FORMAL DEGREES AND LOCAL THETA CORRESPONDENCE

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Contents

1. **Introduction**

Let F be a non-archimedean local field of characteristic 0 and residual characteristic p. Let G be a reductive linear algebraic group over F; we shall identify G with its group of rational points $G(F)$ if there is no cause for confusion. In this introduction, we assume for simplicity that the identity component of the center of G is anisotropic. Then an irreducible unitary representation π of G, with invariant hermitian inner product $(\cdot, \cdot)$, is called a discrete series representation if its matrix coefficients

$$f_{v,v'}(g) = (\pi(g)v, v') \quad \text{for } v, v' \in \pi \text{ and } g \in G$$

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are L^2-functions on G. In this case, the formation of matrix coefficients give a $G \times G$-equivariant embedding

$$i_\pi : \pi \boxtimes \bar{\pi} \hookrightarrow L^2(G),$$

where $\bar{\pi}$ denotes the complex conjugate representation of π. This embedding is not necessarily isometric. Indeed, if we fix a Haar measure dg on G, then there is a positive real number $\deg \pi$ such that

$$\int_G f_{v_1,v_1'}(g) \cdot \overline{f_{v_2,v_2'}(g)}\, dg = \frac{1}{\deg \pi} \cdot (v_1, v_2) \cdot \overline{(v_1', v_2')}$$

for $v_1, v_1', v_2, v_2' \in \pi$. The constant $\deg \pi$ depends on the choice of the Haar measure dg and is called the formal degree of π (relative to the choice of dg). When G is compact, so that π is finite-dimensional, we have

$$\deg \pi = \dim \pi$$

if the Haar measure dg is chosen so that G has volume 1.

In [34], a formula for $\deg \pi$ is conjectured in the framework of the local Langlands correspondence. More precisely, under the local Langlands correspondence, the Langlands parameter (ϕ, η) of π consists of an L-parameter

$$\phi : \mathit{WD}_F \longrightarrow {}^L G$$

and an irreducible character η of $\widetilde{\mathcal{S}}_\phi$. Here, WD_F is the Weil-Deligne group of F, ${}^L G$ is the L-group of G and $\widetilde{\mathcal{S}}_\phi$ is a finite group defined as follows. Let $\widehat{G}$ denote the dual group of G and $\widehat{G}_{\mathrm{sc}}$ the simply-connected cover of the derived group of $\widehat{G}$. Let $\widehat{G}_{\mathrm{ad}} = \widehat{G}/Z_{\widehat{G}}$, where $Z_{\widehat{G}}$ is the center of $\widehat{G}$. We consider the centralizer $\mathcal{S}_\phi$ in $\widehat{G}$ of the image of ϕ. It is expected that $\mathcal{S}_\phi$ is a finite group when π is a discrete series representation. Then $\widetilde{\mathcal{S}}_\phi$ is the preimage in $\widehat{G}_{\mathrm{sc}}$ of the subgroup $\mathcal{S}_\phi / Z_{\widehat{G}}^\Gamma$ of $\widehat{G}_{\mathrm{ad}}$, where $\Gamma = \mathrm{Gal}(\bar{F}/F)$. Now the formal degree conjecture [34] states that:

$$\deg \pi = \frac{\dim \eta}{\# \mathcal{S}_\phi} \cdot |\gamma(0, \mathrm{Ad} \circ \phi, \psi)|$$

for a specific Haar measure dg_ψ on G (which depends on a nontrivial additive character ψ of F). Here, the γ-factor on the right-hand side is defined by

$$\gamma(s, \mathrm{Ad} \circ \phi, \psi) = \epsilon(s, \mathrm{Ad} \circ \phi, \psi) \cdot \frac{L(1-s, \mathrm{Ad} \circ \phi)}{L(s, \mathrm{Ad} \circ \phi)},$$

where Ad is the adjoint representation of ${}^L G$ on its Lie algebra $\mathrm{Lie}({}^L G)$. This conjecture was verified in several cases, such as when $G = \mathrm{GL}_n$, SL_n and their inner forms, as well as when π is a stable discrete series representation of U_3. In general, however, one can only expect to prove the conjecture when the local Langlands correspondence is known for G.

The main purpose of this paper is to investigate the behaviour of the formal degree under some natural instances of Langlands functoriality. As an example, consider the split groups $G = \mathrm{O}_{2n}$ and $G' = \mathrm{Sp}_{2n}$, so that there is a natural inclusion of dual groups

$$\xi : \widehat{G} = \mathrm{O}_{2n}(\mathbb{C}) \longrightarrow \mathrm{SO}_{2n+1}(\mathbb{C}) = \widehat{G}'.$$

Suppose that Π_ϕ is a discrete L-packet of G with L-parameter ϕ such that its functorial lift $\xi_*(\Pi_\phi)$ is a discrete L-packet of G'. Then the L-parameter ϕ' of $\xi_*(\Pi_\phi)$ is given by

$$\phi' = \xi \circ \phi = \phi \oplus \mathbf{1}.$$

Moreover, the component groups $\mathcal{S}_{\phi'}$ and $\mathcal{S}_\phi$ are abelian with

$$\# \mathcal{S}_{\phi'} = \# \mathcal{S}_\phi.$$

Thus, for $\pi \in \Pi_\phi$ and $\pi' \in \Pi_{\phi'}$, the formal degree conjecture [34] predicts that

$$\frac{\deg \pi'}{\deg \pi} = \frac{|\gamma(0, \mathrm{Ad} \circ \phi', \psi)|}{|\gamma(0, \mathrm{Ad} \circ \phi, \psi)|}.$$

Since

$$\mathrm{Ad} \circ \phi' = \mathrm{Ad} \circ \phi \oplus \mathrm{std} \circ \phi,$$

where std denotes the standard representation of $\mathrm{O}_{2n}(\mathbb{C})$, one arrives at the conjectural relation

$$\frac{\deg \pi'}{\deg \pi} = |\gamma(0, \mathrm{std} \circ \phi, \psi)|.$$

A special case of the main result of this paper is the proof of this last identity (suitably reinterpreted) unconditionally, though the formal degree conjecture is not known for classical groups. The identity can be formulated in an unconditional manner because:

- one has a construction of the functorial lift, via the theta correspondence; this is suggested by the results of Mœglin [58];
- one can define the Galois-theoretic γ-factor $\gamma(s, \mathrm{std} \circ \phi, \psi)$ directly in terms of $\pi \in \Pi_\phi$, via the doubling method of Piatetski-Shapiro and Rallis [62].

More precisely, we shall show (see Theorem 15.1(ii)):

Theorem 1.1. *Let π be an irreducible discrete series representation of O_{2n} whose theta lift π' to Sp_{2n} is also an irreducible discrete series representation of Sp_{2n}. Then we have*

$$\frac{\deg \pi'}{\deg \pi} = |\gamma(0, \pi, \psi)|,$$

where $\gamma(s, \pi, \psi)$ is the standard γ-factor of π defined by the doubling method of Piatetski-Shapiro and Rallis.

We briefly mention another instance of our result. Namely, consider the case $G = \mathrm{SO}_{2n+1}$ and $G' = \mathrm{Mp}_{2n}$, where Mp_{2n} is the unique two-fold cover of Sp_{2n}. In this case, if π is an irreducible discrete series representation of G, then it is shown in [13] that the theta lift π' of π to G' is nonzero and is an irreducible discrete series representation of G'. Indeed, the results of [13] place the representation theory of Mp_{2n} in the framework of the Langlands program, and indicate that the dual group of the nonlinear group G' is equal to $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$, such that the theta correspondence realizes the functorial lift with respect to the identity map of dual groups. Further, the results of W. W. Li [53], [54] developing a theory of endoscopy for Mp_{2n} say that G is an endoscopic group of G'. Now we have (see Theorem 15.1(i)):

Theorem 1.2. *Let π be an irreducible discrete series representation of SO_{2n+1} and let π' be its theta lift to Mp_{2n}. Then we have*

$$\deg \pi' = \deg \pi.$$

The full version of our main result, which deals with the functoriality of the formal degree under the theta correspondence for general symplectic-orthogonal and unitary dual pairs, can be found in Theorem 15.1. Our result implies that the formal degree conjecture holds for the representation π if and only if it holds for its theta lift π' (provided that the theta correspondence has the expected effect on L-parameters). Using this, we prove the formal degree conjecture for the groups U_3, Sp_4 and GSp_4, where the local Langlands correspondence is known by [67], [15], [16] (see Theorem 20.11).

Theorem 1.3. *The formal degree conjecture holds for the groups U_3, Sp_4 and GSp_4.*

As we noted before, [34] has established the formal degree conjecture for stable discrete series representations of U_3; the analog of Theorem 1.1 for unitary groups allows us to deduce the conjecture for endoscopic discrete series representations of U_3.

We shall also show how other natural invariants such as standard γ-factors and Plancherel measures behave under the theta correspondence. These results are contained in Theorems 11.3 and 12.1. Indeed, the functoriality of these other invariants are easier to establish, since these invariants have global analogs (such as global standard L-functions and global intertwining operators) which satisfy certain global functional equations. This allows one to deduce the functoriality of these local invariants under the theta correspondence from the global functional equations and the knowledge of the desired functoriality at unramified places and archimedean places.

The formal degree, on the other hand, does not have a global analog, and thus one cannot hope to give a global-to-local argument. The proof of Theorem 1.1 is thus local. However, as we shall now explain, the identity in Theorem 1.1 has a global parallel: the Rallis inner product formula.

More precisely, suppose that π is an irreducible cuspidal automorphic representation of O_{2n} and $\pi' = \Theta(\pi)$ is its global theta lift to Sp_{2n}. Given cusp forms f_1 and f_2 in π and their theta lifts $f_1' = \theta(\phi_1, f_1)$ and $f_2' = \theta(\phi_2, f_2)$ in π' (relative to some elements ϕ_i in the Weil representation), the Rallis inner product formula compares the Petersson inner product $\langle f_1', f_2' \rangle$ with $\langle f_1, f_2 \rangle$. To a first degree of approximation, one has:

$$\frac{\langle f_1', f_2' \rangle}{\langle f_1, f_2 \rangle} \sim L(1, \pi) \sim L(0, \pi).$$

This is the global analog of our Theorem 1.1.

Indeed, the proof of Theorem 1.1 parallels the global proof of the Rallis inner product formula, so it is instructive to recall the latter, if only briefly. One considers the following see-saw diagram:

$$\begin{tikzcd}
\mathrm{O}_{4n} \arrow[d, dash] \arrow[drr, dash] & & \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n} \arrow[d, dash] \arrow[dll, dash] \\
\mathrm{O}_{2n} \times \mathrm{O}_{2n} & & \Delta\mathrm{Sp}_{2n}
\end{tikzcd}$$

Starting with f_1 and f_2 on O_{2n}, and the constant function $\mathbf{1}$ on $\Delta\mathrm{Sp}_{2n}$, the see-saw identity gives (ignoring issues of convergence)

$$\langle f_1', f_2' \rangle = \int_{\mathrm{O}_{2n} \times \mathrm{O}_{2n}} f_1(g_1) \cdot \overline{f_2(g_2)} \cdot \left(\int_{\Delta\mathrm{Sp}_{2n}} \theta(i(g_1, g_2)g') \, dg' \right) dg_1 \, dg_2,$$

where θ is the theta function associated to $\phi_1 \otimes \phi_2$ and $i : \mathrm{O}_{2n} \times \mathrm{O}_{2n} \to \mathrm{O}_{4n}$ is the natural map. Now one invokes the Siegel-Weil formula, which identifies the inner integral with the special value of an Eisenstein series $E(s, g)$ on O_{4n}:

$$\int_{\Delta\mathrm{Sp}_{2n}} \theta(gg') \, dg' = E(\tfrac{1}{2}, g) \quad \text{for } g \in \mathrm{O}_{4n}.$$

Replacing the inner integral by this Eisenstein series, one obtains the doubling zeta integral of Piatetski-Shapiro and Rallis, which represents the partial standard L-function $L^S(s, \pi)$ of π:

$$\int_{\mathrm{O}_{2n} \times \mathrm{O}_{2n}} f_1(g_1) \cdot \overline{f_2(g_2)} \cdot E(s, i(g_1, g_2)) \, dg_1 \, dg_2 \sim L^S(s + \tfrac{1}{2}, \pi) \cdot \langle f_1, f_2 \rangle.$$

In particular, one obtains:

$$\langle f_1', f_2' \rangle \sim L^S(1, \pi) \cdot \langle f_1, f_2 \rangle$$

as desired.

The proof of Theorem 1.1 is the local analog of the above argument. In the context of the above see-saw diagram, we note that:

- the role of the cusp forms f_i is played by the matrix coefficients of the discrete series representation π.
- we consider an explicit local theta lifting of discrete series representations given by an integral of matrix coefficients, i.e., integrating matrix coefficients of π against those of the Weil representation. Such a construction was first considered by J. S. Li [50] in the so-called stable range.
- we formulate a local analog of the Siegel-Weil formula; this is supplied in §17.
- we use the local theory of the doubling zeta integral, as developed by Lapid-Rallis [49].

Finally, we would like to examine the results in Theorems 1.1 and 1.2 from the viewpoint of the theory of endoscopy. Consider first the situation of Theorem 1.2, so that $G = \mathrm{SO}_{2n+1}$ and $G' = \mathrm{Mp}_{2n}$. By the results of [13], a tempered L-packet Π of G gives rise to a tempered L-packet Π' of G'. It is believed that, through the theory of endoscopy for Mp_{2n} developed by W. W. Li [53], [54],

$$\sum_{\pi\in\Pi} \operatorname{trace}\pi \quad \text{transfers to} \quad \sum_{\pi'\in\Pi'} \operatorname{trace}\pi'.$$

By equating the constant terms in the character expansions on both sides, one obtains

$$\sum_{\pi'\in\Pi'} \deg\pi' = \sum_{\pi\in\Pi} \deg\pi.$$

Moreover, endoscopic character identities should imply that the representations in the L-packet have the same formal degrees. Hence, one obtains

$$\deg\pi' = \deg\pi.$$

In other words, the result of Theorem 1.2 should be a consequence of the theory of endoscopy.

On the other hand, for the case $G = \mathrm{O}_{2n}$ and $G' = \mathrm{Sp}_{2n}$, it is not clear to us how Theorem 1.1 would follow from the theory of endoscopy. For a tempered L-packet Π of G, Langlands functoriality gives a tempered L-packet Π' of G' such that

$$\sum_{\pi\in\Pi} \operatorname{trace}\pi \quad \text{transfers to} \quad \sum_{\pi'\in\Pi'} s(\pi')\cdot\operatorname{trace}\pi'$$

with some function $s : \Pi' \to \mathbb{C}^\times$. Indeed, the packet Π' has a structure of a group and s is a nontrivial group homomorphism. By taking the constant terms in the character expansion on both sides, one only obtains

$$\sum_{\pi'\in\Pi'} s(\pi')\cdot\deg\pi' = 0.$$

Hence, it would appear that character identities do not imply Theorem 1.1.

2. **Classical groups**

In this section, we fix some notation and introduce the groups which intervene in this paper.

2.1. **Fields.** Let F be a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\bar{F}$ of F. Let F^{ur} be the maximal unramified extension of F in $\bar{F}$. Let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group of F, W_F the Weil group of F, and $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil-Deligne group of F. Let $\mathfrak{o}_F$ be the maximal compact subring of F, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, ϖ_F a uniformizer of $\mathfrak{o}_F$, and $q = q_F$ the cardinality of $\mathfrak{o}_F/\mathfrak{p}_F$. The absolute value $|\cdot|_F$ on F is normalized so that $|\varpi_F|_F = q^{-1}$. Let

$(\cdot,\cdot)_F$ denote the quadratic Hilbert symbol of F. Let $\mathbb{1} = \mathbb{1}_{F^\times}$ be the trivial character of $F^\times$. We fix a nontrivial additive character ψ of F.

Let E be either F or a quadratic extension of F. Let

$$c = \begin{cases} \text{the identity of } F & \text{if } E = F, \\ \text{the nontrivial element in } \mathrm{Gal}(E/F) & \text{if } [E:F] = 2. \end{cases}$$

Let χ_E be the (possibly trivial) quadratic character of $F^\times$ associated to E/F by class field theory. We define a nontrivial additive character ψ_E of E by $\psi_E = \psi \circ \mathrm{tr}_{E/F}$.

If $[E:F] = 2$, let

$$E^1 = \{x \in E \,|\, x \cdot x^c = 1\} \quad \text{and} \quad E_0 = \{x \in E \,|\, x + x^c = 0\}$$

denote the set of norm 1 elements in E and that of trace 0 elements in E respectively. Let $\mathfrak{d}_E$ be the different of E/F and $\mathfrak{f}_E$ the smallest non-negative integer such that $\chi_E(x) = 1$ for all $x \in \mathfrak{o}_F^\times \cap (1 + \mathfrak{p}_F^{\mathfrak{f}_E})$. Then we have $\mathfrak{d}_E = \mathfrak{p}_E^{\mathfrak{f}_E}$. If $[E:F] = 2$, we fix an element $\daleth \in E^\times$ such that

$$\daleth^c = -\daleth.$$

Let

$$\zeta(s) = (1 - q^{-s})^{-1} \quad \text{and} \quad \zeta_E(s) = (1 - q_E^{-s})^{-1}$$

denote the local zeta functions of F and E respectively.

2.2. Spaces.

In this paper, we shall be interested in certain classical groups. These arise in the following way.

Fix $\epsilon = \pm 1$ in $E^\times$. Let W be a finite dimensional vector space over E of dimension n. Let

$$\langle \cdot,\cdot \rangle : W \times W \longrightarrow E$$

be a non-degenerate ϵ-hermitian c-sesquilinear form on W:

$$\langle au + bv, w \rangle = a\langle u, w\rangle + b\langle v, w\rangle$$
$$\langle v, w \rangle = \epsilon \cdot \langle w, v\rangle^c$$

for $a, b \in E$ and $u, v, w \in W$. Then the different possibilities for $\langle \cdot,\cdot \rangle$ are given in the following table.

