

# EXCEPTIONAL HOWE CORRESPONDENCES OVER FINITE FIELDS

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ABSTRACT. We consider the restriction of the reflection representation to various reductive dual pairs in exceptional groups, and determine the correspondence of generic representations.

## 1. Introduction

Let  $k$  be a finite field with  $q$  elements and of characteristic  $p$ . Recall that if  $Sp$  is the symplectic group over  $k$ , then a pair of reductive subgroups  $(H_1, H_2)$  of  $Sp$  is a dual pair if they are mutual commutants in  $Sp$ . Examples of  $(H_1, H_2)$  are  $(Sp_{2n}, O_m)$ ,  $(GL_n, GL_m)$  and  $(U_n, U_m)$ . The Weil representation [Ge] then defines a correspondence between the irreducible representations of  $H_1$  and  $H_2$ , called the Howe duality correspondence. A first study of the Howe correspondence over finite fields was carried out by Srinivasan [Sr], who determined the decomposition of (the uniform part of) the Weil representation into Deligne-Lusztig characters of the dual pair  $H_1 \times H_2$ . From her results, one sees that the Howe correspondence respects the correspondence of geometric conjugacy classes [D-L] with respect to some natural inclusion of dual groups:

$$i : H_1^\vee \hookrightarrow H_2^\vee.$$

In particular, the unipotent representations correspond. In [AM], it was shown by Adams and Moy that the cuspidal unipotent representations correspond in the case of first occurrence in their terminology. Finally, the Howe correspondence was completely determined by Aubert, Michel and Rouquier in their recent paper [A-M-R], except in the case  $(Sp_{2n}, O_m)$  where they have a conjecture. We remark that the Howe correspondence over local and global fields has also been extensively studied.

In this paper, we study an analogue of the Howe correspondence over finite fields in the exceptional groups. So assume that  $G$  is a split simply-laced simple linear algebraic group over  $k$ , and let  $F$  be the corresponding Frobenius. We first need an analogue of the Weil representation. This is the so-called reflection representation  $\Pi$  [Lu].  $\Pi$  can be characterized in various ways, but for now, it suffices to say that  $\Pi$  is a unipotent principal series representation of  $G^F$ , and its dimension is the smallest among all non-trivial irreducible representations of  $G^F$  (that is, those of dimension greater than 1). Hence it is the analogue of the minimal representation [MS] in the  $p$ -adic case. We recall that the character values of irreducible representations of  $G^F$  on semi-simple elements are known through the work of Lusztig [Lu2]. However, in Section 3, we shall see that the character values of  $\Pi$  on semi-simple elements  $s$  can be interpreted neatly in terms of the motive  $M(s)$ , introduced by Gross [Gr], of the

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connected centralizer  $C_G(s)^0$  of  $s$  in  $G$ . More precisely, for  $s$  a semi-simple element in  $G^F$ , we have:

**Proposition 1.1.**

$$\text{Tr}(s|\Pi) = \text{Tr}(F|M(s)).$$

We shall consider various dual pairs in exceptional groups. As an example, consider the case where  $G$  is a split adjoint group of type  $E_6$ . There is a dual pair [MS]:

$$PGL_3 \times G_2 \subset E_6.$$

This is an interesting dual pair because there is a natural inclusion of dual groups:

$$i : SL_3 \hookrightarrow G_2.$$

For each irreducible representation  $\pi$  of  $PGL_3$ , let  $\Theta(\pi)$  be the set of irreducible representations  $\pi'$  of  $G_2$ , counted with multiplicities, such that  $\pi \otimes \pi'$  occurs in  $\Pi$ , regarded as a  $PGL_3 \times G_2$ -module. Also let  $\Theta_{gen}(\pi) \subset \Theta(\pi)$  be the subset of generic representations. Then we shall determine  $\Theta_{gen}(\pi)$  for each generic representation  $\pi$  of  $PGL_3$ . We first prove:

**Theorem 1.2.** *Let  $U$  be the unipotent radical of a Borel subgroup of  $G_2$ , and let  $\psi$  be a character of  $U$  in general position. Then  $\Pi_{U,\psi}$  is the Gelfand-Graev representation of  $PGL_3$ . In particular,*

$$\#\Theta_{gen}(\pi) = \begin{cases} 1, & \text{if } \pi \text{ is generic;} \\ 0, & \text{otherwise.} \end{cases}$$

Now recall that the generic characters of a connected reductive group with connected center can be parametrized by semi-simple classes in the dual group. For a semi-simple class  $s$  in the dual group, we let  $\chi(s)$  be the corresponding generic representation. Then we shall show:

**Theorem 1.3.**

$$\Theta_{gen}(\chi(s)) = \{\chi(i(s))\}$$

for each semi-simple class  $s$  in  $SL_3$ , the dual group of  $PGL_3$ .

A check on the dimensions shows that this already accounts for a large part of  $\Pi$ . For non-cuspidal representations, this theorem is proved by the computation of Jacquet functors. For cuspidal representations, we resort to the use of base change or Shintani descent, under a suggestion of T. Uzawa. It is interesting to see if it can be adapted to the  $p$ -adic case.

We also have similar results for the dual pairs  $G_2 \times PGSp_6 \subset E_7$  and  $G_2 \times PU_3 \subset {}^2E_6$ . The results of this paper are in a natural way complement to those of Magaard and Savin [MS], who considered the exceptional Howe correspondence over a  $p$ -adic field and showed that the correspondence of tempered spherical representations respects Langlands functoriality.

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## 2. Weyl Group Invariants and Motive of $G$

In this section,  $k$  is any field,  $k^s$  a separable closure of  $k$ , and  $\Gamma = \text{Gal}(k^s/k)$ .

Let  $G$  be a connected quasi-split reductive group over  $k$ , of rank  $l$ . Let  $T_0$  be a maximal torus of  $G$  over  $k$ , which is contained in a Borel subgroup over  $k$ . The Weyl group of  $G$  is  $W = N_G(T_0)/T_0$ . Suppose that  $W$  has rank  $r$  as a reflection group, i.e.  $W$  can be generated by  $r$  simple reflections. Let  $X_\bullet(T_0)$  be the cocharacter group of  $T_0$ , and let  $E = X_\bullet(T_0) \otimes \mathbb{Q}$ , a  $\mathbb{Q}$ -vector space of dimension  $l$ . Then  $W$  acts naturally on  $E$ , and as a  $W$ -module,  $E$  is the direct sum of  $(l-r)$  copies of the trivial representation, and one copy of the natural reflection representation  $E_W$  of  $W$ .

The group  $\Gamma$  also acts naturally as automorphisms on  $E$ , and the action of  $\Gamma$  normalizes that of  $W$ . Hence the semi-direct product  $W \rtimes \Gamma$  acts on  $E$ . Moreover,  $\Gamma$  preserves the decomposition:

$$(2.1) \quad E = E^W \oplus E_W.$$

We shall denote the character of  $E$  as  $\chi_E$ .

Let  $S = \text{Sym}^\bullet(E)$ . Then  $W \rtimes \Gamma$  acts on  $S$ , preserving degrees. Let  $S^W$  be the ring of  $W$ -invariants. As is well-known,  $S^W$  is a polynomial ring, with  $l$  generators:  $S^W = \mathbb{Q}[I_1, \dots, I_l]$ . Let  $d_i$  be the degree of the primitive invariant  $I_i$ . Then the numbers  $d_i$ 's are well-defined and the numbers  $e_i = d_i - 1$  will be called the **exponents of the reductive group  $G$** . Note that exactly  $l-r$  of the  $e_i$ 's, say  $e_{r+1}, \dots, e_l$ , are zero, and  $e_1, \dots, e_r$  are the exponents of the Coxeter group  $W$ .

Note that  $\Gamma$  acts on  $S^W$ . By complexifying  $E$ , we can choose generators  $I_i$ 's of  $S^W \otimes \mathbb{C}$  such that, for a given  $\sigma \in \Gamma$ ,

$$(2.2) \quad \sigma(I_i) = \varepsilon_i(I_i).$$

Now we have the following formula, which can be proved in the similar way as [Ca, Pg. 363]:

**Lemma 2.3.**

$$\frac{1}{|W|} \sum_{w \in W} \frac{\chi(\sigma w)}{\det_E(1 - t\sigma w)} = \frac{\sum_i \varepsilon_i t^{e_i}}{\prod_i (1 - \varepsilon_i t^{e_i+1})}.$$

Now we briefly recall the notion of the motive of  $G$  from [Gr]. Consider the graded  $\Gamma$ -module:

$$(2.4) \quad \mathcal{J}/\mathcal{J}^2 := V = \bigoplus_d V_d$$

where  $\mathcal{J}$  is the ideal of  $S^W$  generated by the invariants of positive degrees. Note that  $V$  is generated by the primitive invariants. It is a theorem of Steinberg [St] that

$$(2.5) \quad E \cong V$$

as a  $\Gamma$ -module.

