } V \text{ over } K,\\ \mathcal{U} = \{u_1, ..., u_t\} \text{ of } K \text{ over } k,\\ \mathcal{V} = \{v_1, ..., v_{t'}\} \text{ of } K^\epsilon \text{ over } k,\end{cases}$$

Let $\mathcal{E}^* = \{e_1^*, ..., e_n^*\}$ be the basis of the dual vector space V^* which is dual to $\mathcal{E}$ and so on. Let $m_{ij}^* : M = V\otimes_K V^* \to K$ be the function $e_i^*\otimes e_j$. Similarly, for $i\neq j$ (respectively $i=j$), we have the function $h_{ij} : H \to K$ (respectively $H \to K^\epsilon$), given by $h \mapsto h(e_i, e_j)$ for $h \in H$.

(10.13) Recall that we have a smooth morphism $f : M^* \to H$ of varieties over k, given by $t \mapsto h_0\circ t$, and $G = f^{-1}(h_0)$. Note that M^* refers to the open subvariety of invertible elements in M, and *not* the dual vector space. Consider the following differential forms on M and H respectively:

$$\omega_{M,\mathcal{E},\mathcal{U}} = \bigwedge_{i,j}\omega_{M,i,j}, \qquad \omega_{M,i,j} = \bigwedge_{k=1}^{t} d(u_k^*\circ m_{ij}^*);$$

$$\omega_{H,\mathcal{E},\mathcal{U},\mathcal{V}} = \bigwedge_{i\geq j}\omega_{H,i,j}, \qquad \omega_{H,i,j} = \begin{cases} \bigwedge_{k=1}^{t} d(u_k^*\circ h_{ij}), & \text{if } i\neq j;\\ \bigwedge_{k=1}^{t'} d(v_k^*\circ h_{ii}), & \text{if } i=j.\end{cases}$$

Then

$$\omega_{\mathcal{E},\mathcal{U},\mathcal{V}} = \omega_{M,\mathcal{E},\mathcal{U}}/f^*\omega_{H,\mathcal{E},\mathcal{U},\mathcal{V}}$$

is a differential of top degree on G, invariant under G°. Now we have the following crucial lemma.

(10.14) Lemma *Suppose that $g \in \mathrm{Aut}_K(V)$, $g' \in \mathrm{Aut}_k(K)$ and $g'' \in \mathrm{Aut}_k(K^\epsilon)$. Let $g \cdot \mathcal{E}$ denote the basis $\{ge_1, ge_2, ..., ge_n\}$ and so on. Then*

$$\frac{\omega_{g\cdot\mathcal{E},g'\cdot\mathcal{U},g''\cdot\mathcal{V}}}{\omega_{\mathcal{E},\mathcal{U},\mathcal{V}}} = \mathrm{Nrd}(g)^{-n(V)} \cdot \det(g')^{-n(n+1)/2} \cdot \det(g'')^n,$$

where $\mathrm{Nrd} : \mathrm{Aut}_K(V) \to k^\times$ is the reduced norm map.

(10.15) Note that the above discussion can be carried out over any local field k_v. Suppose in particular that we are working over a non-archimedean k_v, and that $\mathcal{E}_{L_v}$ is a basis of the lattice L_v over B_v, $\mathcal{U}_{B_v}$ is a basis of B_v over A_v and $\mathcal{V}_{B_v}$ is a basis of B_v^ϵ over A_v. Then we have:

$$\omega_{\mathcal{E}_{L_v},\mathcal{U}_{B_v},\mathcal{V}_{B_v}} = \omega_{L_v}^{\mathrm{ld}}.$$

This, together with Lemma (10.14), implies that:

$$c(L) = \lambda^{-d} \cdot \frac{|\omega_0|}{|\omega_{\mathcal{E},\mathcal{U},\mathcal{V}}|_\infty} \cdot [B(\mathcal{E}) : L]^{n(V)} \cdot [A(\mathcal{U}) : B]^{n(n+1)/2} \cdot [A(\mathcal{V}) : B^\epsilon]^{-n},$$

where $B(\mathcal{E})$ is the B-lattice generated by $\mathcal{E}$, and $A(\mathcal{U})$ (respectively $A(\mathcal{V})$) is the A-lattice generated by $\mathcal{U}$ (respectively $\mathcal{V}$). In particular, we now need to determine the ratio $|\omega_0|/|\omega_{\mathcal{E},\mathcal{U},\mathcal{V}}|_\infty$, which is a purely archimedean computation.