(E,ϵ)	$E = F$, $\epsilon = 1$	$E = F$, $\epsilon = -1$	$[E:F] = 2$, $\epsilon = 1$	$[E:F] = 2$, $\epsilon = -1$
$\langle \cdot,\cdot \rangle$	symmetric bilinear form	symplectic form	hermitian form	skew-hermitian form

We define

$$\mathrm{disc}\, W \in \begin{cases} F^\times/F^{\times 2} & \text{if } E = F, \\ F^\times/\mathrm{N}_{E/F}(E^\times) & \text{if } [E:F] = 2 \text{ and } \epsilon = 1, \\ \daleth^n \cdot F^\times/\mathrm{N}_{E/F}(E^\times) & \text{if } [E:F] = 2 \text{ and } \epsilon = -1, \end{cases}$$

by

$$\mathrm{disc}\, W = \begin{cases} (-1)^{(n-1)n/2} \cdot 2^{-n} \cdot \det(\langle w_i, w_j \rangle) & \text{if } E = F, \\ (-1)^{(n-1)n/2} \cdot \det(\langle w_i, w_j \rangle) & \text{if } [E:F] = 2, \end{cases}$$

where $\{w_i \mid i = 1, \dots, n\}$ is a basis of W over E. Also, we define $\epsilon(W) \in \{\pm 1\}$ by

$$\epsilon(W) = \begin{cases} \text{the Hasse invariant of } W & \text{if } E = F \text{ and } \epsilon = 1, \\ 1 & \text{if } E = F \text{ and } \epsilon = -1, \\ \chi_E(\operatorname{disc} W) & \text{if } [E:F] = 2 \text{ and } \epsilon = 1, \\ \chi_E(\daleth^{-n} \cdot \operatorname{disc} W) & \text{if } [E:F] = 2 \text{ and } \epsilon = -1. \end{cases}$$

Note that $\epsilon(W)$ depends on the choice of $\daleth$ if $[E:F] = 2$, $\epsilon = -1$ and n is odd. Then the isometry class of $\langle \cdot, \cdot \rangle$ is characterized by the invariants given by the following table.

(E, ϵ)	$E = F$, $\epsilon = 1$	$E = F$, $\epsilon = -1$	$[E:F] = 2$, $\epsilon = \pm 1$
invariants	$\dim W$, $\operatorname{disc} W$, $\epsilon(W)$	$\dim W$	$\dim W$, $\epsilon(W)$

2.3. **Classical groups.** Let $G(W)$ be the group of elements g in $\mathrm{GL}(W)$ which preserve the form $\langle \cdot, \cdot \rangle$:

$$\langle gv, gw \rangle = \langle v, w \rangle \quad \text{for } v, w \in W.$$

Then $G(W)$ is a (possibly disconnected) reductive linear algebraic group over F; this is the class of groups we will consider in this paper and the different possibilities for $G(W)$ are given in the following tables.

(E, ϵ)	$E = F$, $\epsilon = 1$	$E = F$, $\epsilon = -1$	$[E:F] = 2$, $\epsilon = \pm 1$
$G(W)$	orthogonal group $\mathrm{O}(W)$	symplectic group $\mathrm{Sp}(W)$	unitary group $\mathrm{U}(W)$

		$G(W)$
$E = F$, $\epsilon = 1$	$\dim W$ odd, $\epsilon(W) = 1$	split
	$\dim W$ odd, $\epsilon(W) = -1$	non-quasi-split
	$\dim W$ even, $\operatorname{disc} W = 1$, $\epsilon(W) = 1$	split
	$\dim W$ even, $\operatorname{disc} W = 1$, $\epsilon(W) = -1$	non-quasi-split
	$\dim W$ even, $\operatorname{disc} W \neq 1$	quasi-split but non-split
$E = F$, $\epsilon = -1$		split
$[E:F] = 2$, $\epsilon = \pm 1$	$\dim W$ odd	quasi-split but non-split
	$\dim W$ even, $\epsilon(W) = 1$	quasi-split but non-split
	$\dim W$ even, $\epsilon(W) = -1$	non-quasi-split

2.4. **Parabolic subgroups.** It will be useful to recall the structure of the parabolic subgroups for $G(W)$. If X is an isotropic subspace of W, then we can write

$$W = X \oplus W' \oplus Y$$

as an orthogonal sum with W' non-degenerate and Y isotropic. The stabilizer of X in $G(W)$ is a maximal parabolic subgroup

$$P(X) = M(X) \cdot U(X)$$

with Levi component

$$M(X) = \mathrm{GL}(X) \times G(W')$$

and unipotent radical $U(X)$. Moreover, every maximal parabolic subgroup of $G(W)$ arises in this way.

More generally, any parabolic subgroup $P = M \cdot U$ of $G(W)$ is the stabilizer of a flag of isotropic subspaces

$$X'_1 \subset \cdots \subset X'_r$$

in W. If we write

$$W = X'_r \oplus W' \oplus Y'_r$$

as an orthogonal sum with W' non-degenerate and Y'_r isotropic, and if we write

$$X'_i = X'_{i-1} \oplus X_i$$

for $1 \le i \le r$ (where $X'_0 = \{0\}$), then the associated parabolic subgroup P has Levi component

$$M = \mathrm{GL}(X_1) \times \cdots \times \mathrm{GL}(X_r) \times G(W').$$

2.5. **Metaplectic groups.** When W is a symplectic space over F, we will need to consider the metaplectic group $\mathrm{Mp}(W)$, which is the unique two-fold cover of $G(W) = \mathrm{Sp}(W)$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

Given an isotropic subspace X of W as above, with associated parabolic subgroup $P(X) = M(X) \cdot U(X)$ of $\mathrm{Sp}(W)$, the metaplectic cover splits uniquely over $U(X)$, which can thus be regarded as a subgroup of $\mathrm{Mp}(W)$. The preimage $\widetilde{P}(X)$ of $P(X)$ in $\mathrm{Mp}(W)$ is then of the form

$$\widetilde{P}(X) = \widetilde{M}(X) \cdot U(X),$$

where

$$\widetilde{M}(X) = \widetilde{\mathrm{GL}}(X) \times_{\boldsymbol{\mu}_2} \mathrm{Mp}(W')$$

and $\widetilde{\mathrm{GL}}(X)$ is a rather innocuous two-fold cover of $\mathrm{GL}(X)$ defined as follows. Consider the set

$$\mathrm{GL}(X) \times \{\pm 1\}$$

with multiplication law

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot (\det g_1, \det g_2)_F)$$

for $g_1, g_2 \in \mathrm{GL}(X)$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. Then $\widetilde{\mathrm{GL}}(X)$ is precisely this two-fold cover of $\mathrm{GL}(X)$.

More generally, if $P = M \cdot U$ is a parabolic subgroup of $\mathrm{Sp}(W)$ with Levi component

$$M = \mathrm{GL}(X_1) \times \cdots \times \mathrm{GL}(X_r) \times \mathrm{Sp}(W'),$$

then the preimage $\widetilde{P}$ of P in $\mathrm{Mp}(W)$ is of the form

$$\widetilde{P} = \widetilde{M} \cdot U,$$

where

$$\widetilde{M} = \widetilde{\mathrm{GL}}(X_1) \times_{\boldsymbol{\mu}_2} \cdots \times_{\boldsymbol{\mu}_2} \widetilde{\mathrm{GL}}(X_r) \times_{\boldsymbol{\mu}_2} \mathrm{Mp}(W').$$

2.6. **Parabolically induced representations.** Let $P = M \cdot U$ be a parabolic subgroup of $G(W)$ with Levi component

$$M = \mathrm{GL}(X_1) \times \cdots \times \mathrm{GL}(X_r) \times G(W').$$

Then an irreducible admissible representation of M is of the form

$$\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi',$$

where τ_i and π' are irreducible admissible representations of $\mathrm{GL}(X_i)$ and $G(W')$ respectively, and one can form a normalized parabolically induced representation

$$I_P^G(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi') := \mathrm{Ind}_P^{G(W)}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi')$$

of $G = G(W)$.

For the metaplectic group $\mathrm{Mp}(W)$, one needs to take irreducible admissible genuine representations of $\widetilde{\mathrm{GL}}(X_i)$ and $\mathrm{Mp}(W')$ in order to form an induced representation. However, it turns out that the genuine representations of $\widetilde{\mathrm{GL}}(X_i)$ can be given in terms of the representations of $\mathrm{GL}(X_i)$.

More precisely, using the additive character ψ, one may define a 1-dimensional genuine character χ_ψ of $\widetilde{\mathrm{GL}}(X)$, where X is an isotropic subspace of W. Indeed, the determinant map

$$\det : \mathrm{GL}(X) \longrightarrow \mathrm{GL}_1$$

has a natural lifting

$$\widetilde{\det} : \widetilde{\mathrm{GL}}(X) \longrightarrow \widetilde{\mathrm{GL}}_1$$

given by

$$\widetilde{\det}(g, \epsilon) = (\det g, \epsilon) \quad \text{for } g \in \mathrm{GL}(X) \text{ and } \epsilon \in \{\pm 1\}.$$

On the other hand, if we fix a nontrivial additive character ψ of F, then there is a natural genuine character of $\widetilde{\mathrm{GL}}_1$ defined by:

$$(a, \epsilon) \longmapsto \epsilon \cdot \gamma_F(a, \psi)^{-1}$$

for $a \in \mathrm{GL}_1$ and $\epsilon \in \{\pm 1\}$. Here,

$$\gamma_F(a, \psi) = \frac{\gamma_F(\psi_a)}{\gamma_F(\psi)},$$

where $\gamma_F(\psi) \in \boldsymbol{\mu}_8$ is the Weil index associated to ψ (see [77], [66, Appendix]) and $\psi_a(x) = \psi(ax)$ for $x \in F$. Composing $\widetilde{\det}$ with this genuine character gives a genuine character χ_ψ of $\widetilde{\mathrm{GL}}(X)$, which satisfies

$$\chi_\psi(g, \epsilon)^2 = (-1, \det g)_F$$

for $g \in \mathrm{GL}(X)$ and $\epsilon \in \{\pm 1\}$. Then the map

$$\tau \longmapsto \tau_\psi := \tau \otimes \chi_\psi$$

defines a bijection between the set of equivalence classes of admissible representations of $\mathrm{GL}(X)$ and the set of equivalence classes of admissible genuine representations of $\widetilde{\mathrm{GL}}(X)$. We stress that this bijection depends on the choice of ψ.

Thus, we see that an irreducible admissible genuine representation of $\widetilde{M}$ is of the form

$$\tau_{1,\psi} \otimes \cdots \otimes \tau_{r,\psi} \otimes \pi',$$

where τ_i is an irreducible admissible representation of $\mathrm{GL}(X_i)$ and π' is an irreducible admissible genuine representation of $\mathrm{Mp}(W')$, and one can form a normalized parabolically induced representation

$$I_P^G(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi') := \mathrm{Ind}_{\widetilde{P}}^{\mathrm{Mp}(W)}(\tau_{1,\psi} \otimes \cdots \otimes \tau_{r,\psi} \otimes \pi')$$

of $G = \mathrm{Mp}(W)$. Since we are fixing ψ, we shall henceforth suppress ψ from the notation.

2.7. **Centers.** The center $Z_{G(W)}$ of the group $G(W)$ is given as follows:

$$Z_{G(W)} = \begin{cases} \boldsymbol{\mu}_2 & \text{if } G(W) = \mathrm{O}(W) \text{ or } \mathrm{Sp}(W), \\ E^1 & \text{if } G(W) = \mathrm{U}(W). \end{cases}$$

For the metaplectic group $\mathrm{Mp}(W)$, the center $Z_{\mathrm{Mp}(W)}$ is a two-fold cover of $Z_{\mathrm{Sp}(W)} = \boldsymbol{\mu}_2$, and thus is a group of size 4. Using the additive character ψ, one can define a genuine character χ_ψ^W of $Z_{\mathrm{Mp}(W)}$ as follows. Let X be a maximal isotropic subspace of W, so that $P(X) = \mathrm{GL}(X) \cdot U(X)$ is a Siegel parabolic subgroup of $\mathrm{Sp}(W)$. Then we have defined the genuine character χ_ψ of $\widetilde{\mathrm{GL}}(X)$. Since $Z_{\mathrm{Mp}(W)} \subset \widetilde{\mathrm{GL}}(X)$, we may set

$$\chi_\psi^W = \text{the restriction of } \chi_\psi \text{ to } Z_{\mathrm{Mp}(W)}.$$

3. Reductive dual pairs

The classical groups introduced in the previous section arise naturally Howe's theory of reductive dual pairs in the symplectic group. In this section, we recall some basic facts about these reductive dual pairs and the splitting of the metaplectic cover over them.

3.1. **Reductive dual pairs.** Let V and W be vector spaces over E equipped with non-degenerate sesquilinear forms

$$(\cdot,\cdot) : V \times V \longrightarrow E \quad \text{and} \quad \langle\cdot,\cdot\rangle : W \times W \longrightarrow E$$

of opposite signs. Put

$$m = \dim V \quad \text{and} \quad n = \dim W.$$

We distinguish the following cases:

(A)	$[E:F] = 2$,	$(\cdot,\cdot)$ is hermitian,	$\langle\cdot,\cdot\rangle$ is skew-hermitian.
(B)	$E = F$,	$(\cdot,\cdot)$ is symplectic,	$\langle\cdot,\cdot\rangle$ is symmetric with n odd.
(C')	$E = F$,	$(\cdot,\cdot)$ is symmetric with m odd,	$\langle\cdot,\cdot\rangle$ is symplectic.
(C")	$E = F$,	$(\cdot,\cdot)$ is symmetric with m even,	$\langle\cdot,\cdot\rangle$ is symplectic.
(D)	$E = F$,	$(\cdot,\cdot)$ is symplectic,	$\langle\cdot,\cdot\rangle$ is symmetric with n even.

We shall refer to Cases C' and C" collectively as Case C. Put

$$l = l_{V,W} := \begin{cases} n - m & \text{in Case A}, \\ n - m - 1 & \text{in Cases BD}, \\ n - m + 1 & \text{in Case C}. \end{cases}$$

Let $G(W)$ and $H(V)$ denote the isometry groups of W and V respectively. Hence, the label for the respective case above refers to the type of the classical group $G(W)$. Let $\mathbb{W} = V \otimes_E W$, regarded as a vector space over F and equip it with a symplectic form

$$\mathrm{tr}_{E/F}((\cdot,\cdot) \otimes \langle\cdot,\cdot\rangle).$$

Then $(G(W), H(V))$ is a reductive dual pair in the symplectic group $\mathrm{Sp}(\mathbb{W})$ and there is a natural map

$$\iota : G(W) \times H(V) \longrightarrow \mathrm{Sp}(\mathbb{W}).$$

Note that this map is not necessarily injective.

3.2. **A pair of characters.** Given the spaces V and W, we fix a pair of characters $\boldsymbol{\chi} = (\chi_V, \chi_W)$ of $E^\times$ defined as follows:

$$\chi_V = \begin{cases} \text{a character of } E^\times \text{ such that } \chi_V|_{F^\times} = \chi_E^m & \text{in Case A,} \\ \text{the trivial character of } F^\times & \text{in Cases BD,} \\ \text{the quadratic character of } F^\times \text{ associated to } F(\sqrt{\operatorname{disc} V})/F & \text{in Case C,} \end{cases}$$

$$\chi_W = \begin{cases} \text{a character of } E^\times \text{ such that } \chi_W|_{F^\times} = \chi_E^n & \text{in Case A,} \\ \text{the quadratic character of } F^\times \text{ associated to } F(\sqrt{\operatorname{disc} W})/F & \text{in Cases BD,} \\ \text{the trivial character of } F^\times & \text{in Case C.} \end{cases}$$

Observe that, except in Case A, the characters χ_V and χ_W are uniquely determined by V and W respectively.

3.3. **Splitting of metaplectic covers.** Consider the metaplectic $\mathbb{C}^1$-cover of $\mathrm{Sp}(\mathbb{W})$:

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathcal{M}p(\mathbb{W}) \longrightarrow \mathrm{Sp}(\mathbb{W}) \longrightarrow 1.$$

In almost all cases, the natural map

$$\iota : G(W) \times H(V) \longrightarrow \mathrm{Sp}(\mathbb{W})$$

can be lifted to a homomorphism to the metaplectic group $\mathcal{M}p(\mathbb{W})$:

$$\begin{tikzcd} & \mathcal{M}p(\mathbb{W}) \arrow[d] \\ G(W) \times H(V) \arrow[ur] \arrow[r] & \mathrm{Sp}(\mathbb{W}) \end{tikzcd}$$

The only exceptions are in Cases B and C', where the dimension of the quadratic space in the pair (V, W) is odd. The lifting, even if it exists, is not necessarily unique. However, one can fix such a splitting using some auxiliary data, following [44] and [27, §1]. We summarize the situation as follows.

(A) In this case, one has a splitting $\iota_{V,W,\boldsymbol{\chi},\psi}$, depending on the data $\boldsymbol{\chi}$ and ψ. In fact, one needs an auxiliary $\daleth$ to define the splitting over the unitary group $H(V)$ of the hermitian space V, but it does not depend on the choice of $\daleth$.

(B) In this case, where $\dim W$ is odd, one has a splitting $\iota_{V,W,\boldsymbol{\chi},\psi}$ over the orthogonal group $G(W)$, but the metaplectic cover does not split over the symplectic group $H(V)$. Henceforth, to simplify notation, we set:

$$H(V) = \mathrm{Mp}(V),$$

which is a subgroup of the $\mathbb{C}^1$-cover $\mathcal{M}p(V)$.

(C') In this case, where $\dim V$ is odd, one has a splitting $\iota_{V,W,\boldsymbol{\chi},\psi}$ over the orthogonal group $H(V)$, but the metaplectic cover does not split over the symplectic group $G(W)$. Henceforth, to simplify notation, we set:

$$G(W) = \mathrm{Mp}(W),$$

which is a subgroup of the $\mathbb{C}^1$-cover $\mathcal{M}p(W)$.

(C") In this case, one has a splitting $\iota_{V,W,\boldsymbol{\chi},\psi}$.

(D) In this case, one has a splitting $\iota_{V,W,\boldsymbol{\chi},\psi}$.

4. Weil representations

In this section, we introduce the Weil representation associated to the reductive dual pair $(G(W), H(V))$.