For a prime number  $l$  not equal to the characteristic of  $k$ , let  $\mathbb{Q}_l(1)$  be the Tate motive (cf. [Gr]) given by the action of  $\Gamma$  on the inverse system formed by the  $l^n$ -th roots of unity in  $k^s$  (as  $n \rightarrow \infty$ ). Then the motive of  $G$  is defined to be the Artin-Tate motive:

$$(2.6) \quad M_G = \bigoplus_d V_d(1-d)$$

where  $V_d(1-d) = V_d \otimes \mathbb{Q}_l(1)^{\otimes(1-d)}$ .

As an example, consider the case when  $k = \mathbb{F}_q$  is a finite field, and  $\Gamma = \langle F \rangle$ . Then (2.5) says that:

$$(2.7) \quad \text{Tr}(F|E) = \text{Tr}(F|V).$$

Also, if we take  $\sigma = F$ , then:

$$(2.8) \quad \text{Tr}(F|M_G) = \sum_i \varepsilon_i q^{e_i}$$

where the  $e_i$ 's are the exponents of the reductive group  $G$ .

### 3. The Reflection Representation

Henceforth, unless otherwise stated, we assume that  $k = \mathbb{F}_q$  is a finite field, of characteristic  $p$ , so that  $\Gamma = \langle F \rangle$ . Also, assume that  $G$  is a split, simply-laced, simple linear algebraic group over  $\mathbb{F}_q$  of rank  $l$ . Let  $F : G \rightarrow G$  be the corresponding Frobenius map, so that  $G^F = G(\mathbb{F}_q)$ . For each  $w \in W$ ,  $T_w$  will denote an  $F$ -stable maximal torus of  $G$  which is obtained from  $T_0$  by twisting with  $w$ .

The reflection representation  $\Pi$  of  $G$  is the unipotent principal series representation corresponding to the reflection representation  $E_W$  of  $W$ . It was shown by Kilmoyer in his thesis that  $\Pi$  is the unique representation of  $G$  satisfying:

$$(3.1) \quad \langle \text{Ind}_P^G 1, \Pi \rangle = l - l'$$

where  $P$  is any parabolic subgroup, and  $l'$  is the semi-simple rank of the Levi factor of  $P$ .

Let  $\chi_\Pi$  be the character of  $\Pi$ . It was shown by Lusztig ([Lu] and [Lu2]) that:

$$(3.2) \quad \chi_\Pi = \frac{1}{|W|} \sum_{w \in W} \chi_E(w) R_w := R_{\chi_E}$$

where  $R_w = R_{T_w,1}$  is the character of Deligne-Lusztig [D-L]. From this, he deduced that if  $s \in G^F$  is regular semi-simple, so that  $s$  is contained in a unique maximal torus  $T_{w_0}$ , then

$$(3.3) \quad \chi_\Pi(s) = \chi_E(w_0).$$

We shall use (3.2) to obtain a formula, which may be well-known already, for  $\chi_\Pi(s)$ , where  $s$  is any semi-simple element.

Let  $C^0(s)$  denote the connected component of the centralizer of  $s$  in  $G$ . The value of  $R_w$  on semisimple elements  $s$  is given by [Ca Pg.233]:

$$\begin{aligned} R_w(s) &= \frac{\varepsilon_{T_w} \varepsilon_{C^0(s)}}{|T_w^F| |C^0(s)^F|_q} \sum_{g \in G^F} 1(g^{-1}sg \in T_w^F) \\ &= \frac{\varepsilon_{T_w} \varepsilon_{C^0(s)}}{|C^0(s)^F|_q} \cdot |W(T_w)^F| \cdot |\Delta(s, w)| \end{aligned}$$

where we let:

$$(3.4) \quad \Delta(s, w) = \{T : T \subset C^0(s) \text{ and } T \text{ is } G^F\text{-conjugate to } T_w\}.$$

Note that  $W(T_w)^F \cong C_W(w)$ , the centralizer of  $w$  in  $W$ .

Now suppose that  $T_{w_0}$  is a maximally split torus in  $C^0(s)$ . The Weyl group  $W_{C^0(s)}$  of  $C^0(s)$  can be identified with a reflection subgroup  $W^0(s)$  of  $W$ , so that the action of  $F$  on  $W_{C^0(s)}$  is given by the action of  $w_0^{-1}$  on  $W^0(s)$  by conjugation. Any  $T \subset C^0(s)$  can be obtained from  $T_{w_0}$  by twisting with an element of  $W^0(s)$ . Hence,  $R_w(s)$  is non-zero only if the conjugacy class  $C_w$  of  $w$  in  $W$  has non-zero intersection with the coset  $w_0W^0(s)$ . Moreover, for  $w_1, w_2 \in W^0(s)$ ,  $w_0w_1$  and  $w_0w_2$  are conjugate in  $W$  if and only if  $w_1$  and  $w_2$  are  $w_0$ -conjugate in  $W$ . Thus we see that there is a uniquely determined  $w_0$ -conjugacy class  $Y_{C_w}$  in  $W$  such that:

$$C_w \cap w_0W^0(s) = w_0(Y_{C_w} \cap W^0(s)).$$

Note that  $Y_{C_w} \cap W^0(s)$  is then a union of  $w_0$ -conjugacy classes of  $W^0(s)$ .

Let  $\Theta$  be the set of conjugacy classes of  $W$  which have non-zero intersection with  $w_0W^0(s)$ . For each class  $C$  in  $\Theta$ ,  $w_C$  will denote an element of  $C$  such that  $T_{w_C} \subset C^0(s)$ . Also, for each  $C \in \Theta$ , let  $\theta_C$  be the set of  $w_0$ -conjugacy classes of  $W^0(s)$  which are contained in  $Y_C \cap W^0(s)$ . For each  $J \in \theta_C$ ,  $w_J$  will denote an element of  $J$ , and  $C_{W^0(s), w_0}(w_J)$  will denote the  $w_0$ -centralizer of  $w_J$  in  $W^0(s)$ .

Now we have:

$$\begin{aligned} R_{\chi_E}(s) &= \frac{1}{|W|} \sum_{C \in \Theta} |C| \cdot \chi(w_C) \cdot \frac{\varepsilon_{T_{w_C}} \varepsilon_{C^0(s)}}{|C^0(s)^F|_q} \cdot |C_W(w_C)| \cdot |\Delta(s, w_C)| \\ &= \sum_{C \in \Theta} \chi(w_C) \cdot \frac{\varepsilon_{T_{w_C}} \varepsilon_{C^0(s)}}{|C^0(s)^F|_q} \cdot \left\{ \sum_{J \in \theta_C} \frac{|C^0(s)^F|}{|N_{C^0(s)}(T_{w_0}w_J)^F|} \right\} \\ &= |C^0(s)^F|_{q'} \cdot \sum_J \det_E(w_J) \cdot \frac{1}{|C_{W^0(s), w_0}(w_J)|} \cdot \frac{\chi(w_0w_J)}{|T_{w_0w_J}^F|} \\ &= |C^0(s)^F|_{q'} \cdot \frac{1}{|W^0(s)|} \sum_{w \in W^0(s)} \det_E(w) \cdot \frac{\chi(w_0w)}{|T_{w_0w}^F|} \\ &= (-1)^l \det_E(w_0) \cdot |C^0(s)^F|_{q'} \cdot \frac{1}{|W^0(s)|} \sum_{w \in W^0(s)} \frac{\chi(w_0w)}{\det_E(1 - qw^{-1}w_0^{-1})}. \end{aligned}$$

Now if we denote  $E_0 = X_{\bullet}(T_{w_0}) \otimes \mathbb{Q}$ , and let  $f_1, \dots, f_l$  be the exponents of the reductive group  $C^0(s)$ , we have:

$$R_{\chi_E}(s) = \prod (1 - \varepsilon_i q^{f_i+1}) \cdot \frac{1}{|W^0(s)|} \cdot \sum_{w \in W^0(s)} \frac{\chi_{E_0}(Fw)}{\det_{E_0}(1 - qFw)}.$$

By Lemma (2.3), and (2.8), we then get:

**Proposition 3.5.**

$$\chi_{\Pi}(s) = R_{\chi_E}(s) = \text{Tr}(F|M(s))$$

where  $M(s)$  is the motive of the reductive group  $C^0(s)$ .