(10.16) Let $\mathcal{U}_\infty = \{u_{\infty,1}, ..., u_{\infty,t}\}$ be a basis of $K \otimes \mathbb{R}$ over $k \otimes \mathbb{R}$ such that

$$\det\left(\mathrm{Tr}(u_{\infty,i} \cdot \sigma(u_{\infty,j}))\right) = (\pm 1)^{r_1+r_2} \in k \otimes \mathbb{R}.$$

Then $\bigwedge_{i=1}^t du_{\infty,i}^*$ induces a measure on $K \otimes \mathbb{R}$, which we call the standard measure. Further,

$$\frac{|\bigwedge_{i=1}^t du_i^*|_\infty}{|\bigwedge_{i=1}^t du_{\infty,i}^*|_\infty} = [A(\mathcal{U}) : B] \cdot d_{K/k}^{-1/2}.$$

Together with Lemma (10.14), this implies that:

$$c(L) = \lambda^{-d} \cdot \frac{|\omega_0|}{|\omega_{\mathcal{E},\mathcal{U}_\infty,\mathcal{V}}|_\infty} \cdot [B(\mathcal{E}) : L]^{n(V)} \cdot [A(\mathcal{V}) : B^\epsilon]^{-n} \cdot d_{K/k}^{n(n+1)/4}.$$

(10.17) Let $\mathcal{E}_\infty = \{e_{\infty,1}, ..., e_{\infty,n}\}$ be a basis of $V \otimes \mathbb{R}$ over $K \otimes \mathbb{R}$ such that

$$\mathrm{Nrd}(\langle e_{\infty,i}, e_{\infty,j}\rangle) = (\pm 1)^{r_1+r_2} \in k \otimes \mathbb{R}.$$

Then by Lemma (10.14) again, we have:

$$c(L) = \lambda^{-d} \cdot \frac{|\omega_0|}{|\omega_{\mathcal{E}_\infty,\mathcal{U}_\infty,\mathcal{V}}|_\infty} \cdot [A(\mathcal{V}) : B^\epsilon]^{-n} \cdot d(L)^{n(V)/2} \cdot d_{K/k}^{n(n+1)/4}.$$

(10.18) We now choose a basis $\mathcal{V}_\infty$ for $(K \otimes \mathbb{R})^\epsilon$ over $k \otimes \mathbb{R}$. . If $\epsilon = 1$, then $(K \otimes \mathbb{R})^\epsilon = k \otimes \mathbb{R}$, and we let $\mathcal{V}_\infty = \{1\}$. If $\epsilon = -1$, consider the exact sequence:

$$0 \to K^\epsilon \to K \to k \to 0,$$

given by the trace map $\mathrm{Tr} : K \to k$. We have already defined standard measures on $K \otimes \mathbb{R}$ and $k \otimes \mathbb{R}$. These induce a standard measure on $(K \otimes \mathbb{R})^\epsilon$, and we let $\mathcal{V}_\infty = \{v_{\infty,1}, ..., v_{\infty,t'}\}$ be a basis such that $\bigwedge_{i=1}^{t'} dv_{\infty,i}^*$ induces this standard measure. Then we have:

$$\frac{|\bigwedge_{i=1}^{t'} dv_i^*|_\infty}{|\bigwedge_{i=1}^{t'} dv_{\infty,i}^*|_\infty} = [A(\mathcal{V}) : B^\epsilon] \cdot \delta_{K,\epsilon},$$

where

$$\delta_{K,\epsilon} = \begin{cases} 1, \text{ if } \epsilon = 1 \text{ or } K = k; \\ d_{K/k}^{-1/2} \cdot [A : \mathrm{Tr}(B)], \text{ if } K = E \text{ and } \epsilon = -1; \\ d_{K/k}^{-1/2}, \text{ if } K = D \text{ and } \epsilon = -1. \end{cases}$$

By Lemma (10.14) again, this implies that:

$$c(L) = \lambda^{-d} \cdot \frac{|\omega_0|}{|\omega_{\mathcal{E}_\infty, \mathcal{U}_\infty, \mathcal{V}_\infty}|} \cdot d(L)^{n(V)/2} \cdot d_{K/k}^{n(n+1)/4} \cdot \delta_{K,\epsilon}^{-n}.$$

Note that

$$[A : \mathrm{Tr}(B)] = \prod_{v|2} [A_v : \mathrm{Tr}(B_v)],$$

and the local factors $[A_v : \mathrm{Tr}(B_v)]$ can be explicitly given as follows:

$$[A_v : \mathrm{Tr}(B_v)] = \begin{cases} 1, \text{ if } v \text{ is split or unramified in } K_v; \\ q_v^{[d_v/2]}, \text{ if } v \text{ is ramified in } K_v, \end{cases}$$

where in the second case, $(\pi_{K_v}^{d_v})$ is the different ideal of K_v/k_v.