4.1. **Weil representations.** Let ω_ψ be the Weil representation of $\mathcal{M}p(\mathbb{W})$ with respect to ψ. Using the splitting

$$\iota_{V,W,\boldsymbol{\chi},\psi} : G(W) \times H(V) \longrightarrow \mathcal{M}p(\mathbb{W})$$

introduced in the previous section, we obtain a representation

$$\omega_{V,W,\boldsymbol{\chi},\psi} = \omega_\psi \circ \iota_{V,W,\boldsymbol{\chi},\psi} \quad \text{of } G(W) \times H(V).$$

We call $\omega_{V,W,\boldsymbol{\chi},\psi}$ the Weil representation of $G(W) \times H(V)$ associated to the auxiliary data above.

4.2. **Doubled spaces.** Let W_- denote the vector space W over E equipped with the non-degenerate sesquilinear form $-\langle \cdot, \cdot \rangle$. Let $\mathbf{W} = W \oplus W_-$ and set:

$$W^\Delta = \{(w, w) \in \mathbf{W} \,|\, w \in W\}, \quad W^\nabla = \{(w, -w) \in \mathbf{W} \,|\, w \in W\}.$$

Let $\mathbf{G}(\mathbf{W})$ denote the isometry group of $\mathbf{W}$ and

$$i : G(W) \times G(W) \longrightarrow \mathbf{G}(\mathbf{W})$$

the natural map. Observe that, here, we have identified $G(W_-)$ with $G(W)$ using the identity map (since both these groups are physically the same subset of $\mathrm{GL}(W)$).

Let $\mathbb{W}_- = V \otimes_E W_-$. Then $(\mathbf{G}(\mathbf{W}), H(V))$ is a reductive dual pair in the symplectic group $\mathrm{Sp}(\mathbb{W} \oplus \mathbb{W}_-)$. Let $\mathcal{M}p(\mathbb{W} \oplus \mathbb{W}_-)$ be the metaplectic $\mathbb{C}^1$-cover of $\mathrm{Sp}(\mathbb{W} \oplus \mathbb{W}_-)$. We lift the natural embedding

$$\mathrm{Sp}(\mathbb{W}) \times \mathrm{Sp}(\mathbb{W}) \hookrightarrow \mathrm{Sp}(\mathbb{W} \oplus \mathbb{W}_-)$$

to a homomorphism

$$\tilde{\imath} : \mathcal{M}p(\mathbb{W}) \times \mathcal{M}p(\mathbb{W}) \longrightarrow \mathcal{M}p(\mathbb{W} \oplus \mathbb{W}_-)$$

which induces

$$(\epsilon_1, \epsilon_2) \longmapsto \epsilon_1 \bar{\epsilon}_2$$

on $\mathbb{C}^1 \times \mathbb{C}^1$. Here, again, we have identified $\mathrm{Sp}(\mathbb{W}_-)$ with $\mathrm{Sp}(\mathbb{W})$ using the identity map, as opposed to using the isomorphism induced by an isometry $\mathbb{W} \cong \mathbb{W}_-$ (which is the outer automorphism given by an element in $\mathrm{GSp}(\mathbb{W})$ with similitude factor -1). Let $\boldsymbol{\omega}_\psi$ be the Weil representation of $\mathcal{M}p(\mathbb{W} \oplus \mathbb{W}_-)$ with respect to ψ. Then we have

$$\boldsymbol{\omega}_\psi \circ \tilde{\imath} = \omega_\psi \otimes \bar{\omega}_\psi$$

as representations of $\mathcal{M}p(\mathbb{W}) \times \mathcal{M}p(\mathbb{W})$.

Let $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ be a complete polarization. Then

$$\mathbb{W} \oplus \mathbb{W}_- = (\mathbb{X} \oplus \mathbb{X}_-) \oplus (\mathbb{Y} \oplus \mathbb{Y}_-)$$

is a complete polarization. We can realize $\boldsymbol{\omega}_\psi$ on the space $S(\mathbb{X} \oplus \mathbb{X}_-)$ of Schwartz-Bruhat functions on $\mathbb{X} \oplus \mathbb{X}_-$. On the other hand,

$$\mathbb{W} \oplus \mathbb{W}_- = (V \otimes W^\Delta) \oplus (V \otimes W^\nabla)$$

is also a complete polarization and hence we can realize $\boldsymbol{\omega}_\psi$ on $S(V \otimes W^\nabla)$. By [51, §2], there exists an isomorphism

$$\delta : S(\mathbb{X} \oplus \mathbb{X}_-) \xrightarrow{\cong} S(V \otimes W^\nabla)$$

as unitary representations of $\mathcal{M}p(\mathbb{W} \oplus \mathbb{W}_-)$ such that

$$\delta(\phi_1 \otimes \bar{\phi}_2)(0) = (\phi_1, \phi_2)$$

for $\phi_1, \phi_2 \in S(\mathbb{X})$. Here $(\cdot, \cdot)$ is the hermitian inner product on $S(\mathbb{X})$.

Let

$$\iota_{V,\mathbf{W},\boldsymbol{\chi},\psi} : \mathbf{G}(\mathbf{W}) \times H(V) \longrightarrow \mathcal{M}p(\mathbb{W} \oplus \mathbb{W}_-)$$

be the splitting given in [44]. We then obtain a representation

$$\omega_{V,\mathbf{W},\boldsymbol{\chi},\psi} = \boldsymbol{\omega}_\psi \circ \iota_{V,\mathbf{W},\boldsymbol{\chi},\psi} \quad \text{of } \mathbf{G}(\mathbf{W}) \times H(V).$$

By [27, Proposition 2.2], we have

$$\omega_{V,\mathbf{W},\boldsymbol{\chi},\psi} \circ i = \omega_{V,W,\boldsymbol{\chi},\psi} \otimes (\bar{\omega}_{V,W,\boldsymbol{\chi},\psi} \cdot \chi_V)$$

as representations of $G(W) \times G(W)$ and

$$(\omega_{V,\mathbf{W},\boldsymbol{\chi},\psi}(h)\varphi)(x) = \chi_W(\det h) \cdot \varphi(h^{-1}x)$$

for $h \in H(V)$ and $\varphi \in S(V \otimes W^\nabla)$. Note that, in the above identity, $\chi_W(\det h) = 1$ except in Case A.

5. **Local theta correspondence**

In this section, we introduce the local theta correspondence induced by the Weil representation $\omega_{V,W,\boldsymbol{\chi},\psi}$ of $G(W) \times H(V)$ and recall some basic general results.

5.1. **Local theta correspondence.** Let π be an irreducible admissible (genuine) representation of $G(W)$. Then the maximal π-isotypic quotient of $\omega_{V,W,\boldsymbol{\chi},\psi}$ is of the form

$$\pi \boxtimes \Theta_{V,W,\boldsymbol{\chi},\psi}(\pi),$$

where $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is a smooth representation of $H(V)$. Let $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ be the maximal semisimple quotient of $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$.

The following theorem summarizes some basic results of Howe, Kudla, Mœglin-Vignéras-Waldspurger and Waldspurger about the local theta correspondence (see [36], [43], [59], [75]). The statement (iii) of the theorem is often referred to as the Howe duality conjecture.

Theorem 5.1. (i) *The representation $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is either zero or of finite length.*

(ii) *If π is supercuspidal, then $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is either zero or irreducible (and thus is equal to $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$). Moreover, for any irreducible supercuspidal representations π and π' of $G(W)$ which occur as quotients of $\omega_{V,W,\boldsymbol{\chi},\psi}$,*

$$\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi) \cong \Theta_{V,W,\boldsymbol{\chi},\psi}(\pi') \Longrightarrow \pi \cong \pi'.$$

(iii) *If $p \neq 2$, then either $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is zero or $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ has a unique irreducible quotient, so that $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is irreducible. Moreover, for any irreducible admissible representations π and π' of $G(W)$ which occur as quotients of $\omega_{V,W,\boldsymbol{\chi},\psi}$,*

$$\theta_{V,W,\boldsymbol{\chi},\psi}(\pi) \cong \theta_{V,W,\boldsymbol{\chi},\psi}(\pi') \Longrightarrow \pi \cong \pi'.$$

A result of Li-Sun-Tian [52] says that $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is multiplicity-free if it is nonzero.

5.2. **Central characters.** Suppose that $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is nonzero and σ is an irreducible constituent of $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$. Let ω_π and ω_σ denote the central characters of π and σ respectively. Observe that in Cases A, C" and D, the centers of $G(W)$ and $H(V)$ are naturally isomorphic, so that one can ask how ω_π and ω_σ are related. We have:

$$\omega_\sigma = \begin{cases} \omega_\pi \cdot \nu_{\boldsymbol{\chi}} & \text{in Case A,} \\ \omega_\pi \cdot \chi_V^{n/2} & \text{in Case C",} \\ \omega_\pi \cdot \chi_W^{m/2} & \text{in Case D.} \end{cases}$$

Here, in Case A, $\nu_{\boldsymbol{\chi}}$ is the character of E^1 defined by

$$\nu_{\boldsymbol{\chi}}(x/x^c) = (\chi_W^m/\chi_V^n)(x) \quad \text{for } x \in E^\times.$$

On the other hand, in Cases B and C', the centers of $G(W)$ and $H(V)$ are not isomorphic. For example, in Case C', the center of the orthogonal group $H(V) = \mathrm{O}(V)$ is $\boldsymbol{\mu}_2$ whereas that of the metaplectic group $G(W) = \mathrm{Mp}(W)$ is a group of size 4. However, as we explain in §2.7, the additive character ψ gives rise to a genuine character χ_ψ^W of the center $Z_{G(W)}$. Since π is genuine, ω_π/χ_ψ^W is a character of $\boldsymbol{\mu}_2$. With this provision, it makes sense to compare ω_π and ω_σ. We have:

$$\omega_\sigma/\chi_\psi^V = \omega_\pi \cdot \chi_W^{m/2} \quad \text{in Case B}$$

and

$$\omega_\sigma \cdot \chi_V^{n/2} = \omega_\pi/\chi_\psi^W \quad \text{in Case C'}.$$

5.3. **Supercuspidal supports.** Let π be an irreducible admissible representation of $G(W)$. Then there exist a parabolic subgroup $P = M \cdot U$ of $G = G(W)$ with Levi component

$$M = \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_r}(E) \times G(W'),$$

and irreducible supercuspidal representations τ_i and π' of $\mathrm{GL}_{n_i}(E)$ and $G(W')$ respectively, such that π is a subquotient of

$$I_P^G(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi').$$

Moreover, the pair

$$(M, \tau_1 \otimes \cdots \otimes \tau_r \otimes \pi')$$

is uniquely determined by π up to conjugacy and is called the supercuspidal support of π. By abuse of notation, we write

$$\{\tau_1, \ldots, \tau_r, \pi'\}$$

for it.

Similarly, for an irreducible admissible representation σ of $H(V)$, let

$$\{\upsilon_1, \ldots, \upsilon_s, \sigma'\}$$

be the supercuspidal support of σ, where υ_i and σ' are irreducible supercuspidal representations of $\mathrm{GL}_{m_i}(E)$ and $H(V')$ respectively.

Put $m' = \dim V'$ and $n' = \dim W'$. Put

$$l' = l_{V',W'} = \begin{cases} n' - m' & \text{in Case A,} \\ n' - m' - 1 & \text{in Cases BD,} \\ n' - m' + 1 & \text{in Case C.} \end{cases}$$

Then we have the following proposition which is called Kudla's supercuspidal support theorem (see [43], [59] and [24]).

Proposition 5.2. *Suppose that* $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ *is nonzero and* σ *is an irreducible constituent of* $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$.

(i) *We have*

$$\sigma' = \theta_{V',W',\boldsymbol{\chi},\psi}(\pi').$$

(ii) *If* $m - m' \leq n - n'$, *then we have*

$$\{\tau_1, \ldots, \tau_r\} = \{\chi_V |\cdot|_E^{(l-1)/2}, \chi_V |\cdot|_E^{(l-3)/2}, \ldots, \chi_V |\cdot|_E^{(l'+1)/2}, \upsilon_1 \chi_V \chi_W^{-1}, \ldots, \upsilon_s \chi_V \chi_W^{-1}\}.$$

If $m - m' \geq n - n'$, *then we have*

$$\{\upsilon_1, \ldots, \upsilon_s\} = \{\chi_W |\cdot|_E^{-(l+1)/2}, \chi_W |\cdot|_E^{-(l+3)/2}, \ldots, \chi_W |\cdot|_E^{-(l'-1)/2}, \tau_1 \chi_V^{-1} \chi_W, \ldots, \tau_r \chi_V^{-1} \chi_W\}.$$

5.4. **Tower property.** Let V_0 be an anisotropic space over E, and for $r \geq 0$, let

$$V_r = V_0 \oplus \mathbb{H}^r$$

with associated isometry groups $H(V_r)$, where $\mathbb{H}$ is the hyperbolic plane. The collection

$$\{V_r \mid r \geq 0\}$$

is called a Witt tower of spaces. We note that any given space V is a member of a unique Witt tower of spaces $\{V_r\}$, where V_0 is the anisotropic kernel of V. One can then consider a tower of the theta correspondence associated to the tower of reductive dual pairs $(G(W), H(V_r))$. For an irreducible admissible representation π of $G(W)$, we thus have the representation $\Theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi)$ of $H(V_r)$. The smallest non-negative integer r_0 such that $\Theta_{V_{r_0},W,\boldsymbol{\chi},\psi}(\pi) \neq 0$ is called the first occurrence index of π for the Witt tower $\{V_r\}$, and the representation $\theta_{V_{r_0},W,\boldsymbol{\chi},\psi}(\pi)$ is called the first occurrence of π for this Witt tower.

The following proposition is often referred to as the tower property of theta correspondence.

Proposition 5.3. *Let r_0 be the first occurrence index of π for the Witt tower $\{V_r\}$.*

(i) *We have $\Theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi) \neq 0$ for all $r \geq r_0$.*

(ii) *If π is supercuspidal, then $\Theta_{V_{r_0},W,\boldsymbol{\chi},\psi}(\pi)$ is irreducible and supercuspidal.*

5.5. **First occurrence.** Harris-Kudla-Sweet [27] and Kudla-Rallis [47] discovered that the first occurrence indices of π for two closely related Witt towers $\{V_r\}$ and $\{V_r'\}$ are not independent of each other. More precisely, in Cases A and C, one considers two Witt towers $\{V_r\}$ and $\{V_r'\}$ such that

$$\begin{cases} \dim V_0 \equiv \dim V_0' \mod 2 & \text{in Case A,} \\ \operatorname{disc} V_0 = \operatorname{disc} V_0' & \text{in Case C.} \end{cases}$$

Hence we may consider the first occurrence indices r_0 (for the tower $\{V_r\}$) and r_0' (for the tower $\{V_r'\}$).

On the other hand, in Cases B and D, there is only a single tower of symplectic spaces $\{V_r\}$. However, since π is a representation of the orthogonal group $G(W)$, we may consider its twist $\pi \otimes \det$. Thus, we have the two towers of theta lifts

$$\Theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi) \quad \text{and} \quad \Theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi \otimes \det),$$

and the corresponding first occurrence indices r_0 (of π) and r_0' (of $\pi \otimes \det$). Henceforth, to simplify notation, we set $V_r' = V_r$ and

$$\Theta_{V_r',W,\boldsymbol{\chi},\psi}(\pi) = \Theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi \otimes \det).$$

Now we have:

Theorem 5.4. *We have*

$$\dim V_{r_0} + \dim V'_{r_0'} \geq 2\dim W + \begin{cases} 2 & \text{in Case A,} \\ 0 & \text{in Cases BD,} \\ 4 & \text{in Cases C.} \end{cases}$$

Moreover, if π is supercuspidal, then equality holds in all cases.

Proof. Case A is due to Harris-Kudla-Sweet [27] and Gong-Grenié [19, Theorem 1.8]. Case C is due to Kudla-Rallis [47, Theorem 3.8]. The proof for the remaining Cases B and D is simpler, as we now explain.

Suppose that W is a quadratic space over F and V_r is the symplectic space over F of dimension $2r$. Let π be an irreducible admissible representation of $G(W)$. Let r_0 and r_0' be the first occurrence indices of π and $\pi \otimes \det$ respectively. Then we have nonzero $G(W)$-equivariant maps

$$\omega_{V_{r_0},W,\boldsymbol{\chi},\psi} \longrightarrow \pi$$

and

$$\omega_{V_{r_0'},W,\boldsymbol{\chi},\psi} \longrightarrow \pi \otimes \det .$$

Since $\pi^\vee \cong \pi$ (see [59, p. 91]), we obtain a nonzero $G(W)$-equivariant map

$$\omega_{V_{r_0+r_0'},W,\boldsymbol{\chi},\psi} \cong \omega_{V_{r_0},W,\boldsymbol{\chi},\psi} \otimes \omega_{V_{r_0'},W,\boldsymbol{\chi},\psi} \longrightarrow \pi \otimes (\pi \otimes \det) \cong \pi \otimes (\pi^\vee \otimes \det) \longrightarrow \det .$$

In other words, the determinant character of $G(W) = \mathrm{O}(W)$ participates in the theta correspondence with

$$H(V_{r_0+r_0'}) = \begin{cases} \mathrm{Mp}(V_{r_0+r_0'}) & \text{in Case B,} \\ \mathrm{Sp}(V_{r_0+r_0'}) & \text{in Case D.} \end{cases}$$

It follows from a result of Rallis [65, p. 399] that

$$\dim V_{r_0} + \dim V_{r_0'} = \dim V_{r_0+r_0'} \geq 2 \dim W,$$

as desired.

Finally, a short and sweet proof of the equality when π is supercuspidal has been given by Mínguez [56]. This completes the proof of the theorem. □

Corollary 5.5. *Let π be an irreducible admissible representation of $G(W)$. Let V and V' be the two spaces in the Witt towers $\{V_r\}$ and $\{V_r'\}$ such that $\dim V = \dim V' = m$. If $l \geq 0$, then at most one of the theta lifts*

$$\begin{cases} \Theta_{V,W,\boldsymbol{\chi},\psi}(\pi) & \textit{to } H(V); \\ \Theta_{V',W,\boldsymbol{\chi},\psi}(\pi) & \textit{to } H(V') \end{cases}$$

is nonzero.

Proof. The corollary follows from Theorem 5.4, noting that

$$2m + 2l + 2 = 2 \dim W + \begin{cases} 2 & \text{in Case A,} \\ 0 & \text{in Cases BD,} \\ 4 & \text{in Cases C.} \end{cases}$$

□

In Corollary 9.2 below, we will strengthen this corollary by showing that exactly one of the theta lifts is nonzero when $l = 0$.