**Examples:** If  $s = 1$ , then  $\chi_\Pi(1) = \sum q^{e_i}$ .

If  $s$  is regular semi-simple, then  $\chi_\Pi(s) = \chi(w_0)$ , as shown in [Lu].

**Remarks:** We can regard the above formula as a  $q$ -deformation of (2.7). Indeed, if we set  $q = 1$  on the right hand side of the above proposition, we get  $Tr(F|V_{C^0(s)})$ . On the other hand, it is "reasonable" to regard the reflection representation  $E_W$  of  $W$  as a degeneration, as  $q \rightarrow 1$ , of the reflection representation  $\Pi$ . Viewing

$$\begin{aligned} \{\text{semi-simple classes in } G^F\} &\longrightarrow \{\text{conjugacy classes in } W\} \\ (s) &\mapsto (w_0) \end{aligned}$$

as a way of deforming semi-simple classes in  $G^F$  to classes in  $W$ , we see that the left hand side of the above Proposition becomes:  $Tr(w_0|E) = Tr(F|E_0)$ . So Proposition (3.5) becomes, on letting  $q = 1$ :

$$Tr(F|E_0) = Tr(F|V_{C^0(s)})$$

which is (2.7) for the reductive group  $C^0(s)$ .

#### 4. Generic Representations

We continue to assume that  $k = \mathbb{F}_q$  is a finite field. The remainder of this paper will be devoted to the study of Howe correspondence. Before that, we review some material about generic representations. In this section,  $G$  will denote any connected reductive group over  $k$ , which has connected center. Then any Levi subgroup of  $G$  will also have connected center [Ca, Pg. 260], and  $G^F$  has a unique Gelfand-Graev representation  $\Delta_G$ . Recall that  $\Delta_G \cong \text{Ind}_U^G \psi$ , where  $\psi$  is a character in general position of a maximal unipotent subgroup  $U$  of  $G$ . An irreducible representation of  $G^F$  is said to be generic if it is a component of  $\Delta_G$ , and then it occurs with multiplicity one. Moreover,  $G^F$  has exactly  $q^l$  generic characters, where  $l$  is the rank of  $G$ .

The following is a result of Rodier [Ca, Pg. 261]:

**Proposition 4.1.** *Let  $P = L \cdot U$  be an  $F$ -stable standard parabolic subgroup. For any character  $\chi_L$  of  $L^F$ , denote by  $\chi_L$  also its lift to  $P^F$ . Then,*

$$\langle \text{Ind}_{P^F}^{G^F} \chi_L, \Delta_G \rangle = \langle \chi_L, \Delta_L \rangle.$$

**Corollary 4.2.** (1) *If  $\chi_L$  is generic, then  $\text{Ind}_{P^F}^{G^F} \chi_L$  has a unique generic summand.*

(2) *If  $\chi$  is a generic character of  $G^F$ , then any irreducible component of  $\chi^U$  is also generic.*

We shall denote the unique generic summand in the above corollary by  $\widetilde{\chi}_L^G$ . By Corollary (4.2),  $\chi_L \mapsto \widetilde{\chi}_L^G$  defines a map (not injective):

$$(4.3) \quad \{\text{generic characters of } L^F\} \longrightarrow \{\text{generic characters of } G^F\}.$$

We can understand this map using the parametrization of Lusztig. From [D-L], we know that the irreducible characters of  $G^F$  can be partitioned into geometric conjugacy classes, which is in turn parametrized by semi-simple conjugacy classes in  $G^{*F^*}$ , where  $(G^*, F^*)$  is the dual group of  $(G, F)$ . In each geometric conjugacy class, there is a unique generic

character. The generic character corresponding to the class of  $s^*$  is denoted  $\chi(s^*)$ . For example,  $\chi(1)$  is the Steinberg representation of  $G^F$ .

If  $L$  is a Levi factor of  $G$ , then there is a Levi factor  $L^* \subset G^*$  which is in duality with  $L$ . Now if  $\chi_L$  is generic with parameter  $s^* \in L^{*F^*}$ , then the parameter of  $\widetilde{\chi}_L^G$  is just the class of  $s^*$  in  $G^{*F^*}$ .

Note that the generic characters are characterized by the fact that their dimensions have the form:

$$(4.4) \quad q^N + (\text{ terms involving lower powers of } q)$$

where  $N$  is the number of positive roots of  $G$ . Hence the generic characters are the biggest representations of  $G^F$  in that their dimensions grow the fastest as  $q$  becomes large. Indeed, if  $s^* \in G^{*F^*}$ , and  $G_{s^*}^*$  the centralizer of  $s^*$  in  $G^*$ , then the dimension of  $\chi(s^*)$  is given by:

$$(4.5) \quad \dim(\chi(s^*)) = q^{\dim(U_{G_{s^*}^*})} \cdot \frac{\det(F-1|M_{G^*}^\vee(1))}{\det(F-1|M_{G_{s^*}^*}^\vee(1))}$$

where  $U_{G_{s^*}^*}$  is the maximal unipotent subgroup of  $G_{s^*}^*$ , and  $M^\vee$  is the motive dual to  $M$ .

## 5. Dual Pairs

Henceforth, we consider the split adjoint groups of type  $E_6$  and  $E_7$ . In [MS], the following dual pairs were constructed:

$$(5.1) \quad G_2 \times PGL_3 \subset E_6$$

$$G_2 \times PGSp_6 \subset E_7$$

and the representation correspondence arising from the restriction of the minimal representation of  $G$  over a non-archimedean local field was studied. The analogue of the minimal representation in the finite field situation is exactly the reflection representation  $\Pi$ . Indeed, the dimension of  $\Pi$  is the smallest among all non-trivial irreducible representations of  $G^F$  (that is, those of dimension greater than 1). Hence, we shall be interested in the restriction of  $\Pi$  to these pairs. In this section, we describe the results we expect.

Consider, for example, the pair  $G_2 \times PGL_3$ . Note that there is a natural inclusion of dual groups:

$$(5.2) \quad SL_3 \longrightarrow G_2.$$

Using the parametrization of generic characters discussed in the previous section, we see that we have a map:

$$(5.3) \quad i_* : \{\text{generic characters of } PGL_3\} \longrightarrow \{\text{generic characters of } G_2\}$$

$$\chi(s)_{PGL_3} \mapsto \chi(i(s))_{G_2}.$$

Note that this map is neither surjective nor injective. Indeed it is usually two-to-one; if  $\chi(s)^* = \chi(s^{-1})$  denotes the contragredient character of  $\chi(s)$ , then  $\chi(i(s)) = \chi(i(s^{-1})) = \chi(i(s))^*$ .

Now we guess that:

$$(5.4) \quad \langle \chi(i(s))_{G_2} \otimes \chi(s)_{PGL_3}, \Pi \rangle_{G_2 \times PGL_3} = 1$$

for any generic character  $\chi(s)_{PGL_3}$  of  $PGL_3$ . Notice that  $PGL_3$  has  $q^2$  generic characters, and their dimensions, given by (4.5), have the form:

$$q^3 + (\text{ terms involving lower powers of } q),$$

whereas the generic characters of  $G_2$  have dimensions:

$$q^6 + (\text{ terms involving lower powers of } q).$$

So if our guess is correct, then we would have accounted for a subspace of dimension:

$$q^{11} + (\text{ terms involving lower powers of } q)$$

of  $\Pi$ , which has dimension  $q^{11} + q^8 + q^7 + q^5 + q^4 + q$ . Hence we would have accounted for a large piece of  $\Pi$ . Indeed, one can check that the dimension of the space unaccounted has leading term  $2q^9$ .