We are finally reduced to the computation of the number $\mu_0 = |\omega_0|/|\omega_{\mathcal{E}_\infty, \mathcal{U}_\infty, \mathcal{V}_\infty}|_\infty$. Notice that this number depends only on the triple $(n, \epsilon, (K \otimes \mathbb{R})/(k \otimes \mathbb{R}))$. The following Proposition shows that in fact it depends only on $(n, \epsilon, (K \otimes \mathbb{C})/(k \otimes \mathbb{C}))$.

(10.19) Proposition *$\mu_0 = \mu^d$, where μ depends only on the triple (n, t, ϵ), and is given by:*

$$\mu = \begin{cases} 2^n, \text{ if } t = 1, \epsilon = 1 \text{ and } n \text{ is even;} \\ 2^{(n+1)/2}, \text{ if } t = 1, \epsilon = 1 \text{ and } n \text{ is odd;} \\ 2^{2n}, \text{ if } t = 4 \text{ and } \epsilon = -1; \\ 1, \text{ otherwise.} \end{cases}$$

PROOF. Write $k\otimes\mathbb{R} = \prod_{v\in S_\infty} k_v$. Then it follows by definition that as Haar measures on $G(k\otimes\mathbb{R}) = \prod_{v\in S_\infty} G(k_v)$,

$$|\omega_0| = \prod_{v\in S_\infty} |\omega_{0,v}|,$$

where $\omega_{0,v}$ is an invariant differential on $G_v = G\times_k k_v$ obtained from the Chevalley model over $\mathbb{Z}$. Note that under any isomorphism $\varphi : G_v \times_{k_v} \mathbb{C} \to G_{v'} \times_{k_{v'}} \mathbb{C}$, we have $\varphi^*(\omega_{0,v'}) = \pm\omega_{0,v}$.

Each element $e_{\infty,i}$ of the basis $\mathcal{E}_\infty$ is an (r_1+r_2)-tuple $(e_{\infty,i,v}) \in \prod_{v\in S_\infty} V\otimes_k k_v$, where for each $v\in S_\infty$,

$$\mathcal{E}_v = \{e_{\infty,1,v}, ..., e_{\infty,n,v}\}$$

is a basis of $V\otimes_k k_v$, with the property that

$$\mathrm{Nrd}_v(\langle e_{\infty,i,v}, e_{\infty,j,v}\rangle) = \pm 1.$$

The analogous statement is true of the bases $\mathcal{U}_\infty$ and $\mathcal{V}_\infty$. Then

$$|\omega_{\mathcal{E}_\infty,\mathcal{U}_\infty,\mathcal{V}_\infty}| = \prod_{v\in S_\infty} |\omega_{\mathcal{E}_v,\mathcal{U}_v,\mathcal{V}_v}|_v$$

as Haar measures on $G(k\otimes\mathbb{R}) = \prod_{v\in S_\infty} G(k_v)$. Now the first statement of the Proposition follows from the observation that under any isomorphism $\varphi : G_v \times_{k_v} \mathbb{C} \to G_{v'} \times_{k_{v'}} \mathbb{C}$,

$$|\varphi^*(\omega_{\mathcal{E}_{v'},\mathcal{U}_{v'},\mathcal{V}_{v'}})| = |\omega_{\mathcal{E}_v,\mathcal{U}_v,\mathcal{V}_v}|,$$

as Haar measures on $G_v(\mathbb{C})$. In particular, $\mu_0 = \mu^d$, with μ equal to the absolute value of the complex number $\omega_{0,v}/\omega_{\mathcal{E}_v,\mathcal{U}_v,\mathcal{V}_v}$, which is independent of $v\in S_\infty$.