6. **Doubling see-saw**

In this section, we consider a useful tool for analyzing the theta correspondence: the so-called doubling see-saw diagram.

6.1. **Doubling see-saw.** Given an irreducible admissible representation π of $G(W)$, one is interested in whether $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is nonzero. To address this question, it is useful to recall the "doubled space"

$$\mathbf{W} = W \oplus W_-$$

introduced in §4.2. Now consider the see-saw diagram:

$$\begin{tikzcd}
\mathbf{G}(\mathbf{W}) \arrow[d, dash] \arrow[dr, dash] & H(V) \times H(V) \arrow[d, dash] \arrow[dl, dash] \\
G(W) \times G(W) & \Delta H(V)
\end{tikzcd}$$

Then the see-saw identity says that:

$$\mathrm{Hom}_{H(V)}(\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi) \otimes \Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)^{\mathrm{MVW}}\chi_W, \chi_W) \cong \mathrm{Hom}_{G(W)\times G(W)}(\Theta_{V,\mathbf{W},\boldsymbol{\chi},\psi}(\chi_W), \pi \otimes \pi^\vee \chi_V),$$

where the superscript MVW refers to the involution on the set of smooth representations of $H(V)$ defined by Mœglin-Vignéras-Waldspurger (see [59]). In particular, we have:

Lemma 6.1. *Let π be an irreducible admissible representation of $G(W)$. Then*

$$\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi) \neq 0 \Longleftrightarrow \mathrm{Hom}_{G(W)\times G(W)}(\Theta_{V,\mathbf{W},\boldsymbol{\chi},\psi}(\chi_W), \pi \otimes \pi^\vee \chi_V) \neq 0.$$

6.2. **Degenerate principal series representations.** In order for the non-vanishing criteria given in Lemma 6.1 to be useful, we need to understand the representation $\Theta_{V,\mathbf{W},\boldsymbol{\chi},\psi}(\chi_W)$ of $\mathbf{G}(\mathbf{W})$ more precisely. For this, we need to describe some degenerate principal series representations of $\mathbf{G}(\mathbf{W})$.

Recall that we have a complete polarization

$$\mathbf{W} = W^\Delta \oplus W^\nabla$$

with a maximal isotropic subspace

$$W^\Delta = \{(w, w) \in \mathbf{W} \mid w \in W\}.$$

The stabilizer of W^Δ in $\mathbf{G}(\mathbf{W})$ is a Siegel parabolic subgroup

$$\mathbf{P} = \mathbf{P}(W^\Delta) = M_{\mathbf{P}} \cdot U_{\mathbf{P}}$$

with Levi component

$$M_{\mathbf{P}} = \mathrm{GL}(W^\Delta)$$

and unipotent radical $U_{\mathbf{P}}$. For $s \in \mathbb{C}$ and a character χ of $E^\times$, let

$$I_{\mathbf{P}}^{\mathbf{G}}(s, \chi) = \mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}(\mathbf{W})}(\chi|\cdot|_E^s)$$

denote the normalized induced representation of $\mathbf{G} = \mathbf{G}(\mathbf{W})$, where we regard $\chi|\cdot|_E^s$ as a character of $\mathbf{P}$ via the natural homomorphism

$$\mathbf{P} \longrightarrow \mathrm{GL}(W^\Delta) \xrightarrow{\det} E^\times.$$

Here, in Case C', where $\mathbf{G}(\mathbf{W}) = \mathrm{Mp}(\mathbf{W})$ is the metaplectic group, we interpret

$$I_{\mathbf{P}}^{\mathbf{G}}(s, \chi) = \mathrm{Ind}_{\widetilde{\mathbf{P}}}^{\mathrm{Mp}(\mathbf{W})}(\chi_\psi \cdot \chi|\cdot|_E^s),$$

where χ_ψ is the genuine character of $\widetilde{\mathrm{GL}}(W^\Delta)$ defined in §2.6. Note that

$$\mathcal{F}(i(g, g)) = \chi(\det g) \cdot \mathcal{F}(1)$$

for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(s, \chi)$ and $g \in G = G(W)$, where $i : G \times G \to \mathbf{G}$ is the natural map.

6.3. **The representation $R_{\mathbf{W},\boldsymbol{\chi},\psi}(V,\chi_W)$.** Now consider the Weil representation $\omega_{V,\mathbf{W},\boldsymbol{\chi},\psi}$ of $\mathbf{G}(\mathbf{W})\times H(V)$, which is realized on $S(V\otimes W^{\nabla})$. Recall that the action of $H=H(V)$ is given by:

$$(\omega_{V,\mathbf{W},\boldsymbol{\chi},\psi}(h)\varphi)(x)=\chi_W(\det h)\cdot\varphi(h^{-1}x)$$

for $h\in H$ and $\varphi\in S(V\otimes W^{\nabla})$. For $\varphi\in S(V\otimes W^{\nabla})$ and $\mathbf{g}\in\mathbf{G}$, put

$$\mathcal{F}_{\varphi}(\mathbf{g})=(\omega_{V,\mathbf{W},\boldsymbol{\chi},\psi}(\mathbf{g})\varphi)(0).$$

Then $\varphi\mapsto\mathcal{F}_{\varphi}$ defines a $\mathbf{G}$-equivariant map

$$\omega_{V,\mathbf{W},\boldsymbol{\chi},\psi}\longrightarrow I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2},\chi_V)$$

which factors through the theta lift $\Theta_{V,\mathbf{W},\boldsymbol{\chi},\psi}(\chi_W)$ of the character χ_W of H to $\mathbf{G}$. Let

$$R_{\mathbf{W},\boldsymbol{\chi},\psi}(V,\chi_W)$$

denote the image of this map. Then we have the following proposition due to Rallis (see [65], [59, p. 70]):

Proposition 6.2. *The map $\varphi\mapsto\mathcal{F}_{\varphi}$ induces an isomorphism*

$$\Theta_{V,\mathbf{W},\boldsymbol{\chi},\psi}(\chi_W)\xrightarrow{\cong}R_{\mathbf{W},\boldsymbol{\chi},\psi}(V,\chi_W)\subset I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2},\chi_V).$$

6.4. **The space $\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(s,\chi_V),\pi\otimes\pi^{\vee}\chi_V)$.** Lemma 6.1 and Proposition 6.2 suggest that one should understand

- the structure of the degenerate principal series representation $I_{\mathbf{P}}^{\mathbf{G}}(s,\chi_V)$;
- the space
$$\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(s,\chi_V),\pi\otimes\pi^{\vee}\chi_V).$$

We shall address the first issue in the next section. For the second issue, we have (see [27, §4]):

Proposition 6.3. *Let π be an irreducible admissible representation of $G=G(W)$. Then we have*

$$\dim_{\mathbb{C}}\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(s,\chi_V),\pi\otimes\pi^{\vee}\chi_V)=1$$

for generic s. If π is supercuspidal, then the equality holds for all s.

7. Degenerate principal series representations

In this section, we recall the structure of the degenerate principal series representation $I_{\mathbf{P}}^{\mathbf{G}}(s,\chi_V)$ due to Kudla-Sweet [48] (Case A), Yamana [78] (Cases B and D), Sweet [73] (Case C') and Kudla-Rallis [46] (Case C"). In fact, the construction of the previous section gives a full description of the module structure of $I_{\mathbf{P}}^{\mathbf{G}}(s,\chi_V)$.

Recall that $\mathbf{G}=\mathbf{G}(\mathbf{W})$ and $\dim\mathbf{W}=2\dim W=2n$.

Proposition 7.1. (i) *The representation $I_{\mathbf{P}}^{\mathbf{G}}(s,\chi)$ is multiplicity-free.*

(ii) *Assume that $s\in\mathbb{R}$ and χ is unitary. Then $I_{\mathbf{P}}^{\mathbf{G}}(s,\chi)$ is reducible precisely in the situations given in the following table.*

Case A	$\chi\vert_{F^\times} = \chi_E^m$ *and* $s = \frac{m-n}{2}$ *with* $0 \le m \le 2n$
Cases BD	$\chi^2 = \mathbb{1}$ *and* $s = \frac{m-n+1}{2}$ *with* m *even and* $0 \le m \le 2n-2$
Case C'	$\chi^2 = \mathbb{1}$ *and* $s = \frac{m-n-1}{2}$ *with* m *odd and* $1 \le m \le 2n+1$
Case C"	$\chi^2 = \mathbb{1}$ *but* $\chi \neq \mathbb{1}$ *and* $s = \frac{m-n-1}{2}$ *with* m *even and* $2 \le m \le 2n$ *or* $\chi = \mathbb{1}$ *and* $s = \frac{m-n-1}{2}$ *with* m *even and* $0 \le m \le 2n+2$

For a given point of reducibility $s = s_{m,n}$ of $I_{\mathbf{P}}^{\mathbf{G}}(s, \chi)$ as in the previous proposition, let V and V' be the spaces described in the following table.

Case A	V and V' are the two hermitian spaces of dimension m with $\epsilon(V) = 1$
Cases BD	V is the symplectic space of dimension m
Case C	V and V' are the two quadratic spaces of dimension m with $\chi_V = \chi_{V'} = \chi$ and $\epsilon(V) = 1$

Observe that the space V' does not exist in Cases B and D, and may not exist in Cases A and C when m is small. In any case, noting that

$$s_{m,n} = -\frac{l}{2}$$

with l given in §3.1, we have constructed in §6.3 the submodules

$$R_{\mathbf{W},\boldsymbol{\chi},\psi}(V, \chi_W) \quad \text{and} \quad R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W)$$

of $I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2}, \chi_V)$. Here, in Cases B and D, where V' does not exist, we interpret

$$R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W) = R_{\mathbf{W},\boldsymbol{\chi},\psi}(V, \chi_W) \otimes \det .$$

Also, if V' does not exist in Cases A and C, then $R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W)$ is interpreted to be 0. The following proposition describes the module structure of $I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2}, \chi_V)$ in terms of these submodules.

Proposition 7.2. *Using the above notation, we have:*

(i) *If* $l \ge 0$, *then* $R_{\mathbf{W},\boldsymbol{\chi},\psi}(V, \chi_W)$ *and* $R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W)$ *are irreducible and unitarizable. Moreover,*

$$R_{\mathbf{W},\boldsymbol{\chi},\psi}(V, \chi_W) \oplus R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W)$$

is the maximal semisimple submodule of $I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2}, \chi_V)$. *When* $l = 0$, *we have*

$$I_{\mathbf{P}}^{\mathbf{G}}(0, \chi_V) = R_{\mathbf{W},\boldsymbol{\chi},\psi}(V, \chi_W) \oplus R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W).$$

(ii) *If* $l < 0$, *then we have*

$$I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2}, \chi_V) = R_{\mathbf{W},\boldsymbol{\chi},\psi}(V, \chi_W) + R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W).$$

Moreover,

$$R_{\mathbf{W},\boldsymbol{\chi},\psi}(V, \chi_W) \cap R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W)$$

is irreducible and is the maximal semisimple submodule of $I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2},\chi_V)$*. When*

$$\begin{cases} m = 2n & \text{in Case A,} \\ m = 2n+1 & \text{in Case C',} \\ m = 2n, \text{ or } m = 2n+2 \text{ and } \chi_V = \mathbb{1} & \text{in Case C'',} \end{cases}$$

we have $I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2},\chi_V) = R_{\mathbf{W},\boldsymbol{\chi},\psi}(V,\chi_W)$*.*

8. Normalized intertwining operators

We now define an intertwining operator

$$M(s,\chi) : I_{\mathbf{P}}^{\mathbf{G}}(s,\chi) \longrightarrow I_{\mathbf{P}}^{\mathbf{G}}(-s,(\chi^c)^{-1})$$

by

$$M(s,\chi)\mathcal{F}(\mathbf{g}) = \int_{U_{\mathbf{P}}} \mathcal{F}(wu\mathbf{g})\,du$$

for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(s,\chi)$ and $\mathbf{g} \in \mathbf{G}$. Here we put $w = i(1,-1)$, where $i : G \times G \to \mathbf{G}$ is the natural map, and fix a Haar measure du on $U_{\mathbf{P}}$. The above integral is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and admits a meromorphic continuation to $\mathbb{C}$. In this section, we shall recall a normalization of $M(s,\chi)$ due to Lapid-Rallis [49].

8.1. The unipotent radical $U_{\mathbf{P}}$.

Recall that $U_{\mathbf{P}}$ is the unipotent radical of the Siegel parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. We have an isomorphism

$$\mathbf{b} : U_{\mathbf{P}} \xrightarrow{\cong} \mathrm{Hom}(W^{\nabla}, W^{\Delta})^{*=-}$$

given by

$$\mathbf{b}(u) = \mathrm{pr}_{W^{\Delta}} \circ u|_{W^{\nabla}} \quad \text{for } u \in U_{\mathbf{P}}.$$

Here

$$\mathrm{Hom}(W^{\nabla}, W^{\Delta})^{*=-} = \{\beta \in \mathrm{Hom}(W^{\nabla}, W^{\Delta}) \,|\, \beta^* = -\beta\},$$

where

$$* : \mathrm{Hom}(W^{\nabla}, W^{\Delta}) \longrightarrow \mathrm{Hom}(W^{\nabla}, W^{\Delta})$$

is defined by requiring that

$$\langle \beta(w), w' \rangle = \langle w, \beta^*(w') \rangle$$

for $\beta \in \mathrm{Hom}(W^{\nabla}, W^{\Delta})$ and $w, w' \in W^{\nabla}$, and $\mathrm{pr}_{W^{\Delta}} : \mathbf{W} = W^{\Delta} \oplus W^{\nabla} \to W^{\Delta}$ is the projection. We put

$$\beta_W = \mathbf{p} \circ \beta \circ \mathbf{i} \in \mathrm{End}(W) \quad \text{for } \beta \in \mathrm{Hom}(W^{\nabla}, W^{\Delta})^{*=-},$$

where

$$\mathbf{i} : W \longrightarrow W^{\nabla} \quad \text{and} \quad \mathbf{p} : W^{\Delta} \longrightarrow W$$

are defined by

$$\mathbf{i}(w) = \frac{1}{2}(w,-w), \quad \mathbf{p}(w,w) = w \quad \text{for } w \in W.$$

8.2. **Normalization of intertwining operators.** Let $\bar{U}_{\mathbf{P}}$ be the unipotent radical of the parabolic subgroup of $\mathbf{G}$ opposite to $\mathbf{P}$. Let $\beta \in \operatorname{Hom}(W^{\nabla}, W^{\Delta})^{*=-}$. Then β determines a character ψ_β of $\bar{U}_{\mathbf{P}}$ by:

$$\psi_\beta(\bar{u}) = \psi(\operatorname{tr}(\bar{\mathbf{b}}(\bar{u}) \circ \beta))$$

for $\bar{u} \in \bar{U}_{\mathbf{P}}$. Here

$$\bar{\mathbf{b}} : \bar{U}_{\mathbf{P}} \xrightarrow{\cong} \operatorname{Hom}(W^{\Delta}, W^{\nabla})^{*=-}$$

is an isomorphism given by $\bar{\mathbf{b}}(\bar{u}) = \mathrm{pr}_{W^{\nabla}} \circ \bar{u}|_{W^{\Delta}}$ for $\bar{u} \in \bar{U}_{\mathbf{P}}$. We note here a typo in [49, p. 323]: ψ_A should be given by $\psi_A(X) = \psi_F(\operatorname{tr} XA)$ if h is hermitian. Now fix a Haar measure $d\bar{u}$ on $\bar{U}_{\mathbf{P}}$ and define a Whittaker functional

$$\ell_\beta : I_{\mathbf{P}}^{\mathbf{G}}(s,\chi) \longrightarrow \mathbb{C}$$

by

$$\ell_\beta(\mathcal{F}) = \int_{\bar{U}_{\mathbf{P}}} \mathcal{F}(\bar{u}) \cdot \psi_\beta(\bar{u})\, d\bar{u}$$

for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(s,\chi)$.

In this subsection, we exclude Case B which is more involved. Assume that β is of rank n. Then the above integral is absolutely convergent for $\operatorname{Re}(s) \gg 0$ and admits an analytic continuation to $\mathbb{C}$. Moreover, ℓ_β is a nonzero element in the space $\operatorname{Hom}_{\bar{U}_{\mathbf{P}}}(I_{\mathbf{P}}^{\mathbf{G}}(s,\chi), \psi_\beta^{-1})$. Since this space is 1-dimensional for all s (see [38]), we obtain:

$$\ell_\beta \circ M(s,\chi) = c_{\beta,\psi}(s,\chi) \cdot \ell_\beta$$

for some rational function $c_{\beta,\psi}(s,\chi)$. Note that $c_{\beta,\psi}(s,\chi)$ depends on the choice of du but not on the choice of $d\bar{u}$.

Now choose β of rank n such that W^{∇} has an isotropic subspace of dimension $[\frac{n}{2}]$ with respect to the sesquilinear form $(w,w') \mapsto \langle \beta(w), w' \rangle$. Following Lapid-Rallis [49], we define a normalized intertwining operator $M_\psi^{\mathcal{LR}}(s,\chi)$ by

$$M_\psi^{\mathcal{LR}}(s,\chi) = c_{\beta,\psi}(s,\chi)^{-1} \cdot M(s,\chi).$$

Note that this normalized intertwining operator is now independent of the choice of the Haar measure du, but it depends on the choice of the additive character ψ.

In Case B, there is a similarly defined normalization, though it is complicated by the fact that one cannot find an element $\beta \in \operatorname{Hom}(W^{\nabla}, W^{\Delta})^{*=-}$ of rank n. Instead one defines a normalization by choosing β of rank $n-1$ and an anisotropic line $\mathfrak{L}$ in $\mathbf{W}$. We refer the reader to Appendix A for details.

For convenience, we shall modify the normalized intertwining operator $M_\psi^{\mathcal{LR}}(s,\chi)$ slightly by setting:

$$M_\psi^{\mathrm{LR}}(s,\chi) := \begin{cases} \gamma(s+\frac{1}{2},\chi,\psi) \cdot M_\psi^{\mathcal{LR}}(s,\chi) & \text{in Case C''}, \\ M_\psi^{\mathcal{LR}}(s,\chi) & \text{in all other cases.} \end{cases}$$

8.3. **Analytic properties.** The analytic properties of the normalized intertwining operator $M_\psi^{\mathrm{LR}}(s,\chi)$ is controlled by the normalizing factor $c_{\beta,\psi}(s,\chi)$. This normalizing factor has been computed by Kudla-Sweet [48] (Case A), Yamana [78] (Cases B and D), Sweet [73] (Case C') and Kudla-Rallis [46] (Case C"); we recall their results in Appendix A. From these results, one deduces the following proposition.