For a generic character  $\chi_{PGL_3}$ , we let:

$$(5.5) \quad \Theta(\chi_{PGL_3}) = \{\chi_{G_2} : \langle \Pi, \chi_{PGL_3} \otimes \chi_{G_2} \rangle \neq 0\}.$$

Here, the representations  $\chi_{G_2}$  are counted with multiplicities. Also, let  $\Theta_{gen}(\chi_{PGL_3})$  be the subset of generic representations in  $\Theta(\chi_{PGL_3})$ . Thus, (5.4) says that:

$$(5.6) \quad \Theta_{gen}(\chi(s)_{PGL_3}) = \{\chi(i(s))_{G_2}\}.$$

Similarly, in the case  $E_7$ , we have a natural inclusion of dual groups:

$$(5.7) \quad i : G_2 \longrightarrow Spin_7$$

Again, we guess that:

$$(5.8) \quad \Theta_{gen}(\chi(s)_{G_2}) = \{\chi(i(s))_{PGSp_6}\}$$

for every semi-simple class  $(s)$  in  $G_2$ , where the set  $\Theta_{gen}(\chi(s)_{G_2})$  is similarly defined.

Notice that  $G_2$  has  $q^2$  generic characters of dimension  $(q^6 + \dots)$ , whereas the generic characters of  $PGSp_6$  have dimension  $(q^9 + \dots)$ . Hence, if our guess is true, we would have accounted for a subspace of dimension  $(q^{17} + \dots)$  in  $\Pi$ , which has dimension  $q^{17} + q^{13} + q^{11} + q^9 + q^7 + q^5 + q$ .

**Remark:** The inclusion (5.7) is realized by regarding  $G_2$  as the stabilizer of a non-isotropic vector in the eight-dimensional Spin representation of  $Spin_7$ . If we work over an algebraic closure  $\bar{k}$  of  $k$ , all such embeddings, which a priori depend on the choice of the non-isotropic vector, are in fact conjugate in  $Spin_7$ . This is, however, not true over  $k$ , since the norm  $\langle v, v \rangle$  of the non-isotropic vector  $v$  is an invariant, with values in  $k^\times / k^{\times 2}$ , of the conjugacy class of an embedding. Fortunately, for our purposes, this does not matter, since any two semi-simple elements of  $Spin_7(k)$  which are conjugate over  $\bar{k}$  are already conjugate over  $k$ .

The remainder of this paper is devoted to proving (5.6) and (5.8). We shall first show that, in each case,  $\Theta_{gen}(\chi(s))$  is a singleton set, by computing Whittaker vectors. We then proceed to check that it is really what we expect. This is accomplished for non-cuspidal representations by computing Jacquet functors. The case of cuspidal generic representations is settled by using base change, or Shintani descent.

### 6. Restriction to Maximal Parabolics

Henceforth, we assume that  $p \geq 5$ , so that  $p$  is a good prime for  $G = E_6$  or  $E_7$ , and the Killing form is non-degenerate. For the purposes of computing Whittaker vectors and Jacquet functors, we need to know the restriction of  $\Pi$  to various maximal parabolic subgroups of  $E_6$  and  $E_7$ .

In each case, there is a maximal parabolic subgroup  $P_0 = M_0 \ltimes N_0$ , whose unipotent radical  $N_0$  is abelian. In the case of  $E_6$ , the derived group of  $M_0$  is of type  $D_5$ , and  $N_0$  is a Spin representation of  $D_5$  via the adjoint action of  $M_0$ . In the case of  $E_7$ , the derived group of  $M_0$  is of type  $E_6$  and  $N_0$  is one of the two 27-dimensional representations of  $E_6$  via adjoint action, and we can identify  $N_0$  with the exceptional Jordan algebra over  $k = \mathbb{F}_q$ . Note that this Jordan algebra is split.

Since  $N_0$  is abelian, we can identify its character group  $N_0^\vee$  with  $\overline{N}_0$ , the unipotent radical of the opposite parabolic  $\overline{P}_0$ , as follows. If

$$(6.1) \quad \langle \cdot, \cdot \rangle : N_0 \times \overline{N}_0 \longrightarrow k$$

is the pairing induced by the Killing form, which is non-degenerate by assumption, and  $\psi : k \longrightarrow \mathbb{C}^*$  is a fixed non-trivial additive character, then

$$(6.2) \quad \psi_x = \psi(\langle -, x \rangle) : N_0 \longrightarrow \mathbb{C}^*$$

is a character of  $N_0$ , and the map  $x \mapsto \psi_x$  gives an identification of  $N_0^\vee$  with  $\overline{N}_0$ . Note that the minimal non-trivial  $M_0$ -orbit  $\omega$  in  $\overline{N}_0$  is the orbit of the highest weight vector.

The following proposition, which describes the restriction of  $\Pi$  to  $P_0$ , is the finite field analogue of Theorem (1.1) in [MS].

**Proposition 6.3.**

$$\Pi \downarrow_{P_0} \cong \Pi^{N_0} \oplus V$$

where,

$$\Pi^{N_0} \cong 1 \oplus \Pi(M_0)$$

and as a  $N_0$ -module,

$$V \cong \bigoplus_{x \in \omega} \mathbb{C}\psi_x$$

and  $M_0$  acts on  $V$  by its permutation action on  $\omega$ .

Now consider the Heisenberg maximal parabolic  $P = M \cdot N$ , so-called because  $N$  is a Heisenberg group.  $P$  corresponds to the unique vertex  $\alpha$  joined to the negative of the highest root in the extended Dynkin diagram. So for  $E_6$ ,  $M$  has derived group of type  $A_5$ , and for  $E_7$ , it is of type  $D_6$ .

Let  $Z$  be the one-dimensional center of  $N$ , and  $\overline{Z}$  that of  $\overline{N}$ . Then the Killing form induces a non-degenerate pairing:

$$(6.4) \quad \langle \cdot, \cdot \rangle : N/Z \times \overline{N}/\overline{Z} \longrightarrow k$$

With the fixed character  $\psi$ , we can identify the character group of  $N/Z$  with  $\overline{N}/\overline{Z}$  as before. Let  $\Omega$  be the minimal non-trivial  $M$ -orbit in  $\overline{N}/\overline{Z}$ .

The following Proposition is the finite field analogue of Theorem (6.1) in [MS].

**Proposition 6.5.**

$$\Pi^Z \cong \Pi^N \oplus V$$

where,

$$\Pi^N \cong 1 \oplus \Pi(M)$$

and, as a  $N/Z$ -module,

$$V \cong \bigoplus_{x \in \Omega} \mathbb{C}\psi_x$$

and  $M$  acts on  $V$  via its permutation action on  $\Omega$ .

## 7. Whittaker Vectors

In this section, let  $U$  be the unipotent radical of a Borel subgroup of  $G_2$  (respectively  $PGLSp_6$ ), and let  $\psi$  be a generic character of  $U$ . We shall compute  $\Pi_{U,\psi}$  for  $E_6$  (respectively  $E_7$ ). It is a pleasure to thank G. Savin for his suggestion to do this computation.

The main result of this section is:

**Theorem 7.1.**

- (1) *If  $\Pi$  is the reflection representation of  $E_6$ , and  $\psi$  a generic character of  $U$ , then  $\Pi_{U,\psi}$  is the Gelfand-Graev representation of  $PGL_3$ .*
- (2) *Similarly, if  $\Pi$  is the reflection representation of  $E_7$ , and  $\psi$  a generic character of  $U$ , then  $\Pi_{U,\psi}$  is the Gelfand-Graev representation of  $G_2$ .*

*Proof.* Let us consider the case of  $E_6$  first. Let  $P_2 = L_2U_2$  be the Heisenberg parabolic of  $G_2$ . Write  $U = U_2 \rtimes U'$ , with  $U' \cong k$ . Then we can denote  $\psi$  by  $\psi = (\phi, \varphi)$ , with  $\phi := \psi|_{U_2}$ , and  $\varphi := \psi|_{U'}$ .

Now it was shown in [MS] that there is an embedding of the dual pair  $G_2 \times PGL_3$  such that  $G_2 \cap P = P_2$ . Then  $\phi$  is a degenerate character of  $U_2$  in the smallest  $L_2$  orbit. We first compute  $\Pi_{U_2,\phi}$ . By Proposition (6.5), we need to compute  $V_{U_2,\phi}$ . The same considerations as in [GrS, Prop. 2.8, Section VI] show that:

$$\Pi_{U_2,\phi} = \mathbb{C}[\Omega_\phi]$$

as a representation of  $PGL_3 \times L_{2,\phi}$ , where  $\Omega_\phi$  is the set of nilpotent 3-by-3 matrices, and  $L_{2,\phi}$  is the stabilizer of  $\phi$  in  $L_2$ . In particular,  $U' \subset L_{2,\phi}$ .