To compute μ, it suffices to work over $\mathbb{C}$. Hence, let $\mathcal{E}$, $\mathcal{U}$ and $\mathcal{V}$ be bases of V, K and K^ϵ chosen as before, where now K is $\mathbb{C}$, $\mathbb{C}\times\mathbb{C}$ or $M_2(\mathbb{C})$. To compare the invariant differentials ω_0 and $\omega = \omega_{\mathcal{E},\mathcal{U},\mathcal{V}}$ on G, it suffices to compare them on the tangent space of the identity element e of G. Let $df : T_eM^* = M \to T_{h_0}H = H$ be the map on tangent spaces induced by f. Then $\omega = w_{M,\mathcal{E},\mathcal{U}}/(df)^*(\omega_{H,\mathcal{E},\mathcal{U},\mathcal{V}})$ is a differential on the kernel of df.

We treat the case when G is the orthogonal group in some details. If n is even, one choose the basis $\mathcal{E}$ of V such that the form h_0 is represented by the matrix A, where $A_{ij} = \delta_{i,n+1-j}$. With respect to this basis, one identifies M and H with the space $M_n(\mathbb{C})$ of $n\times n$-matrices and the space $\mathrm{Sym}_n(\mathbb{C})$ of symmetric matrices respectively. The differentials $\omega_{M,\mathcal{E},\mathcal{U}}$ and $\omega_{H,\mathcal{E},\mathcal{U},\mathcal{V}}$ are then the standard ones on $M_n(\mathbb{C})$ and $\mathrm{Sym}_n(\mathbb{C})$, and the map df is given by:

$$X \mapsto {}^tXA + AX.$$

From this, one can write down ω explicitly. On the other hand, a Chevalley basis of $\mathrm{Lie}(G) = \mathrm{Ker}(df)$ was given explicitly by Bourbaki [B, Ch. VIII, §13.4, Pg. 211], which allows one to write down ω_0. Comparing, one finds that:

$$\omega = \frac{1}{2^n}\cdot\omega_0,$$

which is the result sought for in this case.

The case for odd n is slightly more subtle. Let $\mathcal{E}'$ be a basis of V such that h_0 is represented by the matrix A', with

$$A'_{ij} = \begin{cases} 2, \text{ if } i = j = \frac{n+1}{2}; \\ \delta_{i,n+1-j}, \text{ otherwise.} \end{cases}$$

Using $\mathcal{E}'$, Bourbaki described in [B, Ch. VIII, §13.2, Pg. 199-200] an explicit Chevalley basis on $\mathrm{Ker}(df)$, from which a direct computation gives

$$\omega_{\mathcal{E}'} = \frac{1}{2^{n+1}} \cdot \omega_0.$$

On the other hand, by Lemma (10.14),

$$\omega_{\mathcal{E}'} = \frac{1}{2^{(n+1)/2}} \cdot \omega.$$

Hence the result follows in this case.

The remaining cases when $K = k$ or E follow by a similar computation, using the results in [B, Ch. VIII, §13], and are more straightforward than the case of orthogonal groups. The cases when $K = D$ then follow by Morita context, as in Section 8. The Proposition is proved. ∎

We summarize the above discussion in the following theorem.

(10.20) Theorem

$$\mathrm{Mass}(L) = c(L) \cdot \frac{\tau(G) \cdot d_k^{\dim(G)/2}}{\prod_{v \text{ finite}} \beta_{L_v}},$$

where

$$c(L) = \left(\lambda^{-1}\mu\right)^d \cdot [L^\perp : L]^{n(V)/2} \cdot d_{K/k}^{n(n+1)/4} \cdot \delta_{K,\epsilon}^{-n}.$$

Here, $\tau(G)$ is given in (10.9), λ is given in (10.6), μ is given in Proposition (10.19), $n(V)$ is given in Lemma (10.11), $d_{K/k}$ is given in (10.2) and $\delta_{K,\epsilon}$ is given in (10.18).

§11. An Example: Quaternionic Hermitian Spaces

As a result of Theorem (10.20), Theorem (7.3) and the remarks in Section 9, we can give an exact formula for the mass of an arbitrary lattice in a quaternionic hermitian space V over any number field k. In this section, we write down the various quantities which appear in Theorem (10.20) as explicitly as possible.