Proposition 8.1. *Assume that $s \in \mathbb{R}$ and χ is unitary. Then the normalized intertwining operator $M_\psi^{\mathrm{LR}}(s,\chi)$ is holomorphic at s unless $s > 0$ and $I_{\mathbf{P}}^{\mathbf{G}}(s,\chi)$ is reducible, in which case $M_\psi^{\mathrm{LR}}(s,\chi)$ has a simple pole. More explicitly, the set of poles of $M_\psi^{\mathrm{LR}}(s,\chi)$ is given by the following table.*

Case A	$\chi\|_{F^\times} = \chi_E^n$ *and* $s = 1, 2, \dots, [\frac{n}{2}]$ *or* $\chi\|_{F^\times} = \chi_E^{n+1}$ *and* $s = \frac{1}{2}, \frac{3}{2}, \dots, [\frac{n-1}{2}] + \frac{1}{2}$
Cases BC'	$\chi^2 = \mathbb{1}$ *and* $s = 1, 2, \dots, [\frac{n}{2}]$
Cases C"D	$\chi^2 = \mathbb{1}$ *and* $s = \frac{1}{2}, \frac{3}{2}, \dots, \frac{n-1}{2}$ *and* $\chi = \mathbb{1}$ *and* $s = \frac{n+1}{2}$ *in Case C"*

8.4. **Kernels and images.** The following proposition describes the residue

$$M^{\mathrm{LR}}_{\psi,-1}(s_0, \chi) := \operatorname*{Res}_{s=s_0} M^{\mathrm{LR}}_{\psi}(s, \chi)$$

of $M^{\mathrm{LR}}_{\psi}(s,\chi)$ at $s = s_0$ for the simple poles described in the above proposition.

Proposition 8.2. *Assume that*

$$\begin{cases} n < m \le 2n & \textit{in Case A,} \\ n-1 < m \le 2n-2 & \textit{in Cases BD,} \\ n+1 < m \le 2n+1, \textit{ or } m = 2n+2 \textit{ and } \chi_V = \mathbb{1} & \textit{in Case C,} \end{cases}$$

so that $l < 0$, $I^{\mathbf{G}}_{\mathbf{P}}(-\frac{l}{2}, \chi_V)$ *is reducible and*

$$M^{\mathrm{LR}}_{\psi,-1}(-\tfrac{l}{2}, \chi_V) : I^{\mathbf{G}}_{\mathbf{P}}(-\tfrac{l}{2}, \chi_V) \longrightarrow I^{\mathbf{G}}_{\mathbf{P}}(\tfrac{l}{2}, \chi_V)$$

is a nonzero $\mathbf{G}$*-equivariant map. Then*

$$\ker M^{\mathrm{LR}}_{\psi,-1}(-\tfrac{l}{2}, \chi_V) = \textit{the maximal semisimple submodule of } I^{\mathbf{G}}_{\mathbf{P}}(-\tfrac{l}{2}, \chi_V)$$

and

$$\operatorname{im} M^{\mathrm{LR}}_{\psi,-1}(-\tfrac{l}{2}, \chi_V) = \textit{the maximal semisimple submodule of } I^{\mathbf{G}}_{\mathbf{P}}(\tfrac{l}{2}, \chi_V).$$

These maximal semisimple submodules were described in Proposition 7.2.

9. Doubling zeta integrals

We saw that the space

$$\operatorname{Hom}_{G\times G}(I^{\mathbf{G}}_{\mathbf{P}}(s,\chi), \pi \otimes \pi^\vee \chi)$$

is 1-dimensional for generic s. We shall see that the doubling zeta integral of Piatetski-Shapiro and Rallis [62] defines a nonzero element in this space for each s. In this section, we introduce this doubling zeta integral.

9.1. **Doubling zeta integrals.** Let π be an irreducible admissible representation of $G = G(W)$ and $\pi^\vee$ the contragredient representation of π. For a holomorphic section $\mathcal{F}$ of $I^{\mathbf{G}}_{\mathbf{P}}(s,\chi)$ and a matrix coefficient f of $\pi^\vee$, put

$$Z(s, \mathcal{F}, f) = \int_G \mathcal{F}(i(g,1)) \cdot f(g)\, dg.$$

If π is supercuspidal, then this integral is absolutely convergent. The following theorem summarizes some basic properties of zeta integrals (see [45], [49]).

Theorem 9.1. (i) *There exists a real number c such that $Z(s,\mathcal{F},f)$ is absolutely convergent for all $\mathcal{F}$ and f whenever* $\mathrm{Re}(s) > c$.

(ii) *If $\mathcal{F}$ is standard, then $Z(s,\mathcal{F},f)$ is a rational function of q^{-s} (when* $\mathrm{Re}(s) > c$*) and thus admits a meromorphic continuation to $\mathbb{C}$.*

(iii) *For each $s_0 \in \mathbb{C}$, there exist $\mathcal{F}$ and f such that $Z(s_0,\mathcal{F},f)$ is absolutely convergent and nonzero.*

(iv) *There exists a non-negative integer d (depending on s_0) such that $(s-s_0)^d \cdot Z(s,\mathcal{F},f)$ is holomorphic at $s = s_0$ for all $\mathcal{F}$ and f.*

(v) *Let $Z^*(s_0,\cdot,\cdot)$ denote the leading term of the Laurent expansion of $Z(s,\cdot,\cdot)$ at $s = s_0$. Then $Z^*(s_0,\cdot,\cdot)$ is a nonzero element in* $\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(s_0,\chi)\otimes\pi^\vee\otimes\pi\chi^{-1},\mathbb{C})$. *In particular, we have*

$$\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(s,\chi)\otimes\pi^\vee\otimes\pi\chi^{-1},\mathbb{C}) \neq 0$$

for all s.

9.2. **Theta dichotomy.**

As a consequence, we deduce the following dichotomy which was established in [27] for Case A and in [10, §11], [13] for Cases B and C'.

Corollary 9.2. *Let π be an irreducible admissible representation of $G(W)$. Let V and V' be the two spaces in the Witt towers $\{V_r\}$ and $\{V'_r\}$ such that* $\dim V = \dim V' = m$. *If $l = 0$, then exactly one of the theta lifts*

$$\begin{cases} \Theta_{V,W,\boldsymbol{\chi},\psi}(\pi) & \textit{to } H(V); \\ \Theta_{V',W,\boldsymbol{\chi},\psi}(\pi) & \textit{to } H(V') \end{cases}$$

is nonzero.

Proof. By Proposition 7.2(i), we have

$$\begin{aligned} &\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(0,\chi_V)\otimes\pi^\vee\otimes\pi\chi_V^{-1},\mathbb{C}) \\ &= \mathrm{Hom}_{G\times G}(R_{\mathbf{W},\boldsymbol{\chi},\psi}(V,\chi_W)\otimes\pi^\vee\otimes\pi\chi_V^{-1},\mathbb{C}) \oplus \mathrm{Hom}_{G\times G}(R_{\mathbf{W},\boldsymbol{\chi},\psi}(V',\chi_W)\otimes\pi^\vee\otimes\pi\chi_V^{-1},\mathbb{C}) \end{aligned}$$

when $l = 0$. In view of Corollary 5.5, Lemma 6.1, Proposition 6.2 and Theorem 9.1(v), the desired assertion follows. ☐

In Theorem 11.1 below, we will determine exactly which of the theta lifts is nonzero.

9.3. **Tempered case.**

We can be more precise about the range of convergence of the doubling zeta integral when π is tempered. Let r be the F-rank of $G = G(W)$ and put

$$\rho = \begin{cases} \frac{n}{2} & \text{in Case A,} \\ \frac{n-1}{2} & \text{in Cases BD,} \\ \frac{n+1}{2} & \text{in Case C.} \end{cases}$$

Fix a maximal split torus A_0 of G and a minimal parabolic subgroup P_0 of G containing A_0. Let δ_0 be the modulus character of P_0 and

$$A_0^+ = \{a \in A_0 \,|\, |\alpha(a)|_F \le 1 \text{ for all simple roots of } A_0 \text{ in } P_0\}.$$

We may identify A_0 with $(F^\times)^r$ so that

$$\delta_0(a) = \prod_{i=1}^{r} |a_i|_E^{2\rho+1-2i}$$

for $a = (a_1,\dots,a_r) \in A_0 \cong (F^\times)^r$ and

$$A_0^+ \cong \{(a_1,\dots,a_r) \in (F^\times)^r \,|\, |a_1|_F \le \dots \le |a_r|_F \le 1\}.$$

Fix a maximal compact subgroup K of G such that $G^0 \cap K$ is the stabilizer of a special point of the apartment associated to A_0 in the building of G^0, where G^0 is identity component of G.

Fix a height function $\| \cdot \|$ on G such that

$$\|g\| \geq 1, \quad \|g_1 g_2\| \leq \|g_1\| \|g_2\|, \quad \|k_1 g k_2\| = \|g\|$$

for $g, g_1, g_2 \in G$ and $k_1, k_2 \in K$.

Let Ξ denote the Harish-Chandra function on G defined in [76, §II.1]. We recall some properties of Ξ which we will need.

- The function Ξ is positive and K-bi-invariant.
- There exists $d \in \mathbb{N}$ such that
$$\Xi(a) \ll \delta_0(a)^{1/2} (1 + \log \|a\|)^d$$
for $a \in A_0^+$ (see [76, Lemme II.1.1]).
- We have
$$\int_K \Xi(g_1 k g_2) \, dk = \Xi(g_1) \Xi(g_2)$$
for $g_1, g_2 \in G$, where the Haar measure dk on K is normalized so that $\mathrm{vol}(K) = 1$ (see [76, Lemme II.1.3]).
- We have $\Xi(g^{-1}) = \Xi(g)$ for $g \in G$ (see [76, Lemme II.1.4]).

Note that though [76] only treats connected reductive linear algebraic groups, there is no difficulty in including the disconnected orthogonal groups or the nonlinear metaplectic groups in the discussion. For example, when $G = \mathrm{Mp}(W)$ is the metaplectic group, one simply takes the pull-back of the Harish-Chandra function on $\mathrm{Sp}(W)$ to $\mathrm{Mp}(W)$.

Now we have:

Lemma 9.3. *We have*

$$|\mathcal{F}(i(a, 1))| \ll \prod_{i=1}^{r} |a_i|_E^{\mathrm{Re}(s) + \rho}$$

for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(s, \chi)$ *and* $a = (a_1, \dots, a_r) \in A_0^+$.

Proof. Let W_{an} be the anisotropic kernel of W. Put

$$W' = (W_{\mathrm{an}})^{\perp} \subset W, \quad \mathbf{W}' = W' \oplus W'_{-} \subset \mathbf{W}$$

and $n' = \dim W'$. Let $\mathbf{G}' = \mathbf{G}(\mathbf{W}')$ and $\mathbf{P}' = \mathbf{G}' \cap \mathbf{P}$. Then we have

$$\mathcal{F}|_{\mathbf{G}'} \in I_{\mathbf{P}'}^{\mathbf{G}'}(s + \tfrac{n - n'}{2}, \chi)$$

for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(s, \chi)$. Hence we may assume that $W = W'$, in which case the lemma follows from [62, Proposition 6.4]. We remark that the analog of [62, Proposition 6.4] holds in Case A. □

Lemma 9.4. (i) *Assume that* π *is square integrable. If* $\mathrm{Re}(s) \geq -\frac{1}{2}$, *then* $Z(s, \mathcal{F}, f)$ *is absolutely convergent.*

(ii) *Assume that* π *is tempered. If* $\mathrm{Re}(s) > -\frac{1}{2}$, *then* $Z(s, \mathcal{F}, f)$ *is absolutely convergent.*

Proof. We first assume that π is square integrable. Then for any $d \gg 0$, we have

$$|f(g)| \ll \Xi(g) \cdot (1 + \log \|g\|)^{-d}$$

for $g \in G$. By Lemma 9.3, we have

$$\begin{aligned} |Z(s,\mathcal{F},f)| &\ll \int_{A_0^+} \prod_{i=1}^r |a_i|_E^{\mathrm{Re}(s)+\rho} \cdot \delta_0(a)^{1/2}(1+\log\|a\|)^{-d} \cdot \delta_0(a)^{-1}\, da \\ &= \int_{A_0^+} \prod_{i=1}^r |a_i|_E^{\mathrm{Re}(s)+i-1/2} \cdot (1+\log\|a\|)^{-d}\, da. \end{aligned}$$

If $\mathrm{Re}(s) \geq -\frac{1}{2}$, then this integral is absolutely convergent. The case when π is tempered is similar and we omit the details. □

10. **Standard γ-factors**

In this section, we recall the definition of the standard γ-factors of representations of $G(W)$, following Lapid-Rallis [49].

10.1. **Definition.** Let π be an irreducible admissible representation of $G = G(W)$ and χ a character of $E^\times$. By Proposition 6.3, one knows that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(s,\chi) \otimes \pi^\vee \otimes \pi\chi^{-1}, \mathbb{C}) = 1$$

for generic s. Since both

$$Z(s,\cdot,\cdot) \quad \text{and} \quad Z(-s,\cdot,\cdot) \circ (M_\psi^{\mathrm{LR}}(s,\chi) \boxtimes \mathrm{id}_{\pi^\vee \boxtimes \pi\chi})$$

belong to the space $\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(s,\chi) \otimes \pi^\vee \otimes \pi\chi^{-1}, \mathbb{C})$, they must be proportional. In [49], Lapid-Rallis defined the standard γ-factor $\gamma(s,\pi,\chi^c,\psi)$ by the local functional equation

$$Z(-s, M_\psi^{\mathrm{LR}}(s,\chi)\mathcal{F}, f) = \varepsilon \cdot \omega_\pi(-1) \cdot \chi(\det \beta_W) \cdot |\det \beta_W|_E^s \cdot \gamma(s+\tfrac{1}{2},\pi,\chi^c,\psi) \cdot Z(s,\mathcal{F},f)$$

for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(s,\chi)$ and a matrix coefficient f of $\pi^\vee$, where

$$\varepsilon = \begin{cases} \epsilon(W)^{n+1} & \text{in Case A,} \\ \epsilon(W) & \text{in Case B,} \\ \chi_\psi^W(-1)^{-1} & \text{in Case C',} \\ 1 & \text{in Case C'',} \\ \epsilon(\frac{1}{2},\chi_W,\psi)^{-1} & \text{in Case D.} \end{cases}$$

Here, we remark that:

- in Case A, the above definition differs from that in [49] by $\epsilon(W)$ when n is even. This modification is necessary to remedy an error in the formula for $c(s,\omega,A,\psi)$ in [49, §9.8]. In fact, they had $\gamma_{\mathbb{R}}(-1,\psi) = i$ in [49, §9.8] if $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$ (where $i = \sqrt{-1}$), but the correct formula should be $\gamma_{\mathbb{R}}(-1,\psi) = -i$ and hence
$$c(s,\omega,A,\psi) = \frac{\Gamma_{\mathbb{R}}(2s-1)\Gamma_{\mathbb{R}}(2s+1)}{\Gamma_{\mathbb{R}}(-2s+2)^2}.$$
One sees that the above definition leads to the desired properties of the standard γ-factors, in particular the minimal cases of the Ten Commandments.
- in Case B, as we noted in §8.2, we have $\det \beta_W = 0$. The normalization of the intertwining operator $M(s,\chi)$ requires additional choices of an anisotropic line $\mathfrak{L}$ in $\mathbf{W}$ and an element $\tilde{\beta} = \tilde{\beta}_{\mathfrak{L}} \in \mathrm{Hom}(W^\nabla, W^\Delta)$ of rank n, so that we should replace $\det \beta_W$ by $\det \tilde{\beta}_W$. See Appendix A for details.
- in Case C', the definition of the standard γ-factors is carried out in [9].

- the above definition seems to differ from that in [49], where the above local functional equation is used to define the standard γ-factor $\gamma(s,\pi^\vee,\chi,\psi)$ except in Case C'. However, it was shown in [49, Theorem 4] that $\gamma(s,\pi^\vee,\chi,\psi)=\gamma(s,\pi,\chi^c,\psi)$, so that the above definition actually agrees with that in [49]. The somewhat unorthodox choice is made here so that the somewhat subtle Case C' can be included in the discussion.

10.2. **Ten Commandments.** The standard γ-factor $\gamma(s,\pi,\chi,\psi)$ satisfies a list of expected properties, affectionately known as the Ten Commandments. We shall not reproduce this list of properties here, but refer the reader to [49, Theorem 4] for Cases A, B, C" and D and [9, Theorem 8.1] for Case C'. For convenience, we recall some properties which we will use.