To finish the computation, we need to know the action of  $U' \cong k$  on  $\Omega_\phi$ . If  $\lambda \in k$  and  $z \in \Omega_\phi$ , then the action of  $\lambda$  on  $z$  is given by:

$$\lambda : z \mapsto z + 2\lambda e \times (z \times z)$$

where  $e$  is the identity matrix, and  $z \times z$  is the adjoint matrix of  $z$ . Hence, we see that  $U'$  fixes each  $z$  of rank less than 2, and acts freely on the rank 2 elements.

Now the action of  $PGL_3$  on  $\Omega_\phi$  is by conjugation, and so  $PGL_3$  has 3 orbits on  $\Omega_\phi$ , characterized by rank. Since  $U'$  acts trivially on the elements of rank less than 2, we have:

$$\Pi_{U,\psi} = \mathbb{C}[\{z \in \Omega_\phi : \text{rank}(z) = 2\}]_{U',\varphi}$$

Let  $z \in \Omega_\phi$  be given by:

$$z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the  $U'$ -orbit of  $z$  are elements of the form:

$$\begin{pmatrix} 0 & 1 & \lambda \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the stabilizer of  $z$  in  $PGL_3$  is :

$$H = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : b, c \in k \right\}.$$

So as a representation of  $PGL_3 \times U'$ ,

$$\mathbb{C}[\{z \in \Omega_\phi : \text{rank}(z) = 2\}] \cong \text{Ind}_{U_{PGL_3 \times k}}^{PGL_3 \times k} \mathbb{C}[k]$$

where the action of  $k$  is by right translation, and the action of  $U_{PGL_3}$ , the group of upper triangular unipotent matrices, factors through the quotient:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto a - b \in k.$$

Hence, we deduce that:

$$\Pi_{U,\psi} = \text{Ind}_{U_{PGL_3}}^{PGL_3} \mathbb{C}[k]_{U',\varphi} = \text{Ind}_{U_{PGL_3}}^{PGL_3} \varphi_0$$

where  $\varphi_0$  is the character of  $U_{PGL_3}$  given by:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \varphi(b - a).$$

Hence,  $\Pi_{U,\psi}$  is indeed the Gelfand-Graev representation of  $PGL_3$ .

Now we consider the case of  $E_7$ . Let  $P_0$  be the maximal parabolic of  $E_7$  considered in the previous section. Then  $P_0 \cap PGSp_6 = GL_3 \rtimes U_0$  is the Siegel parabolic of  $PGSp_6$ , and  $U_0$  can be identified with the set of all 6-by-6 matrices of the form:

$$\begin{pmatrix} I_3 & B \\ 0 & I_3 \end{pmatrix}, \quad B^t = B$$

As before, write  $U = U_0 \rtimes U'$ , where  $U'$  is the unipotent radical of  $GL_3$ . Also, let  $\phi := \psi|_{U_0}$ , and  $\varphi := \psi|_{U'}$ .

Note that:

$$\phi : \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \mapsto \phi_0(B_{33})$$

where  $\phi_0$  is a non-trivial character of  $k$ .

By Proposition (6.3), we see that:

$$\Pi_{U,\psi} = \mathbb{C}[\omega]_{U,\psi}.$$

We first compute  $\mathbb{C}[\omega]_{U_0, \phi}$ . As before, we have:

$$\Pi_{U_0, \phi} = \mathbb{C}[\omega_\phi]$$

where  $\omega_\phi$  is the set of all  $X$  in  $\omega$  of the form:

$$X = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 1 \end{pmatrix}$$

with  $x, y, z$  elements of the octonion algebra  $\mathbb{O}$  satisfying:

$$\begin{aligned} \text{tr}(x) = \text{tr}(y) = \text{tr}(z) = 0, x^2 = y^2 = z^2 = 0, \\ xz = yz = 0, xy = -z. \end{aligned}$$

Now we need to consider the action of  $U'$  on  $\omega_\phi$ . Note that  $U'$  is a Heisenberg group of dimension 3. The  $U'$ -orbit of an  $X$  as above consists of all elements of  $\omega_\phi$  of the form:

$$\begin{pmatrix} 0 & z & -(y + cz) \\ 0 & x + ay + bz \\ 1 \end{pmatrix}, \quad a, b, c \in k.$$

Now,  $G_2 \times U'$  has various orbits in  $\omega_\phi$ . The generic orbit is the one consisting of all those  $X$ 's such that the  $(x, y, z)$  of  $X$  span a 3-dimensional space. Call this orbit  $\mathcal{O}$ . Then  $U'$  acts freely on  $\mathcal{O}$ . One checks that only  $\mathbb{C}[\mathcal{O}]$  will contribute to the space of Whittaker vectors, i.e.

$$\Pi_{U, \psi} = \mathbb{C}[\mathcal{O}]_{U', \varphi}$$

Now as a representation of  $G_2 \times U'$ ,

$$\mathbb{C}[\mathcal{O}] = \text{Ind}_{U_{G_2} \times U'}^{G_2 \times U'} \mathbb{C}[U'].$$

The action of  $U_{G_2}$  on  $\mathbb{C}[\mathcal{O}]$  is as follows. Note that  $U_{G_2}$  is a 5-step nilpotent group:

$$U_{G_2} \supset U_{G_2}^{(1)} \supset U_{G_2}^{(2)} \supset U_{G_2}^{(3)} \supset U_{G_2}^{(4)} \supset \{1\}$$

and  $U_{G_2}/U_{G_2}^{(2)} \cong U'$ . Hence the action of  $U_{G_2}$  is via the quotient:  $U_{G_2} \longrightarrow U_{G_2}/U_{G_2}^{(2)}$ . Hence, we see that:

$$\mathbb{C}[\mathcal{O}]_{U', \varphi} \cong \text{Ind}_{U_{G_2}}^{G_2} \varphi^{-1}$$

which is the Gelfand-Graev representation of  $G_2$ . □

**Corollary 7.2.**

- (1) For any generic representation  $\pi$  of  $PGL_3$ ,  $\Theta_{gen}(\pi)$  is a singleton set.
- (2) For any generic representation  $\pi$  of  $G_2$ ,  $\Theta_{gen}(\pi)$  is a singleton set.

**Corollary 7.3.**

- (1) If  $\sigma$  is a non-generic representation of  $PGL_3$ , then

$$\Theta_{gen}(\sigma) = \emptyset.$$

- (2) If  $\sigma$  is a non-generic representation of  $G_2$ , then

$$\Theta_{gen}(\sigma) = \emptyset.$$

## 8. Jacquet Functors

By the previous corollaries, it remains to check that the correspondence of the parameters of generic representations is given by the natural inclusions of dual groups. For non-cuspidal representations, this can be checked by computing Jacquet functors. Most of what we need have been computed in [MS]; so we begin by transferring their results to the finite field situation.

Let us denote our two dual pairs by:  $G_2 \times H$ , and let  $P_0$  be the maximal parabolic introduced in Section 6. Recall that the unipotent radical of  $P_0$  is abelian. Also, we have:

$$(8.1) \quad (G_2 \times H) \cap P_0 = G_2 \times Q_0$$

where  $Q_0 = L_0 \cdot U_0$  is a maximal parabolic of  $H$ . In the case of  $E_6$ ,  $L_0 = GL_2$ , and for  $E_7$ ,  $Q_0$  is the Siegel parabolic of  $PGSp_6$ , so that  $L_0 = GL_3$ . We also denote by  $P_1 = L_1 \cdot U_1$  (respectively  $P_2 = L_2 \cdot U_2$ ) the maximal parabolic of  $G_2$  whose Levi factor  $L_1$  (respectively  $L_2$ ) is spanned by the long (respectively short) simple root.

The following Propositions give the structure of the  $G_2 \times L_0$ -module  $\Pi^{U_0}$ , and are the finite field analogues of Theorem 4.3 and Theorem 5.3 in [MS].