(11.1) Let S_D be the finite set of finite places of k for which D_v is ramified. The group G is a form of the symplectic group in $2n$ variables. Hence, we have:

$$\begin{aligned}\tau(G) &= 1,\\ \lambda^{-1} &= \prod_{j=1}^{n} \frac{(2j-1)!}{(2\pi)^{2j}},\\ \mu &= 1,\\ d_{K/k} &= \prod_{v\in S_D} q_v^2,\\ \delta_{K,\epsilon} &= 1.\end{aligned}$$

Therefore, the mass of a lattice L is given by:

$$\mathrm{Mass}(L) = d_k^{n(n+\frac{1}{2})} \cdot \left(\prod_{v\in S_D} q_v\right)^{\frac{n(n+1)}{2}} \cdot \left(\prod_{j=1}^{n} \frac{(2j-1)!}{(2\pi)^{2j}}\right)^{d} \cdot [L^{\perp} : L]^{\frac{1}{2}(n-\frac{1}{2})} \cdot \prod_{v \text{ finite}} \beta_{L_v}^{-1},$$

where β_{L_v} is given explicitly by Theorem (7.3) for all non-archimedean v.

(11.2) As a check for the above result, we consider the case when k is totally real, D is ramified at all real places of k, the hermitian space V is totally definite, and L is a maximal lattice in V. The mass of such an L was recently obtained by Shimura [Sh]. In the terminology of [Sh], L is maximal with respect to the property that $\langle -,- \rangle$ is B-valued on L. For each finite place v of k, L_v is self-dual, and hence $[L^{\perp} : L] = 1$. Further,

$$\beta_v = \begin{cases} \prod_{j=1}^{n}(1-q_v^{-2j}), \text{ if } v \notin S_D;\\ \\ \prod_{j=1}^{n}(1-(-q_v)^{-j}), \text{ if } v \in S_D.\end{cases}$$

As a result,

$$\mathrm{Mass}(L) = d_k^{n(n+\frac{1}{2})} \cdot \prod_{j=1}^{n} \left[\zeta_k(2j) \left(\frac{(2j-1)!}{(2\pi)^{2j}}\right)^{d} \cdot \prod_{v\in S_D} (q_v^j + (-1)^j)\right],$$

which agrees with the formula in [Sh].

(11.3) On the other hand, one can consider a different notion of maximality. Let L be a lattice such that $L^{\perp}$ is maximal with respect to the property that $\langle x,x \rangle \in A$ for all $x \in L^{\perp}$. The mass of such a lattice L was obtained in [GHY, Section 9] from a general mass formula in [GrG, Section 10]. For such an L, L_v is self-dual if $v \notin S_D$, but for $v \in S_D$,

$$L_v = \begin{cases} (L_v)_1, \text{ if } n \text{ is even};\\ (L_v)_0 \oplus (L_v)_1, \text{ if } n \text{ is odd},\end{cases}$$

with $(L_v)_0$ having rank 1. Here, $(L_v)_i$ is as defined in Corollary (4.3). Hence, we have:

$$[L^\perp : L] = \prod_{v \in S_D} q_v^{4[n/2]},$$

and for $v \in S_D$,

$$\beta_{L_v} = \begin{cases} q_v^{n^2} \cdot \prod_{j=1}^{n/2}(1 - q_v^{-4j}), & \text{if } n \text{ is even;} \\ \\ q_v^{(n-1)^2} \cdot (1 + q_v^{-1}) \cdot \prod_{j=1}^{(n-1)/2}(1 - q_v^{-4j}), & \text{if } n \text{ is odd.} \end{cases}$$

As a result,

$$\text{Mass}(L) = d_k^{n(n+\frac{1}{2})} \cdot \prod_{j=1}^{n} \zeta_k(2j) \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \cdot \prod_{v \in S_D} \lambda_v,$$

with

$$\lambda_v = \begin{cases} \prod_{j=1}^{n/2}(q_v^{4j-2} - 1), & \text{if } n \text{ is even;} \\ \\ (q_v + 1)^{-1} \cdot \prod_{j=1}^{(n+1)/2}(q_v^{4j-2} - 1), & \text{if } n \text{ is odd,} \end{cases}$$

which agrees with [GHY, Proposition 9.4].

(11.4) It should be noted that in [Sh], Shimura obtained the mass of the maximal lattice over an arbitrary number field. However, in the case when $\langle -, - \rangle$ is not totally definite, his definition of the mass of L differs from ours. Indeed, the mass in [Sh] was defined using the symmetric space $G(k \otimes \mathbb{R})/U_\infty$ where U_∞ is the maximal compact subgroup of $G(k \otimes \mathbb{R})$. The invariant measure used on the symmetric space is, up to a precise power of 2, the quotient of the Haar measures on $G(k \otimes \mathbb{R})$ and U_∞ coming from the split form of the groups. In view of the precise relation between $|\omega_0|$ and $|\omega_c|$ given in (10.6), it should not be difficult to translate his formula for the mass of the maximal lattice to our formulation.

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