- (Multiplicativity) Suppose that π' is an irreducible subrepresentation of $I_P^{G'}(\tau\otimes\pi)$, where P is a maximal parabolic subgroup of $G'=G(\mathbb{H}^k\oplus W)$ with Levi component $\mathrm{GL}_k(E)\times G$, τ is an irreducible admissible representation of $\mathrm{GL}_k(E)$, and π is an irreducible admissible representation of G. Then we have
$$\gamma(s,\pi',\chi,\psi)=\gamma^{\mathrm{GJ}}(s,\tau\otimes\chi,\psi_E)\cdot\gamma^{\mathrm{GJ}}(s,(\tau^c)^\vee\otimes\chi,\psi_E)\cdot\gamma(s,\pi,\chi,\psi),$$
where γ^{GJ} refers to the γ-factors of Godement-Jacquet [18].
- (Functional equation) We have
$$\gamma(s,\pi,\chi,\psi)\cdot\gamma(1-s,\pi^\vee,\chi^{-1},\bar{\psi})=1.$$
- (Duality) We have
$$\gamma(s,\pi^\vee,\chi,\psi)=\begin{cases}\gamma(s,\pi,\chi^c,\psi) & \text{in Cases ABC"D,}\\ \gamma(s,\pi,\chi,\bar{\psi}) & \text{in Case C'.}\end{cases}$$
- (Variation of ψ) For $a\in F^\times$, we have
$$\gamma(s,\pi,\chi,\psi_a)=\gamma(s,\pi,\chi,\psi)\times\begin{cases}\chi(a)^n\cdot|a|_E^{n(s-1/2)} & \text{in Case A,}\\ \chi(a)^{n-1}\cdot|a|_F^{(n-1)(s-1/2)} & \text{in Case B,}\\ \chi(a)^{n+1}\cdot|a|_F^{(n+1)(s-1/2)} & \text{in Case C",}\\ \chi_W(a)\cdot\chi(a)^n\cdot|a|_F^{n(s-1/2)} & \text{in Case D,}\end{cases}$$
and
$$\gamma(s,\pi,\chi,\psi_a)=\gamma(s,\pi,\chi\chi_a,\psi)\cdot\chi(a)^n\cdot|a|_F^{n(s-1/2)}\quad\text{in Case C'.}$$
Here $\psi_a(x)=\psi(ax)$ for $x\in F$ and $\chi_a(x)=(a,x)_F$ for $x\in F^\times$.
- (Global property) Suppose that $\mathbb{F}$ is a number field with ring of adeles $\mathbb{A}=\mathbb{A}_\mathbb{F}$ and $\mathbb{W}$ is a space over $\mathbb{E}$, where either $\mathbb{E}=\mathbb{F}$ or $\mathbb{E}$ is a quadratic extension of $\mathbb{F}$. Let Π be an irreducible cuspidal automorphic representation of $G(\mathbb{W})(\mathbb{A})$, $\boldsymbol{\chi}$ a Hecke character of $\mathbb{A}_\mathbb{E}^\times$ and Ψ a nontrivial additive character of $\mathbb{F}\backslash\mathbb{A}$. Then we have
$$L^S(s,\Pi,\boldsymbol{\chi})=\prod_{v\in S}\gamma(s,\Pi_v,\boldsymbol{\chi}_v,\Psi_v)\cdot L^S(1-s,\Pi^\vee,\boldsymbol{\chi}^{-1})$$
for a sufficiently large finite set S of places of $\mathbb{F}$.

10.3. **Analytic properties.** The analytic properties of the standard γ-factor $\gamma(s,\pi,\chi,\psi)$ is, to a certain extent, controlled by the analytic properties of the normalized intertwining operator $M_\psi^{\mathrm{LR}}(s,\chi)$. As a consequence of Proposition 8.1, we deduce:

Proposition 10.1. *Let π be an irreducible admissible representation of G. Assume that $s \in \mathbb{R}$ and χ is unitary.*

(i) *If π is supercuspidal, then $\gamma(s, \pi, \chi, \psi)$ is holomorphic at s unless $s - \frac{1}{2}$ belongs to the distinguished set of points highlighted in Proposition 8.1, in which case $\gamma(s, \pi, \chi, \psi)$ has at most a simple pole.*

(ii) *If π is square integrable, then $\gamma(s, \pi, \chi, \psi)$ is holomorphic for $s < 1$ and has at most a simple pole at $s = 1$.*

(iii) *If π is tempered, then $\gamma(s, \pi, \chi, \psi)$ is holomorphic for $s < 1$ and is nonzero for $s > 0$.*

Proof. (i) Consider the local functional equation which defines $\gamma(s, \pi, \chi^c, \psi)$:

$$Z(-s, M_\psi^{\mathrm{LR}}(s, \chi)\mathcal{F}, f) = \alpha(s) \cdot \gamma(s + \tfrac{1}{2}, \pi, \chi^c, \psi) \cdot Z(s, \mathcal{F}, f)$$

where $\alpha(s)$ is a nonzero exponential function of s. By Theorem 9.1(iii), we can choose $\mathcal{F}$ and f such that $Z(s_0, \mathcal{F}, f) = 1$ for any given $s_0 \in \mathbb{C}$. On the other hand, since π is supercuspidal, the order of pole of $Z(-s, M_\psi^{\mathrm{LR}}(s, \chi)\mathcal{F}, f)$ at $s = s_0$ is bounded above by the order of pole of $M_\psi^{\mathrm{LR}}(s, \chi)$ at $s = s_0$. The desired assertion then follows from Proposition 8.1.

(ii) The same argument as in the proof of (i) works when π is square integrable as long as $\mathrm{Re}(s_0) \le \frac{1}{2}$, in view of Lemma 9.4(i).

(iii) The same argument as in the proof of (i) works when π is tempered as long as $\mathrm{Re}(s_0) < \frac{1}{2}$, in view of Lemma 9.4(ii), from which the first assertion follows. The second assertion follows from the local functional equation:

$$\gamma(s, \pi, \chi, \psi) \cdot \gamma(1 - s, \pi^\vee, \chi^{-1}, \bar{\psi}) = 1.$$

□

11. Standard γ-factors and local theta correspondence

In this section, we study the behaviour of the standard γ-factor defined in the previous section under the theta correspondence.

11.1. Epsilon dichotomy.

We first recall the following dichotomy which was established in [27, Theorem 6.1] for Case A and in [13, Theorem 1.4] for Cases B and C'.

Theorem 11.1. *Assume that $l = 0$. Let π be an irreducible admissible representation of $G(W)$. Assume further that π is tempered in Case A. Then $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi) \neq 0$ if and only if*

$$\epsilon(\tfrac{1}{2}, \pi, \chi_V^{-1}, \psi) = \begin{cases} \omega_\pi(-1) \cdot \chi_V(\daleth)^n \cdot \epsilon(V) \cdot \epsilon(W) & \textit{in Case A,} \\ \omega_\pi(-1) \cdot \epsilon(W) & \textit{in Case B,} \\ \omega_\pi(-1) \cdot \chi_\psi^W(-1)^{-1} \cdot \chi_V(-1)^{n/2} \cdot \epsilon(V) & \textit{in Case C'.} \end{cases}$$

Here $\epsilon(s, \pi, \chi_V^{-1}, \psi)$ is the standard ϵ-factor defined by the doubling method (see [49, §10]*).*

We remark that in Case A, the convention in [27] differs from ours, and [27] only treats the case when π is supercuspidal (or more generally, does not occur in the boundary). We shall transport and extend their result in Appendix A. Also, in Cases B and C', [13] only treats the case when the discriminant of the quadratic space is trivial, but this implies the general case.

11.2. **Poles of standard γ-factors.** When π is supercuspidal, we saw in Proposition 10.1 that $\gamma(s,\pi,\chi,\psi)$ has at most simple poles at certain positive integers or half-integers (depending on the case). The following proposition characterizes the existence of these poles in terms of the non-vanishing of theta lifts.

Recall that one may consider the theta correspondence associated to two Witt towers $\{V_r\}$ and $\{V_r'\}$ (though strictly speaking, there is only a single tower of spaces in Cases B and D). Suppose that V and V' are the two spaces in these Witt towers such that $\dim V = \dim V' = m$. Let l be as in §3.1. Then we have the following characterization, whose special case can be found in [27].

Proposition 11.2. *Assume that $l > 0$. Let π be an irreducible tempered representation of $G(W)$. Consider the theta lifts*

$$\begin{cases} \Theta_{V,W,\boldsymbol{\chi},\psi}(\pi) & \text{to } H(V); \\ \Theta_{V',W,\boldsymbol{\chi},\psi}(\pi) & \text{to } H(V'). \end{cases}$$

(i) *If one of $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ and $\Theta_{V',W,\boldsymbol{\chi},\psi}(\pi)$ is nonzero, then $\gamma(s,\pi,\chi_V^{-1},\psi)$ has a pole at $s=\frac{l+1}{2}$.*

(ii) *Suppose that either π is supercuspidal, or $l=1$ and π is square integrable. Then the converse of* (i) *also holds.*

Proof. (i) Suppose that $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi) \neq 0$ but $\gamma(s,\pi,\chi_V^{-1},\psi)$ is holomorphic at $s=\frac{l+1}{2}$; we shall derive a contradiction. By Proposition 10.1(iii), we have $\gamma(\frac{l+1}{2},\pi,\chi_V^{-1},\psi)\neq 0$. Hence we deduce from the local functional equation

$$\gamma(s,\pi,\chi_V^{-1},\psi)\cdot\gamma(1-s,\pi^\vee,\chi_V,\bar{\psi}) = 1$$

that $\gamma(s,\pi,\chi_V^c,\psi)$ is holomorphic and nonzero at $s=\frac{-l+1}{2}$.

Now consider the local functional equation

$$Z(-s, M_\psi^{\mathrm{LR}}(s,\chi_V)\mathcal{F}, f) = \alpha(s)\cdot\gamma(s+\tfrac{1}{2},\pi,\chi_V^c,\psi)\cdot Z(s,\mathcal{F},f)$$

at $s=-\frac{l}{2}$, where $\alpha(s)$ is a nonzero exponential function of s. Since π is tempered, $Z(\frac{l}{2},\mathcal{F},f)$ is absolutely convergent by Lemma 9.4(ii) and hence

$$Z(\tfrac{l}{2},\cdot,\cdot)\in \mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(\tfrac{l}{2},\chi_V)\otimes\pi^\vee\otimes\pi\chi_V^{-1},\mathbb{C}).$$

Also, $M_\psi^{\mathrm{LR}}(s,\chi_V)$ is holomorphic at $s=-\frac{l}{2}$ by Proposition 8.1. Thus we obtain

$$Z(\tfrac{l}{2},\cdot,\cdot)\circ M_\psi^{\mathrm{LR}}(-\tfrac{l}{2},\chi_V)\in \mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2},\chi_V)\otimes\pi^\vee\otimes\pi\chi_V^{-1},\mathbb{C}).$$

On the other hand, by Theorem 9.1(iii), we can choose $\mathcal{F}$ and f such that $Z(-\frac{l}{2},\mathcal{F},f)=1$, so that the right-hand side of the local functional equation is holomorphic and nonzero at $s=-\frac{l}{2}$. Hence $Z(\frac{l}{2},\cdot,\cdot)\circ M_\psi^{\mathrm{LR}}(-\frac{l}{2},\chi_V)$ is nonzero.

Thus, we see that the restriction of $Z(\frac{l}{2},\cdot,\cdot)$ to the image of $M_\psi^{\mathrm{LR}}(-\frac{l}{2},\chi_V)$ is nonzero. But this image is precisely

$$R_{\mathbf{W},\boldsymbol{\chi},\psi}(V\oplus\mathbb{H}^l,\chi_W)\cap R_{\mathbf{W},\boldsymbol{\chi},\psi}(V'\oplus\mathbb{H}^l,\chi_W),$$

where $\mathbb{H}$ is the hyperbolic plane. In view of Lemma 6.1 and Proposition 6.2, we conclude that

$$\Theta_{V'\oplus\mathbb{H}^l,W,\boldsymbol{\chi},\psi}(\pi)\neq 0,$$

so that

$$\begin{aligned}\dim V_{r_0} + \dim V'_{r'_0} &\le \dim V + (\dim V' + 2l) = 2m + 2l \\ &= 2\dim W + \begin{cases} 0 & \text{in Case A,} \\ -2 & \text{in Cases BD,} \\ 2 & \text{in Case C,} \end{cases}\end{aligned}$$

where r_0 and r'_0 are the first occurrence indices for the two Witt towers $\{V_r\}$ and $\{V'_r\}$ respectively. This contradicts Theorem 5.4 and the desired assertion follows.

(ii) For the converse, suppose that $\gamma(s, \pi, \chi_V^{-1}, \psi) = \gamma(s, \pi, \chi_V^c, \psi)$ has a pole at $s = \frac{l+1}{2}$. Consider the local functional equation

$$Z(-s, M_\psi^{\mathrm{LR}}(s, \chi_V)\mathcal{F}, f) = \alpha(s) \cdot \gamma(s + \tfrac{1}{2}, \pi, \chi_V^c, \psi) \cdot Z(s, \mathcal{F}, f)$$

at $s = \frac{l}{2}$. Under the assumption that either π is supercuspidal, or $l = 1$ and π is square integrable, $Z(-\frac{l}{2}, \mathcal{F}, f)$ is absolutely convergent by Lemma 9.4(i) and hence

$$Z(-\tfrac{l}{2}, \cdot, \cdot) \in \mathrm{Hom}_{G \times G}(I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2}, \chi_V) \otimes \pi^\vee \otimes \pi\chi_V^{-1}, \mathbb{C}).$$

Also, $M_\psi^{\mathrm{LR}}(s, \chi_V)$ has a simple pole at $s = \frac{l}{2}$ by Proposition 8.1. Thus we obtain

$$Z(-\tfrac{l}{2}, \cdot, \cdot) \circ M_{\psi,-1}^{\mathrm{LR}}(\tfrac{l}{2}, \chi_V) \in \mathrm{Hom}_{G \times G}(I_{\mathbf{P}}^{\mathbf{G}}(\tfrac{l}{2}, \chi_V) \otimes \pi^\vee \otimes \pi\chi_V^{-1}, \mathbb{C}).$$

On the other hand, by Theorem 9.1(iii), we can choose $\mathcal{F}$ and f such that $Z(\frac{l}{2}, \mathcal{F}, f) = 1$, so that the right-hand side of the local functional equation has a simple pole at $s = \frac{l}{2}$. Hence $Z(-\frac{l}{2}, \cdot, \cdot) \circ M_{\psi,-1}^{\mathrm{LR}}(\frac{l}{2}, \chi_V)$ is nonzero.

Thus, we see that the restriction of $Z(-\frac{l}{2}, \cdot, \cdot)$ to the image of $M_{\psi,-1}^{\mathrm{LR}}(\frac{l}{2}, \chi_V)$ is nonzero. But this image is precisely

$$R_{\mathbf{W},\boldsymbol{\chi},\psi}(V, \chi_W) \oplus R_{\mathbf{W},\boldsymbol{\chi},\psi}(V', \chi_W).$$

In view of Lemma 6.1 and Proposition 6.2, the desired assertion follows. □

11.3. **Behaviour of standard γ-factors.**

The following theorem describes how the standard γ-factor behaves under the theta correspondence.

Theorem 11.3. *Let π and σ be irreducible admissible representations of $G(W)$ and $H(V)$ respectively, such that σ is an irreducible constituent of $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$. Let χ be a character of $E^\times$.*

(i) *If $l \ge 0$, then we have*

$$\frac{\gamma(s, \pi, \chi\chi_W, \psi)}{\gamma(s, \sigma, \chi\chi_V, \psi)} = \prod_{i=1}^{l} \gamma(s + \tfrac{l+1}{2} - i, \chi\chi_V\chi_W, \psi_E).$$

(ii) *If $l \le 0$, then we have*

$$\frac{\gamma(s, \sigma, \chi\chi_V, \psi)}{\gamma(s, \pi, \chi\chi_W, \psi)} = \prod_{i=1}^{-l} \gamma(s + \tfrac{-l+1}{2} - i, \chi\chi_V\chi_W, \psi_E).$$

Here the product on the right-hand side is interpreted to be 1 when $l = 0$.

As an immediate consequence of Propositions 10.1, 11.2 and Theorem 11.3, we deduce:

Corollary 11.4. *Assume that* $l > 0$. *Let* π *be an irreducible discrete series representation of* $G(W)$ *such that its theta lift* $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ *to* $H(V)$ *is nonzero and let* σ *be an irreducible constituent of* $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$. *Suppose that either* π *is supercuspidal or* $l = 1$. *Then* $\gamma(s, \sigma, \chi_W^{-1}, \psi)$ *is holomorphic and nonzero at* $s = \frac{l+1}{2}$, *and has a simple pole at* $s = \frac{l+1}{2} - i$ *for* $1 \leq i \leq [\frac{l-1}{2}]$.

This corollary ensures that the special value of $\gamma(s, \sigma, \chi_W^{-1}, \psi)$ in Theorem 15.1 below makes sense.

We note that if the desired identity in Theorem 11.3 holds for some ψ, then it holds for all ψ by the Ten Commandments. The rest of this section is devoted to the proof of Theorem 11.3.

11.4. **Representations with Iwahori-fixed vectors.**

We first consider the case when π has nonzero Iwahori-fixed vectors, so that π is a submodule of a principal series representation induced from a minimal parabolic subgroup P_0 of $G = G(W)$. Here, by an Iwahori subgroup, we mean the pointwise stabilizer of a fundamental chamber in the building of G.

If we write $W = \mathbb{H}^r \oplus W_{\mathrm{an}}$ with W_{an} anisotropic, then P_0 has Levi component $\mathrm{GL}_1(E)^r \times G(W_{\mathrm{an}})$, and we have

$$\pi \subset I_{P_0}^G(\mu_1 \otimes \cdots \otimes \mu_r \otimes \mathbf{1}_{G(W_{\mathrm{an}})})$$

for some unramified characters μ_i of $\mathrm{GL}_1(E)$, where $\mathbf{1}_{G(W_{\mathrm{an}})}$ denotes the trivial representation of $G(W_{\mathrm{an}})$. Similarly, if we write $V = \mathbb{H}^s \oplus V_{\mathrm{an}}$ with V_{an} anisotropic, then a minimal parabolic subgroup Q_0 of $H = H(V)$ has Levi component $\mathrm{GL}_1(E)^s \times H(V_{\mathrm{an}})$.

By Kudla's supercuspidal support theorem (Proposition 5.2), the supercuspidal support of σ is determined by the first occurrence of $\mathbf{1}_{G(W_{\mathrm{an}})}$ for the Witt tower containing V. We note the following lemma, which follows from case-by-case considerations.