**Proposition 8.2.** *For  $E_6$ ,*

$$\Pi^{U_0} \cong \Pi^{N_0} \oplus \text{Ind}_{P_2 \times GL_2}^{G_2 \times GL_2} \mathbb{C}[GL_2] \oplus \text{Ind}_{P_1 \times B}^{G_2 \times GL_2} \mathbb{C}[GL_1].$$

Here,  $B$  is the Borel subgroup of  $GL_2$ , and the actions of  $P_1, P_2$  and  $B$  on the appropriate spaces are via the quotients:

$$\begin{aligned} P_2 &\twoheadrightarrow L_2 \cong GL_2, \\ P_1 &\twoheadrightarrow L_1 \cong GL_2 \longrightarrow GL_1 \text{ (determinant map)}, \\ B &\twoheadrightarrow GL_1 \times GL_1 \twoheadrightarrow GL_1 \text{ (projection onto first factor)}. \end{aligned}$$

**Proposition 8.3.** *For  $E_7$ ,*

$$\Pi^{U_0} \cong \Pi^{N_0} \oplus \text{Ind}_{P_2 \times Q_2}^{G_2 \times GL_3} \mathbb{C}[GL_2] \oplus \text{Ind}_{P_1 \times Q_1}^{G_2 \times GL_3} \mathbb{C}[GL_1]$$

Here,  $Q_1 = M_1 \cdot V_1$  (respectively  $Q_2 = M_2 \cdot V_2$ ) is the maximal parabolic of  $GL_3$  which stabilizes a line (respectively a plane) in  $\mathbb{C}^3$ , and the actions of  $P_i$  and  $Q_i$  on the appropriate spaces are via the quotients:

$$\begin{aligned} P_2 &\twoheadrightarrow L_2, \\ P_1 &\twoheadrightarrow L_1 \longrightarrow GL_1 \text{ (determinant map)}, \\ Q_2 &\twoheadrightarrow M_2 \cong GL_2 \times GL_1 \longrightarrow GL_2 \text{ (projection onto first factor)}, \\ Q_1 &\twoheadrightarrow M_1 \cong GL_1 \times GL_2 \longrightarrow GL_1 \text{ (projection onto first factor)}. \end{aligned}$$

Now we consider the Heisenberg parabolic  $P$ . It was shown in [MS] that there is an embedding of the dual pairs such that:

$$(8.4) \quad (G_2 \times H) \cap P = P_2 \times H$$

where  $P_2$  is as defined before and is the Heisenberg parabolic of  $G_2$ . The following Proposition gives the structure of the  $L_2 \times H$ -module  $\Pi^{U_2}$  and is the finite field analogue of Theorem 7.6 in [MS].

**Proposition 8.5.** *For  $E_7$ ,*

$$\Pi^{U_2} \cong \Pi^N \oplus \text{Ind}_{L_2 \times Q'_2}^{L_2 \times PGSp_6} \mathbb{C}[GL_2] \oplus \text{Ind}_{B \times Q}^{L_2 \times PGSp_6} \mathbb{C}[GL_1]$$

Here,  $Q$  is the maximal parabolic subgroup of  $PGSp_6$  corresponding to the middle vertex in the Dynkin diagram, and  $Q'_2$  is the minimal parabolic of  $PGSp_6$  which intersects the Levi factor of the Siegel parabolic in  $Q_2$ .

For our purposes, we also need to compute the Jacquet functor of  $\Pi(E_7)$  with respect to the maximal parabolic  $P_1 = L_1 U_1$  of  $G_2$ . Over a  $p$ -adic field, this is computed in Proposition 6.8 in [SG]. The result over finite fields can be checked along similar lines, but to state it, we need to introduce some more notations.

Let  $P'$  be the maximal parabolic of  $E_7$  corresponding to the unique vertex  $\beta$  joined to  $\alpha$  in the Dynkin diagram. Then,

$$(G_2 \times PGSp_6) \cap P' = P_1 \times PGSp_6.$$

Let  $Q = L \cdot U$  be the maximal parabolic subgroup of  $PGSp_6$  corresponding to the middle vertex in the Dynkin diagram of type  $C_3$ . Hence, its Levi factor is:

$$L \cong (GL_2 \times GL_2)/k^\times$$

with  $k^\times$  embedded via:  $a \mapsto (a, a^{-1})$ . Let

$$R_0 = \{(g_1, g_2, g_3) \in L_1 \times GL_2 \times GL_2 : \det(g_1 g_2 g_3) = 1\}.$$

Then  $R_0$  has a natural Weil representation  $W$  which can be realized on  $\mathbb{C}[M_2(k)]$ , where  $M_2(k)$  is the space of 2-by-2 matrices over  $k$ , which we describe below. Let  $\{\cdot, \cdot\}$  be the standard non-degenerate symplectic form on  $k^2$ , and let  $V = k^2 \otimes k^2 \otimes k^2$  with symplectic form:

$$\{u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3\} = \prod_{i=1}^3 \{u_i, v_i\}.$$

Then  $R_0 \rightarrow Sp(V)$  under the natural action of  $GL_2$  on  $k^2$ . Now, it is well-known that  $Sp(V)$  has a natural Weil representation [Ge], which can be realized on the space of  $\mathbb{C}$ -valued functions on the maximal isotropic subspace  $ke_1 \otimes k^2 \otimes k^2$ . Then the representation  $W$  is the pullback of the Weil representation of  $Sp(V)$  to  $R_0$ . In particular, the action of  $SL_2 \subset L_1$  is via the usual Weil representation formulas [Ge], whereas the action of the subgroup:

$$R_1 = \{(g, h) : \det(gh) = 1\} \subset GL_2 \times GL_2$$

is geometric, and we see that  $W$  is actually a representation of the quotient

$$R = \{(g, h) \in L_1 \times L : \det(gh) = 1\}$$

of  $R_0$ . Let

$$(8.6) \quad \tilde{W} = \text{Ind}_R^{L_1 \times L} W.$$

Then the following proposition is the finite field analogue of Proposition 6.8 of [SG]:

**Proposition 8.7.** *In the case  $E_7$ ,*

$$\Pi^{U_1} \cong \Pi^{N'} \oplus \text{Ind}_{B \times Q}^{GL_2 \times PGSp_6} \mathbb{C}[GL_1] \oplus \text{Ind}_{GL_2 \times Q}^{GL_2 \times PGSp_6} \tilde{W}.$$

### 9. Shintani Descent

The results of the last section will allow us to determine the correspondence for non-cuspidal representations. Before doing that, we review the results about Shintani descent that we need for the correspondence of cuspidal representations. We refer the reader to the article of Digne [D] for a quick introduction.

Let  $k_m := \mathbb{F}_{q^m}$  and  $G(q^m) := G^{F^m}$ . There is a norm map:

$$\mathbb{N}_m : G(q^m) \longrightarrow G(q)$$

which induces a bijection of  $F$ -conjugacy classes in  $G(q^m)$  and conjugacy classes in  $G(q)$ . Hence, given any  $F$ -class function  $\psi$  of  $G(q^m)$ ,  $Sh_m(\psi) := \psi \circ \mathbb{N}_m^{-1}$  is a class function on  $G(q)$ , and  $Sh_m$  is an isomorphism of the vector space of  $F$ -class function on  $G(q^m)$  and the vector space of class functions on  $G(q)$ . Moreover, it is an isometry for the natural inner products on the two vector spaces:

$$\langle \psi_1, \psi_2 \rangle_{G(q^m)} = \langle Sh_m(\psi_1), Sh_m(\psi_2) \rangle_{G(q)}.$$

We will call  $\psi$  the base change of  $Sh_m(\psi)$  or equivalently,  $Sh_m(\psi)$  the Shintani descent of  $\psi$ . Note that our  $Sh_m$  is equal to the map  $i \circ Sh_{\mathbb{F}_{q^m}/\mathbb{F}_q}$  in [D].

If  $\chi$  is (the character of) an  $F$ -stable representation of  $G(q^m)$ , then  $\chi$  extends (non-uniquely) to a representation of  $G(q^m) \rtimes \langle F \rangle$ , denoted  $\tilde{\chi}$ . If  $\chi$  is irreducible, then  $\tilde{\chi}$  is well-determined up to an  $m$ -th root of unity. In any case, the function  $g \mapsto \tilde{\chi}(Fg)$  is an  $F$ -class function on  $G(q^m)$ , and so we can consider  $Sh_m(\tilde{\chi}(F\cdot))$ . For ease of notation, we shall denote this class function on  $G(q)$  simply by  $Sh_m(\tilde{\chi})$ .

Now, the results we need are summarized in the the following Propositions, which are special cases of results of Digne-Michel [D] and Digne [D2] respectively.

**Proposition 9.1.** *There is an extension  $\widetilde{\Pi}_m$  of the reflection representation  $\Pi_m$  of  $G(q^m)$  (for  $G$  of type  $E_n$ ) such that:*

$$Sh_m(\widetilde{\Pi}_m) = \Pi.$$

**Remarks:** We note that  $\Pi_m$  is  $F$ -stable by the characterization of the reflection representation in Section 3.

**Proposition 9.2.** *Assume that  $p$  is a good prime for  $G$ , and  $(m, p) = 1$ . Let  $\theta$  be a character of  $T(q)$ , where  $T$  is an  $F$ -stable maximal torus of  $G$ . Let  $R(\theta)_m := R_{T(q^m)}^{G(q^m)}(\theta \circ \mathbb{N}_m)$ , a virtual character. We shall write  $R(\theta)$  for  $R(\theta)_1$ . Then there is an extension of  $R(\theta)_m$  to a virtual character  $\widetilde{R(\theta)_m}$  of  $G(q^m) \rtimes \langle F \rangle$  such that:*

$$Sh_m(\widetilde{R(\theta)_m}) = R(\theta).$$

**Remarks:** In our applications, we shall only need to consider  $m = 2$  or  $3$ . Hence  $p \geq 5$  will suffice.

### 10. The Dual Pair $G_2 \times PGL_3$

In this section, we consider the restriction of  $\Pi$  to  $G_2 \times PGL_3 \subset E_6$ . We shall first show that (5.4) holds for non-cuspidal generic characters of  $PGL_3$ . Suppose  $\chi(s)_{PGL_3}$  is a

non-cuspidal generic character. Then we can assume that  $s \in L_0^* \cong GL_2$ , so that

$$\chi(s)_{PGL_3} \hookrightarrow \text{Ind}_{Q_0}^{PGL_3} \chi(s)_{GL_2}.$$

By Frobenius reciprocity,

$$\langle \Pi, \chi(i(s))_{G_2} \otimes \text{Ind}_{Q_0}^{PGL_3} \chi(s)_{GL_2} \rangle = \langle \Pi^{U_0}, \chi(i(s))_{G_2} \otimes \chi(s)_{GL_2} \rangle.$$

This is non-zero, since by Proposition 8.2,  $\chi(i(s))_{G_2} \otimes \chi(s)_{GL_2}$  occurs in the second summand of  $\Pi^{U_0}$ . By Corollaries 7.2 and 7.3, we have:

**Proposition 10.1.** *For any non-cuspidal generic representation  $\chi(s)_{PGL_3}$ ,*

$$\Theta_{gen}(\chi(s)_{PGL_3}) = \{\chi(i(s))_{G_2}\}.$$

**Example:** Since the Steinberg character has parameter the trivial class 1,

$$\Theta_{gen}(St_{PGL_3}) = \{St_{G_2}\}.$$

Now we have:

**Theorem 10.2.** *For  $p \geq 5$*

$$\Theta_{gen}(\chi(s)_{PGL_3}) = \{\chi(i(s))_{G_2}\}.$$

*Proof.* After Proposition 10.1, it remains to prove this for cuspidal representations. Let  $G = PGL_3 \times G_2$ . Note that if  $\chi(s)_{PGL_3}$  is cuspidal, then  $s$  must be regular in  $SL_3$ , the dual group of  $PGL_3$ . So any cuspidal representation of  $PGL_3$  is of the form  $R_T(\theta)$ , where  $T(q) = k_3^\times/k^\times$ , and  $\theta$  is a regular character of  $T(q)$ . This has parameter:

$$s = \begin{pmatrix} a & & \\ & a^q & \\ & & a^{q^2} \end{pmatrix} \in SL_3(q^3)$$

with all elements on the diagonal distinct. Now  $i(s) \in G_2$  is also regular and is contained in an elliptic torus. So it corresponds to a cuspidal generic representation  $R_{T'}(\theta')$  of  $G_2$ . Hence, we need to show that:

$$\langle \Pi, R(\theta) \otimes R(\theta') \rangle = 1$$

Now, by Propositions 9.1 and 9.2, we can find extensions such that:

$$\begin{aligned} Sh_3(\widetilde{\Pi}_3) &= \Pi, \\ Sh_3(\widetilde{R(\theta)}_3) &= R(\theta), \\ Sh_3(\widetilde{R(\theta')}_3) &= R(\theta'). \end{aligned}$$

Note that since  $T$  and  $T'$  both split over  $k_3$ ,  $R(\theta)_3$  and  $R(\theta')_3$  are irreducible principal series representations of  $PGL_3(q^3)$  and  $G_2(q^3)$  respectively, with parameters  $s$  and  $i(s)$  in  $SL_3(q^3)$  and  $G_2(q^3)$  respectively. Hence, by Proposition 10.1, their tensor product occurs in  $\Pi_3$  with multiplicity 1. Let us denote  $R(\theta)_3 \otimes R(\theta')_3$  by  $\Pi(\theta)_3$ , for simplicity. Note that the extension of  $\Pi_3$  above induces an extension of  $\Pi(\theta)_3$ , but this may or may not be the same as the extension of  $\Pi(\theta)_3$  chosen above.

Now, we have:

$$\sum_{i=0}^2 \frac{1}{3} \langle \widetilde{\Pi}_3(F^i \cdot), \widetilde{\Pi}(\theta)_3(F^i \cdot) \rangle_{G(q^3)} = \langle \widetilde{\Pi}_3, \widetilde{\Pi}(\theta)_3 \rangle_{G(q^3) \rtimes \langle F \rangle} \in \mathbb{Z}.$$

But we know, by Proposition 10.1, that:

$$\langle \Pi_3, \Pi(\theta)_3 \rangle_{G(q^3)} = 1.$$

Moreover, since  $Sh_m$  is an isometry, for  $i = 1$  or  $2$ :

$$\langle \widetilde{\Pi}_3(F^i \cdot), \widetilde{\Pi}(\theta)_3(F^i \cdot) \rangle_{G(q^3)} = \langle \Pi, \Pi(\theta) \rangle_{G(q)} \in \mathbb{N}.$$

Thus we see that the only possibility is that:

$$\langle \Pi, \Pi(\theta) \rangle = 1.$$

This completes the proof of the theorem.  $\square$

### 11. The Dual Pair $G_2 \times PGSp_6$

In this section, we consider the dual pair  $G_2 \times PGSp_6$  in  $E_7$ . As before, we have the set  $\Theta_{gen}(\chi_{G_2})$ , and we shall prove (5.8). Before that, we prove a lemma concerning the Weil representation.

**Lemma 11.1.** *Let  $\tilde{W}$  denote the representation  $Ind_{R_0}^{GL_2 \times GL_2 \times GL_2} W$  as in (8.6), with:*

$$R_0 = \{(g_1, g_2, g_3) : \det(g_1 g_2 g_3) = 1\}.$$

*Then for any generic cuspidal representation  $\pi$  of  $GL_2$ ,*

$$\langle \tilde{W}, \pi \otimes \pi \otimes \pi \rangle = 1.$$

*Proof.* First, we claim that to prove the lemma, it suffices to show that as a representation of  $GL_2 \times GL_2$  (the last two copies),

$$\langle \tilde{W}, \pi \otimes \sigma \rangle_{GL_2 \times GL_2} = \begin{cases} \dim(\pi), & \text{if } \sigma = \pi; \\ 0, & \text{if } \sigma \neq \pi. \end{cases}$$

where  $\pi$  is a generic cuspidal representation of  $GL_2$ . This is because we could use the maximal isotropic subspace  $k^2 \otimes ke_1 \otimes k^2$ , or  $k^2 \otimes k^2 \otimes ke_1$  to define  $W$ .