Lemma 11.5. *Using the above notation, we have:*

(i) *In Cases B, C and D, the theta lift of* $\mathbf{1}_{G(W_{\mathrm{an}})}$ *to* $H(V_{\mathrm{an}})$ *is nonzero.*

(ii) *In Case A, consider the two Witt towers* $\{V_r^+\}$ *and* $\{V_r^-\}$ *such that* $\operatorname{disc} V_r^+ = 1$, *so that* $H(V_r^+)$ *is quasi-split. Let* V^+ *and* V^- *be the spaces at which the first occurrences of* $\mathbf{1}_{G(W_{\mathrm{an}})}$ *for these Witt towers occur. Then the following table describes* $(\dim V^+, \dim V^-)$ *in the various cases.*

$\dim W_{\mathrm{an}}$	$\dim V^{\pm}$	$(\dim V^+, \dim V^-)$
0	*odd*	$(1, 1)$
0	*even*	$(0, 2)$
1	*odd*	$(1, 3)$ *or* $(3, 1)$
1	*even*	$(0, 4)$
2	*odd*	$(3, 3)$
2	*even*	$(0, 6)$ *or* $(4, 2)$

In particular, except in the case when $\dim W_{\mathrm{an}} = 2$ *and* $\dim V^{\pm}$ *is odd, at least one of the two spaces* V^+ *and* V^- *is anisotropic.*

Now we have:

Lemma 11.6. *Assume that the theta lift of* $\mathbf{1}_{G(W_{\mathrm{an}})}$ *to* $H(V_{\mathrm{an}})$ *is nonzero, where* W_{an} *and* V_{an} *are the anisotropic kernels of* W *and* V *respectively. (This is only an assumption in Case A.) If* π *has nonzero Iwahori-fixed vectors, then the desired identity in Theorem 11.3 holds.*

Proof. In all cases, one knows that

$$\sigma_{\mathrm{an}} := \Theta_{V_{\mathrm{an}},W_{\mathrm{an}},\boldsymbol{\chi},\psi}(\mathbf{1}_{G(W_{\mathrm{an}})})$$

is a 1-dimensional character of $H(V_{\mathrm{an}})$. By Proposition 5.2, we have

$$\sigma \subset I^H_{Q_0}(\nu_1 \otimes \cdots \otimes \nu_s \otimes \sigma_{\mathrm{an}})$$

for some characters ν_i of $\mathrm{GL}_1(E)$. Moreover, putting $l_{\mathrm{an}} := l_{V_{\mathrm{an}},W_{\mathrm{an}}}$ as in §3.1, we see that if $r \geq s$, then

$$\{\mu_1,\dots,\mu_r\} = \{\chi_V|\cdot|_E^{(l-1)/2}, \chi_V|\cdot|_E^{(l-3)/2}, \dots, \chi_V|\cdot|_E^{(l_{\mathrm{an}}+1)/2}, \nu_1\chi_V\chi_W^{-1}, \dots, \nu_s\chi_V\chi_W^{-1}\},$$

whereas if $s \geq r$, then

$$\{\nu_1,\dots,\nu_s\} = \{\chi_W|\cdot|_E^{-(l+1)/2}, \chi_W|\cdot|_E^{-(l+3)/2}, \dots, \chi_W|\cdot|_E^{-(l_{\mathrm{an}}-1)/2}, \mu_1\chi_V^{-1}\chi_W, \dots, \mu_r\chi_V^{-1}\chi_W\}.$$

Using this identity of supercuspidal supports, the multiplicativity of standard γ-factors and the computation of $\gamma(s, \mathbf{1}_{G(W_{\mathrm{an}})}, \chi, \psi)$ and $\gamma(s, \sigma_{\mathrm{an}}, \chi, \psi)$ in [49, Theorem 4(7)], one can deduce the desired identity in Theorem 11.3. □

11.5. **Reduction to the supercuspidal case.** By a similar application of Kudla's supercuspidal support theorem and the multiplicativity of standard γ-factors as in the proof of the above lemma, we have the following reductions:

Lemma 11.7. (i) *Consider the Witt tower* $\{V_r\}$ *containing the space* V *and the associated theta lifts* $\theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi)$. *If the desired identity in Theorem 11.3 holds for* π *and* $\theta_{V_{r_0},W,\boldsymbol{\chi},\psi}(\pi)$ *for some* r_0 *such that* $\theta_{V_{r_0},W,\boldsymbol{\chi},\psi}(\pi) \neq 0$, *then it holds for* π *and* $\theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi)$ *for all* r *such that* $\theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi) \neq 0$.

(ii) *If the desired identity in Theorem 11.3 holds for all cases when* π *and* σ *are supercuspidal, then it holds in general.*

We shall henceforth assume that π and σ are supercuspidal. In Case C', we shall switch the roles of W and V. Then the desired identity in Theorem 11.3 will be proved by a global argument. For this, we globalize every object in sight.

11.6. **Globalization.** Consider first Cases B, C" and D. We can find:

- a totally imaginary number field $\mathbb{F}$ with ring of adeles $\mathbb{A} = \mathbb{A}_{\mathbb{F}}$ such that $\mathbb{F}_{v_0} = F$ for some place v_0;
- a nontrivial additive character Ψ of $\mathbb{F}\backslash\mathbb{A}$, and we set $\psi := \Psi_{v_0}$;
- a space $\mathbb{W}$ over $\mathbb{F}$ such that $\mathbb{W}_{v_0} = W$, with associated isometry group $G(\mathbb{W})$ and Hecke character $\boldsymbol{\chi}_W$ of $\mathbb{A}^\times$;
- a space $\mathbb{V}$ over $\mathbb{F}$ such that $\mathbb{V}_{v_0} = V$, with associated isometry group $H(\mathbb{V})$ and Hecke character $\boldsymbol{\chi}_V$ of $\mathbb{A}^\times$;
- an irreducible cuspidal automorphic representation Π of $G(\mathbb{W})(\mathbb{A})$ such that
  - $\Pi_{v_0} = \pi$,
  - Π_v has nonzero Iwahori-fixed vectors for all finite places $v \neq v_0$,

  via a construction by Henniart [31, Appendice 1] using Poincaré series;
- a Hecke character $\boldsymbol{\chi}$ of $\mathbb{A}^\times$ such that $\boldsymbol{\chi}_{v_0} = \chi$.

Now consider the more troublesome Case A. Recall that there is an exceptional case in Lemma 11.5, i.e., the case when $\dim W$ is even and $\dim V$ is odd. In this exceptional case, we shall switch the roles of W and V. Hence, when $\dim W$ and $\dim V$ are of opposite parity, we may assume without loss of generality that $\dim W$ is odd and $\dim V$ is even.

Then we can find:

- a totally real number field $\mathbb{F}$ such that $\mathbb{F}_{v_0} = F$ for some place v_0;
- a totally imaginary quadratic extension $\mathbb{E}$ of $\mathbb{F}$ such that $\mathbb{E}_{v_0} = E$;
- a nontrivial additive character Ψ of $\mathbb{F}\backslash\mathbb{A}$, and we set $\psi := \Psi_{v_0}$;
- a pair of Hecke characters $\boldsymbol{\chi}_V$ and $\boldsymbol{\chi}_W$ of $\mathbb{A}_{\mathbb{E}}^\times$ such that
$$\boldsymbol{\chi}_V|_{\mathbb{A}^\times} = \boldsymbol{\chi}_{\mathbb{E}}^m \quad \text{and} \quad \boldsymbol{\chi}_W|_{\mathbb{A}^\times} = \boldsymbol{\chi}_{\mathbb{E}}^n$$
and
$$\boldsymbol{\chi}_{V,v_0} = \chi_V \quad \text{and} \quad \boldsymbol{\chi}_{W,v_0} = \chi_W,$$
where $\boldsymbol{\chi}_{\mathbb{E}}$ is the quadratic character of $\mathbb{A}^\times$ associated to $\mathbb{E}/\mathbb{F}$ by class field theory;
- a space $\mathbb{W}$ over $\mathbb{E}$ such that
  - $\mathbb{W}_{v_0} = W$,
  - $\mathbb{W}_v$ is anisotropic for all archimedean places v;
- a space $\mathbb{V}$ over $\mathbb{E}$ such that
  - $\mathbb{V}_{v_0} = V$,
  - for all finite places $v \neq v_0$, the theta lift (with respect to Ψ_v, $\boldsymbol{\chi}_{V,v}$ and $\boldsymbol{\chi}_{W,v}$) of the trivial representation $\mathbf{1}_{G(\mathbb{W}_{v,\mathrm{an}})}$ of $G(\mathbb{W}_{v,\mathrm{an}})$ to $H(\mathbb{V}_{v,\mathrm{an}})$ is nonzero, where $\mathbb{W}_{v,\mathrm{an}}$ and $\mathbb{V}_{v,\mathrm{an}}$ are the anisotropic kernels of $\mathbb{W}_v$ and $\mathbb{V}_v$ respectively.

  This is possible since we do not need to consider the exceptional case in Lemma 11.5. We also note that given a collection $\{V_v\}$ of local hermitian spaces for all but one place v of $\mathbb{F}$, one can find a global hermitian space $\mathbb{V}$ with these localizations.
- an irreducible cuspidal automorphic representation Π of $G(\mathbb{W})(\mathbb{A})$ such that
  - $\Pi_{v_0} = \pi$,
  - Π_v has nonzero Iwahori-fixed vectors for all finite places $v \neq v_0$;
- a Hecke character $\boldsymbol{\chi}$ of $\mathbb{A}_{\mathbb{E}}^\times$ such that $\boldsymbol{\chi}_{v_0} = \chi$.

11.7. **Global-to-local argument.** In all cases, we consider the global Witt tower $\{\mathbb{V}_r\}$ containing the space $\mathbb{V}$ and the associated global theta lift
$$\Theta_{\mathbb{V}_r,\mathbb{W}}(\Pi)$$
(with respect to Ψ, $\boldsymbol{\chi}_V$ and $\boldsymbol{\chi}_W$) of Π to $H(\mathbb{V}_r)$. We know that $\Theta_{\mathbb{V}_r,\mathbb{W}}(\Pi)$ is zero if $\dim \mathbb{V}_r < m$, since the first occurrence of π for the local Witt tower containing the space V occurs at V. Let r_0 be the smallest non-negative integer such that $\Theta_{\mathbb{V}_{r_0},\mathbb{W}}(\Pi)$ is nonzero. Then $\Theta_{\mathbb{V}_{r_0},\mathbb{W}}(\Pi)$ is cuspidal by the tower property of global theta correspondence. Let $l_0 := l_{\mathbb{V}_{r_0},\mathbb{W}}$ be as in §3.1. For simplicity, we assume that $l_0 > 0$.

Now let Σ be an irreducible constituent of $\Theta_{\mathbb{V}_{r_0},\mathbb{W}}(\Pi)$. By Lemma 11.6 and [55], for all finite places $v \neq v_0$, we have
$$\frac{\gamma(s, \Pi_v, \boldsymbol{\chi}_v \boldsymbol{\chi}_{W,v}, \Psi_v)}{\gamma(s, \Sigma_v, \boldsymbol{\chi}_v \boldsymbol{\chi}_{V,v}, \Psi_v)} = \prod_{i=1}^{l_0} \gamma(s + \tfrac{l_0+1}{2} - i, \boldsymbol{\chi}_v \boldsymbol{\chi}_{V,v} \boldsymbol{\chi}_{W,v}, \Psi_v).$$
Moreover, for all archimedean places v, either $\mathbb{F}_v = \mathbb{C}$ in Cases B, C" and D, or $G(\mathbb{W}_v)$ is compact in Case A. In such cases, the theta correspondence is completely understood, and can be described in terms of the local Langlands correspondence (see [2], [39], [41]). From this, one can deduce the above identity for archimedean places as well.

Consider the global functional equation of standard L-functions. We have:

$$\begin{aligned}
&\prod_{v\in S}\gamma(s,\Pi_v,\boldsymbol{\chi}_v\boldsymbol{\chi}_{W,v},\Psi_v)\cdot\frac{L^S(1-s,\Pi^\vee,\boldsymbol{\chi}^{-1}\boldsymbol{\chi}_W^{-1})}{L^S(s,\Pi,\boldsymbol{\chi}\boldsymbol{\chi}_W)}=1\\
&=\prod_{v\in S}\left(\gamma(s,\Sigma_v,\boldsymbol{\chi}_v\boldsymbol{\chi}_{V,v},\Psi_v)\cdot\prod_{i=1}^{l_0}\gamma(s+\tfrac{l_0+1}{2}-i,\boldsymbol{\chi}_v\boldsymbol{\chi}_{V,v}\boldsymbol{\chi}_{W,v},\Psi_v)\right)\\
&\qquad\times\frac{L^S(1-s,\Sigma^\vee,\boldsymbol{\chi}^{-1}\boldsymbol{\chi}_V^{-1})}{L^S(s,\Sigma,\boldsymbol{\chi}\boldsymbol{\chi}_V)}\cdot\prod_{i=1}^{l_0}\frac{L^S(1-s-\tfrac{l_0+1}{2}+i,\boldsymbol{\chi}^{-1}\boldsymbol{\chi}_V^{-1}\boldsymbol{\chi}_W^{-1})}{L^S(s+\tfrac{l_0+1}{2}-i,\boldsymbol{\chi}\boldsymbol{\chi}_V\boldsymbol{\chi}_W)}
\end{aligned}$$

for a sufficiently large finite set S of places of $\mathbb{F}$. Using this identity and our knowledge that the desired identity holds for all places outside v_0, we conclude that the desired identity holds for $\Pi_{v_0}\cong\pi$ and $\Sigma_{v_0}\cong\Theta_{\mathbb{V}_{r_0,v_0},W,\boldsymbol{\chi},\psi}(\pi)$. In view of Lemma 11.7, this completes the proof of Theorem 11.3.

12. **Plancherel measures and local theta correspondence**

In this section, we study the behaviour of another representation-theoretic invariant, the so-called Plancherel measure, under the theta correspondence. We briefly recall the definition of the Plancherel measures.

12.1. **Plancherel measures.**

Let G be a reductive linear algebraic group over F. Let $P=MU$ be a parabolic subgroup of G with Levi component M and unipotent radical U. Let π be an irreducible admissible representation of M on the space V_π. Consider the normalized induced representation

$$I_P^G(\pi):=\mathrm{Ind}_P^G(\pi)$$

of G, which is realized on the space of smooth functions

$$f:G\longrightarrow V_\pi$$

such that

$$f(mug)=\delta_P(m)^{1/2}\pi(m)f(g)\quad\text{for } m\in M,\ u\in U,\ g\in G.$$

Here δ_P is the modulus character of P.

We define an intertwining operator

$$J_{\bar{P}|P}(\pi):I_P^G(\pi)\longrightarrow I_{\bar{P}}^G(\pi)$$

by

$$J_{\bar{P}|P}(\pi)f(g)=\int_{\bar{U}}f(\bar{u}g)\,d\bar{u}$$

for $f\in I_P^G(\pi)$ and $g\in G$. Here $\bar{P}=M\bar{U}$ is the parabolic subgroup of G opposite to P and $\bar{U}$ is its unipotent radical. Then there exists a rational function μ of π such that

$$J_{P|\bar{P}}(\pi)\circ J_{\bar{P}|P}(\pi)=\mu(\pi)^{-1}.$$

We remark that the above definition of μ differs from that in [76] by a constant. At this point, the function μ depends on the choice of Haar measures on U and $\bar{U}$. In Appendix B, we describe exactly our choice of the Haar measures on U and $\bar{U}$, and recall some basic properties of Plancherel measures which we will need. For the rest of this section, we shall freely use the results in Appendix B.

12.2. **Behaviour of Plancherel measures.** Suppose that

$$W' = \mathbb{H}^k \oplus W \quad \text{and} \quad V' = \mathbb{H}^k \oplus V,$$

and let $G' = G(W')$ and $H' = H(V')$ be the isometry groups of W' and V' respectively. We consider the maximal parabolic subgroups $P = M_P U_P$ and $Q = M_Q U_Q$ of G' and H' with Levi components

$$M_P = \mathrm{GL}_k(E) \times G(W) \quad \text{and} \quad M_Q = \mathrm{GL}_k(E) \times H(V)$$

respectively.

The following theorem describes how the Plancherel measure behaves under the theta correspondence.

Theorem 12.1. *Let π and σ be irreducible admissible representations of $G(W)$ and $H(V)$ respectively, such that σ is an irreducible constituent of $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$. Let τ be an irreducible admissible representation of $\mathrm{GL}_k(E)$ and put $\tau_s = \tau|\det|_E^s$ for $s \in \mathbb{C}$. Then we have:*

$$\frac{\mu(\tau_s\chi_V \otimes \pi)}{\mu(\tau_s\chi_W \otimes \sigma)} = \gamma(s - \tfrac{l-1}{2}, \tau, \psi_E) \cdot \gamma(-s - \tfrac{l-1}{2}, \tau^\vee, \bar{\psi}_E).$$

Observe that when $l = 0$, the product of the γ-factors on the right-hand side is equal to 1 by the functional equation of γ-factors.

Corollary 12.2. *Let π and σ be irreducible supercuspidal representations of $G(W)$ and $H(V)$ respectively, such that $\sigma = \Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$. Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_k(E)$. Let $s \in \mathbb{R}$. If $l \neq 0$, $k = 1$ and $\tau = \mathbb{1}_{E^\times}$, we assume that $s \notin \{\pm\frac{l-1}{2}, \pm\frac{l+1}{2}\}$. Then $I_P^{G'}(\tau_s\chi_V \otimes \pi)$ is reducible if and only if $I_Q^{H'}(\tau_s\chi_W \otimes \sigma)$ is reducible.*

Proof. Since τ is supercuspidal, we have

$$L(s,\tau) = \begin{cases} (1 - \tau(\varpi_E) \cdot q_E^{-s})^{-1} & \text{if } \tau \text{ is an unramified character of } \mathrm{GL}_1(E), \\ 1 & \text{otherwise.} \end{cases}$$

Thus the corollary follows from Theorem 12.1 and Proposition B.6, noting that $\gamma(s, \tau, \psi_E)$ is equal to

$$\frac{L(1-s, \tau^\vee)}{L(s,\tau)}$$

up to a nonzero exponential function of s. □

The proof of Theorem 12.1 is similar to that of Theorem 11.3, though one must exercise care in globalizing the local objects. We simply indicate the necessary modifications here.

12.3. **Representations with Iwahori-fixed vectors.** We first need the analog of Lemma 11.6, which proves Theorem 12.1 when π and τ have nonzero Iwahori-fixed vectors. We shall assume here that $G(W)$ is quasi-split, so that if we write $W = \mathbb{H}^r \oplus W_{\mathrm{an}}$ with W_{an} anisotropic, then we have

$$\dim W_{\mathrm{an}} \leq \begin{cases} 1 & \text{in Case A,} \\ 2 & \text{in Cases BD,} \\ 0 & \text{in Case C.} \end{cases}$$

As in §11.4, we have

$$\pi \subset I_{P_0}^G(\mu_1 \otimes \cdots \otimes \mu_r \otimes \mathbf{1}_{G(W_{\mathrm{an}})}),$$

where P_0 is a Borel subgroup of $G = G(W)$. Similarly, τ is a submodule of a principal series representation induced from a Borel subgroup of $\mathrm{GL}_k(E)$. Then by the multiplicativity of Plancherel measures (Propositions B.3 and B.4), one can reduce the computation of the Plancherel measure

$\mu(\tau_s\chi_V \otimes \pi)$ to that for the rank 1 groups SL_2, SO_3, SU_3 and Mp_2. For these groups, the Plancherel measure has been computed explicitly (see [40], [73]), and can be expressed as a product of Tate's γ-factors (see Proposition B.5). Thus, one can express $\mu(\tau_s\chi_V \otimes \pi)$ explicitly as a product of Tate's γ-factors.