Now, one sees that, as a representation of  $GL_2 \times GL_2$  (the last two copies),

$$\begin{aligned} \tilde{W} &\cong Ind_{R_1}^{GL_2 \times GL_2} \mathbb{C}[M_2(k)] \\ &\cong (Ind_{R_1}^{GL_2 \times GL_2} 1) \otimes \mathbb{C}[M_2(k)] \\ &\cong (\oplus_{\psi \in \hat{k}^\times} \mathbb{C}\psi) \otimes \mathbb{C}[M_2(k)]. \end{aligned}$$

From this, the lemma follows easily.  $\square$

Now we can determine the correspondence for non-cuspidal representations:

**Proposition 11.2.** *If  $\chi(s)_{G_2}$  is a non-cuspidal generic representation of  $G_2$ , then*

$$\Theta_{gen}(\chi(s)_{G_2}) = \{\chi(i(s))_{PGSp_6}\}.$$

*Proof.* The argument is essentially the same as in the proof of Proposition 10.1. For generic representations of  $G_2$  which are induced from (not necessarily cuspidal) representations of  $P_2$ , we use Proposition 8.5. For generic representations induced from cuspidal representations of  $P_1$ , we use Proposition 8.7, and the previous lemma.  $\square$

Now we have:

**Theorem 11.3.** *If  $p \geq 5$ , then*

$$\Theta_{gen}(\chi(s)_{G_2}) = \{\chi(i(s))_{PGSp_6}\}.$$

*Proof.* Parts of the proof involve the same sort of considerations as in the previous sections and so we shall be brief on those parts. Also, after the previous proposition, it remains to prove this for cuspidal generic representations only.

Now, if the parameter  $s$  is contained in  $SL_3 \subset G_2$ , then since the induced representation  $Ind_{Q_0}^{PGSp_6} \chi(s)_{PGL_3}$  is always irreducible for  $s$  in the elliptic torus of  $SL_3(q)$ , we have the required result, using Proposition 8.3. Hence we are left with those cuspidal generic representations which do not come from  $PGL_3$ . Again, these have parameters  $s$  which are regular in  $G_2^* = G_2$ . Moreover,  $s$  lies in a unique maximal torus  $T_w^*$ , where  $w$  is either the non-trivial element in the center of the Weyl group, or the class of elements of order 6. Let us consider the latter case first. Then the regular parameter  $s$  looks like:

$$\begin{pmatrix} a & & \\ & a^{-q} & \\ & & a^{q^2} \end{pmatrix}$$

with  $a \in k_6 = \mathbb{F}_{q^6}$ , and  $a^{q^2-q+1} = 1$ . It is straightforward to check that in  $Spin_7$ , the parameter  $i(s)$  is still regular. Hence we have:

$$\begin{aligned} \chi(s) &= R_T(\theta) \\ \chi(i(s)) &= R_{T'}(\theta') \end{aligned}$$

with  $\theta, \theta'$  regular characters of  $T(q)$  and  $T'(q)$  respectively.

By going to  $k_2$ , the torus  $T_w^*$  becomes conjugate to the elliptic torus in  $SL_3(q^2)$ . Thus both  $\chi(s)$  and  $\chi(i(s))$  lift to cuspidal representations with parameters in the elliptic torus in  $SL_3(q^2)$ , and since we already know the result for representations associated to such tori, a base change argument as in the proof of Theorem 10.2 gives the result in this case.

Finally, if  $w$  is the non-trivial element in the center of the Weyl group of  $G_2$ , then the regular parameter  $s$  looks like:

$$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$$

with  $a, b, c \in k_2^\times$ ,  $a^{q+1} = b^{q+1} = c^{q+1} = 1$ , and  $abc = 1$ . Now  $i(s)$  is still regular in  $Spin_7$ , and by base change to  $k_2$ , we obtain the required result.  $\square$

## 12. The Outer Form of $E_6$

We have seen that about half the generic characters of  $G_2$  can be obtained as lifts from  $PGL_3$ . These are the generic representations whose parameter lies in  $SL_3$ , i.e. is of the form  $i(s)$  for some  $s \in SL_3$ . For  $i(s) \in G_2$  regular, these can be characterized by their dimensions, which has the form  $(q^3 + 1)\dim(\chi(s)_{PGL_3}) = (q^3 + 1)P_{w(s)}(q)$ , where  $P_{w(s)}$  is a polynomial with integer coefficients, and depends only on the class of  $w(s)$  in the Weyl group. Here,  $w(s)$  is such that  $G_s^* = T_{w(s)}$ .

In this section, we see that the other generic representations are obtained as lifts from  $PU_3$ , the outer form of  $PGL_3$ , by using the dual pair  $G_2 \times PU_3 \subset {}^2E_6$ , and the reflection representation  $\Pi$  of  ${}^2E_6$ . Before proceeding, we have to say what we mean by the reflection representation of  $G^F := {}^2E_6$ . As before,  $\Pi$  is a unipotent principal series representation. The Weyl group  $W^F$  of  $G^F$  is of type  $F_4$ , and so the irreducible components of  $Ind_{B^F}^{G^F} 1$  are parametrized by the irreducible characters of the Weyl group of  $F_4$  (here,  $B^F$  is the Borel subgroup of  $G^F$ ). In this case, however,  $\Pi$  does not correspond to the reflection representation of the Weyl group. Instead it corresponds to a two-dimensional representation which is denoted  $\phi'_{2,4}$  in the notation of [C]. Hence, in particular, the space of  $B^F$ -fixed vectors in  $\Pi$  has dimension 2. Note, however, that the dimension of  $\Pi$  is  $q^{11} - q^8 + q^7 + q^5 - q^4 + q$ , which is the smallest among the irreducible representations of  $G^F$  of dimension greater than 1. Hence it might be more appropriate to call  $\Pi$  the minimal representation in this case. Now we can state the result:

**Theorem 12.1.** *Assume that  $p \geq 5$ , as before. Let  $i : SU_3 \rightarrow G_2$  be the natural embedding of dual groups (by regarding  $SU_3$  as the Galois group of  $\mathbb{F}_{q^2}$  in  $\mathbb{O}$ , the octonion algebra). Then,*

$$\Theta_{gen}(\chi(s)) = \{\chi(i(s))\}.$$

The proof of this is similar to the other dual pairs, and will be omitted. Notice that if  $i(s) \in G_2$  is regular, then the dimension of  $\chi(i(s))$  is given by  $(q^3 - 1)|P_{w(s)}(-q)|$ . This concludes our study of the Howe correspondence in the finite field situation.

## REFERENCES

- [AM] J. Adams and A. Moy, *Unipotent Representations and Reductive Dual Pairs*, Transactions of AMS 340(1993), Pg. 309-321.
- [A-M-R] M.F. Aubert, J. Michel and R. Rouquier, *Correspondance de Howe pour les Groupes Reductifs sur les Corps Finis*, Duke Math. Journal 83(1996), Pg. 353-397.
- [C] R.Carter, *Finite Groups of Lie Type*, Wiley (1985).
- [D] F. Digne, *Shintani Descent and L-functions on Deligne-Lusztig Varieties*, Proceedings of Symposia in Pure Math. 47 part 1, Pg. 61-68.
- [D2] F. Digne, *Descente de Shintani et Restriction des Scalaires*, Preprint(1996).
- [D-L] P. Deligne and G. Lusztig, *Representations of Reductive Groups over Finite Fields*, Ann. of Math. 103(1976), Pg. 103-161.
- [Ge] P. Geradin, *Weil Representations associated to Finite Fields*, Journal of Algebra 46(1977), Pg. 54-101
- [Gr] B.H. Gross, *On the Motive of a Reductive Group*, to appear in Invent. Math(1997).
- [GrS] B. H. Gross and G. Savin, *Motives with Galois Group of Type  $G_2$ : an Exceptional Theta Correspondence*, to appear in Compositio Math.

- [Lu] G. Lusztig, *On the Reflection Representation of finite Chevalley Groups*, London Mathematics Society Lecture Notes Series 34(1978) Pg. 325-337.
- [Lu2] G. Lusztig, *The Characters of Reductive Groups over a Finite Field*, Ann. of Math. Studies 107(1984).
- [MS] K. Magaard and G. Savin, *Exceptional Theta Correspondences*, Composition Math. 107 (1997), Pg. 89-123.
- [SG] G. Savin and W. T. Gan, *The Dual Pair  $G_2 \times PU_3(D)$  ( $p$ -adic case)*, preprint(1998).
- [Sr] B. Srinivasan, *Weil Representations of Finite Classical Groups*, Invent. Math 51(1979), Pg. 143-153.
- [St] R. Steinberg, *Endomorphisms of Linear Algebraic Groups*, Memoirs of AMS 80(1968).

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