Now suppose that the theta lift of $\mathbf{1}_{G(W_{\mathrm{an}})}$ to $H(V_{\mathrm{an}})$ is nonzero (as in Lemma 11.6), where V_{an} is the anisotropic kernel of V. If $H(V)$ is quasi-split, then σ is also a submodule of a principal series representation induced from a Borel subgroup of $H(V)$ and one can also compute the Plancherel measure $\mu(\tau_s\chi_W \otimes \sigma)$ explicitly. From this and Kudla's supercuspidal support theorem, one can compare $\mu(\tau_s\chi_V \otimes \pi)$ and $\mu(\tau_s\chi_W \otimes \sigma)$, and deduce the desired identity in Theorem 12.1. In other words, we have:

Lemma 12.3. *Assume that $G(W)$ and $H(V)$ are quasi-split. Assume further that the theta lift of $\mathbf{1}_{G(W_{\mathrm{an}})}$ to $H(V_{\mathrm{an}})$ is nonzero, where W_{an} and V_{an} are the anisotropic kernels of W and V respectively. (This is only an assumption in Case A.) If π and τ have nonzero Iwahori-fixed vectors, then the desired identity in Theorem 12.1 holds.*

12.4. **Reduction to the supercuspidal case.** By an application of Kudla's supercuspidal support theorem and the multiplicativity of Plancherel measures (Propositions B.3 and B.4), we have the following analog of Lemma 11.7:

Lemma 12.4. (i) *Consider the Witt tower $\{V_r\}$ containing the space V and the associated theta lifts $\theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi)$. If the desired identity in Theorem 12.1 holds for π and $\theta_{V_{r_0},W,\boldsymbol{\chi},\psi}(\pi)$ for some r_0 such that $\theta_{V_{r_0},W,\boldsymbol{\chi},\psi}(\pi) \neq 0$, then it holds for π and $\theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi)$ for all r such that $\theta_{V_r,W,\boldsymbol{\chi},\psi}(\pi) \neq 0$.*

(ii) *If the desired identity in Theorem 12.1 holds for all cases when π, σ and τ are unitary and supercuspidal, then it holds in general.*

We shall henceforth assume that π, σ and τ are unitary and supercuspidal. In Case C', we shall switch the roles of W and V.

12.5. **Globalization.** As in §11.6, we globalize the various objects to give a global-to-local argument. However, since Lemma 12.3 is more restrictive than Lemma 11.6, one must exercise care.

In Cases B, C" and D, we can find:

- a totally imaginary number field $\mathbb{F}$ such that $\mathbb{F}_{v_1} = \mathbb{F}_{v_2} = F$ for some two places v_1 and v_2;
- a nontrivial additive character Ψ of $\mathbb{F}\backslash\mathbb{A}$ such that $\Psi_{v_1} = \Psi_{v_2}$;
- a space $\mathbb{W}$ over $\mathbb{F}$ such that
  - $\mathbb{W}_{v_1} = \mathbb{W}_{v_2} = W$,
  - $G(\mathbb{W}_v)$ is quasi-split for all finite places $v \notin \{v_1, v_2\}$;
- a space $\mathbb{V}$ over $\mathbb{F}$ such that
  - $\mathbb{V}_{v_1} = \mathbb{V}_{v_2} = V$,
  - $H(\mathbb{V}_v)$ is quasi-split for all finite places $v \notin \{v_1, v_2\}$;
- an irreducible cuspidal automorphic representation Π of $G(\mathbb{W})(\mathbb{A})$ such that
  - $\Pi_{v_1} = \Pi_{v_2} = \pi$,
  - Π_v has nonzero Iwahori-fixed vectors for all finite places $v \notin \{v_1, v_2\}$;
- an irreducible cuspidal automorphic representation $\mathfrak{T}$ of $\mathrm{GL}_k(\mathbb{A})$ such that
  - $\mathfrak{T}_{v_1} = \mathfrak{T}_{v_2} = \tau$,

  - $\mathfrak{T}_v$ has nonzero Iwahori-fixed vectors for all finite places $v \notin \{v_1, v_2\}$.

In Case A, we can find:

- a totally real number field $\mathbb{F}$ such that
  - $\mathbb{F}_{v_1} = \mathbb{F}_{v_2} = F$ for some two places v_1 and v_2,
  - $[\mathbb{F} : \mathbb{Q}] = \mathbf{d}$ for any given integer $\mathbf{d} \geq [F : \mathbb{Q}_p]$,

  by [11, Lemma 15.3];
- a totally imaginary quadratic extension $\mathbb{E}$ of $\mathbb{F}$ such that $\mathbb{E}_{v_1} = \mathbb{E}_{v_2} = E$;
- a nontrivial additive character Ψ of $\mathbb{F}\backslash\mathbb{A}$ such that $\Psi_{v_1} = \Psi_{v_2}$;
- a pair of Hecke characters $\boldsymbol{\chi}_V$ and $\boldsymbol{\chi}_W$ of $\mathbb{A}_{\mathbb{E}}^\times$ such that

$$\boldsymbol{\chi}_V|_{\mathbb{A}^\times} = \boldsymbol{\chi}_{\mathbb{E}}^m \quad \text{and} \quad \boldsymbol{\chi}_W|_{\mathbb{A}^\times} = \boldsymbol{\chi}_{\mathbb{E}}^n$$

  and

$$\boldsymbol{\chi}_{V,v_1} = \boldsymbol{\chi}_{V,v_2} = \chi_V \quad \text{and} \quad \boldsymbol{\chi}_{W,v_1} = \boldsymbol{\chi}_{W,v_2} = \chi_W;$$

- a space $\mathbb{W}$ over $\mathbb{E}$ such that
  - $\mathbb{W}_{v_1} = \mathbb{W}_{v_2} = W$,
  - $G(\mathbb{W}_v)$ is quasi-split for all finite places $v \notin \{v_1, v_2\}$,
  - $\mathbb{W}_v$ is anisotropic for all archimedean places v.

  This is possible unless $\dim W \equiv 2 \mod 4$ and $[\mathbb{F} : \mathbb{Q}]$ is odd. However, since we may assume that $[\mathbb{F} : \mathbb{Q}]$ is even, this is irrelevant.
- a space $\mathbb{V}$ over $\mathbb{E}$ such that
  - $\mathbb{V}_{v_1} = \mathbb{V}_{v_2} = V$,
  - for all finite places $v \notin \{v_1, v_2\}$, $H(\mathbb{V}_v)$ is quasi-split and the theta lift (with respect to Ψ_v, $\boldsymbol{\chi}_{V,v}$ and $\boldsymbol{\chi}_{W,v}$) of the trivial representation $\mathbf{1}_{G(\mathbb{W}_{v,\mathrm{an}})}$ of $G(\mathbb{W}_{v,\mathrm{an}})$ to $H(\mathbb{V}_{v,\mathrm{an}})$ is nonzero, where $\mathbb{W}_{v,\mathrm{an}}$ and $\mathbb{V}_{v,\mathrm{an}}$ are the anisotropic kernels of $\mathbb{W}_v$ and $\mathbb{V}_v$ respectively.

  This is possible by Lemma 11.5 and the assumption that $G(\mathbb{W}_v)$ is quasi-split.
- an irreducible cuspidal automorphic representation Π of $G(\mathbb{W})(\mathbb{A})$ such that
  - $\Pi_{v_1} = \Pi_{v_2} = \pi$,
  - Π_v has nonzero Iwahori-fixed vectors for all finite places $v \notin \{v_1, v_2\}$;
- an irreducible cuspidal automorphic representation $\mathfrak{T}$ of $\mathrm{GL}_k(\mathbb{A}_{\mathbb{E}})$ such that
  - $\mathfrak{T}_{v_1} = \mathfrak{T}_{v_2} = \tau$,
  - $\mathfrak{T}_v$ has nonzero Iwahori-fixed vectors for all finite places $v \notin \{v_1, v_2\}$.

12.6. **Global-to-local argument.** Now we repeat the global-to-local argument in §11.7, using the global functional equation of intertwining operators.

Let Σ be an irreducible constituent of the nonzero cuspidal automorphic representation $\Theta_{\mathbb{V}_{r_0},\mathbb{W}}(\Pi)$ of $H(\mathbb{V}_{r_0})(\mathbb{A})$ as in §11.7. By Lemma 12.3 and [55], for all finite places $v \notin \{v_1, v_2\}$, we have

$$\frac{\mu(\mathfrak{T}_{v,s}\boldsymbol{\chi}_{V,v} \otimes \Pi_v)}{\mu(\mathfrak{T}_{v,s}\boldsymbol{\chi}_{W,v} \otimes \Sigma_v)} = \gamma(s - \tfrac{l_0-1}{2}, \mathfrak{T}_v, \Psi_v) \cdot \gamma(-s - \tfrac{l_0-1}{2}, \mathfrak{T}_v^\vee, \bar{\Psi}_v),$$

where $l_0 = l_{\mathbb{V}_{r_0},\mathbb{W}}$. Moreover, for all archimedean places v, the Plancherel measure has been computed explicitly (see [3], [26]) and the theta correspondence is completely understood (see [2], [39], [41]). From this, one can deduce the above identity for archimedean places as well.

Suppose that

$$\mathbb{W}' = \mathbb{H}^k \oplus \mathbb{W} \quad \text{and} \quad \mathbb{V}' = \mathbb{H}^k \oplus \mathbb{V}_{r_0},$$

and let $\mathbb{G}' = G(\mathbb{W}')$ and $\mathbb{H}' = H(\mathbb{V}')$ be the isometry groups of $\mathbb{W}'$ and $\mathbb{V}'$ respectively. We consider the maximal parabolic subgroups $\mathbb{P} = \mathbb{M}_{\mathbb{P}}\mathbb{U}_{\mathbb{P}}$ and $\mathbb{Q} = \mathbb{M}_{\mathbb{Q}}\mathbb{U}_{\mathbb{Q}}$ of $\mathbb{G}'$ and $\mathbb{H}'$ with Levi components

$$\mathbb{M}_{\mathbb{P}} = \mathrm{GL}_k(\mathbb{E}) \times G(\mathbb{W}) \quad \text{and} \quad \mathbb{M}_{\mathbb{Q}} = \mathrm{GL}_k(\mathbb{E}) \times H(\mathbb{V}_{r_0})$$

respectively, where $\mathbb{E} = \mathbb{F}$ in Cases B, C" and D. Let $r_{\mathbb{P}}$ and $r_{\mathbb{Q}}$ be the adjoint representations of ${}^L\mathbb{M}_{\mathbb{P}}$ and ${}^L\mathbb{M}_{\mathbb{Q}}$ on $\mathrm{Lie}({}^L\mathbb{G}')/\mathrm{Lie}({}^L\mathbb{M}_{\mathbb{P}})$ and $\mathrm{Lie}({}^L\mathbb{H}')/\mathrm{Lie}({}^L\mathbb{M}_{\mathbb{Q}})$ respectively. Then we have the functional equation of global intertwining operators (Proposition B.7)

$$\left(\prod_{v\in S} J_{\mathbb{P}_v|\bar{\mathbb{P}}_v}(\mathfrak{T}_{v,s}\boldsymbol{\chi}_{V,v} \otimes \Pi_v) \circ J_{\bar{\mathbb{P}}_v|\mathbb{P}_v}(\mathfrak{T}_{v,s}\boldsymbol{\chi}_{V,v} \otimes \Pi_v)\right) \cdot \frac{L^S(0, \mathfrak{T}_s\boldsymbol{\chi}_V \otimes \Pi, r_{\mathbb{P}})}{L^S(1, \mathfrak{T}_s\boldsymbol{\chi}_V \otimes \Pi, r_{\mathbb{P}}^{\vee})} = 1, \tag{12.1}$$

$$\left(\prod_{v\in S} J_{\mathbb{Q}_v|\bar{\mathbb{Q}}_v}(\mathfrak{T}_{v,s}\boldsymbol{\chi}_{W,v} \otimes \Sigma_v) \circ J_{\bar{\mathbb{Q}}_v|\mathbb{Q}_v}(\mathfrak{T}_{v,s}\boldsymbol{\chi}_{W,v} \otimes \Sigma_v)\right) \cdot \frac{L^S(0, \mathfrak{T}_s\boldsymbol{\chi}_W \otimes \Sigma, r_{\mathbb{Q}})}{L^S(1, \mathfrak{T}_s\boldsymbol{\chi}_W \otimes \Sigma, r_{\mathbb{Q}}^{\vee})} = 1, \tag{12.2}$$

and the functional equation of automorphic L-functions
(12.3)

$$\left(\prod_{v\in S} \gamma(s - \tfrac{l_0-1}{2}, \mathfrak{T}_v, \Psi_v)^{-1} \cdot \gamma(-s - \tfrac{l_0-1}{2}, \mathfrak{T}_v^{\vee}, \bar{\Psi}_v)^{-1}\right) \cdot \frac{L^S(s - \frac{l_0-1}{2}, \mathfrak{T})}{L^S(-s + \frac{l_0+1}{2}, \mathfrak{T}^{\vee})} \cdot \frac{L^S(-s - \frac{l_0-1}{2}, \mathfrak{T}^{\vee})}{L^S(s + \frac{l_0+1}{2}, \mathfrak{T})} = 1$$

for a sufficiently large finite set S of places of $\mathbb{F}$.

Observing that $1 = 1 \times 1$, so that

$$(12.1) = (12.2) \times (12.3),$$

and using our knowledge that the desired identity holds for all places outside $\{v_1, v_2\}$, we obtain

$$\frac{\mu(\tau_s\chi_V \otimes \pi)^2}{\mu(\tau_s\chi_W \otimes \sigma_0)^2} = \gamma(s - \tfrac{l_0-1}{2}, \tau, \psi_E)^2 \cdot \gamma(-s - \tfrac{l_0-1}{2}, \tau^{\vee}, \bar{\psi}_E)^2,$$

where $\sigma_0 = \Sigma_{v_1}$ and $\psi = \Psi_{v_1}$. Using the non-negativity of Plancherel measures on the imaginary axis and the observation that

$$\begin{aligned} \gamma(s - \tfrac{l_0-1}{2}, \tau, \psi_E) \cdot \gamma(-s - \tfrac{l_0-1}{2}, \tau^{\vee}, \bar{\psi}_E) &= \gamma(s - \tfrac{l_0-1}{2}, \tau, \psi_E) \cdot \overline{\gamma(s - \tfrac{l_0-1}{2}, \tau, \psi_E)} \\ &= |\gamma(s - \tfrac{l_0-1}{2}, \tau, \psi_E)|^2 \end{aligned}$$

for $\mathrm{Re}(s) = 0$, we conclude that the desired identity holds for π and $\sigma_0 \cong \Theta_{\mathbb{V}_{r_0,v_1},W,\boldsymbol{\chi},\psi}(\pi)$. In view of Lemma 12.4, this completes the proof of Theorem 12.1.

13. **Formal degrees**

We now come to the main object of interest in this paper: the formal degree of a discrete series representation. In this section, we recall the definition of the formal degrees and investigate the behaviour of the formal degree when one passes from an orthogonal group to a special orthogonal group, or from a similitude group to an isometry group.

13.1. **Haar measures.** Let G be a reductive linear algebraic group over F and G^0 its identity component. Let $\mathcal{G}$ be the split form of G^0/A_G, where A_G is the split component of the center of G. We extend $\mathcal{G}$ to a Chevalley group scheme over $\mathfrak{o}_F$. Choose an isomorphism

$$\eta : G^0/A_G \longrightarrow \mathcal{G} \quad \text{over } \bar{F}$$

and a differential form $\omega_{\mathcal{G}}$ of top degree on $\mathcal{G}$ over $\mathfrak{o}_F$ with nonzero reduction, and put $\omega_G = \eta^*(\omega_{\mathcal{G}})$. Let $dg = dg_\psi$ denote the Haar measure on G^0/A_G determined by ω_G and the self-dual measure on F with respect to ψ. Then dg does not depend on the choice of η and $\omega_{\mathcal{G}}$ (see [21, §5]) and extends naturally to a Haar measure on G/A_G.

Let G' be another reductive linear algebraic group over F such that $A_{G'} = \{1\}$ and dg' the Haar measure on G' defined above. Suppose that $f : G \to G'$ is a central isogeny defined over F. If $\mathcal{U}$ is a compact open subset of G such that $f|_{\mathcal{U}} : \mathcal{U} \to f(\mathcal{U})$ is a homeomorphism, then we have

$$\mathrm{vol}(f(\mathcal{U}), dg') = |\#N(\bar{F})|_F \cdot \mathrm{vol}(\mathcal{U}, dg),$$

where $N = \ker f$.

When W is a symplectic space over F, we take the Haar measure on $\mathrm{Mp}(W)$ such that its push-forward by the projection $\mathrm{pr} : \mathrm{Mp}(W) \to \mathrm{Sp}(W)$ is the Haar measure on $\mathrm{Sp}(W)$ defined above, i.e., if $\mathcal{U}$ is a compact open subset of $\mathrm{Sp}(W)$, then we have

$$\mathrm{vol}(\mathrm{pr}^{-1}(\mathcal{U})) = \mathrm{vol}(\mathcal{U}).$$

13.2. **Definition.** Let π be an irreducible discrete series representation of G. We fix an invariant hermitian inner product $(\cdot,\cdot)$ on π. Then the formal degree of π with respect to the Haar measure dg on G/A_G defined above is a positive real number $\deg \pi$ defined by the Schur orthogonality relation:

$$\int_{G/A_G} (\pi(g)v_1, v_2) \cdot \overline{(\pi(g)v_3, v_4)}\, dg = \frac{1}{\deg \pi} \cdot (v_1, v_3) \cdot \overline{(v_2, v_4)}$$

for $v_1, \dots, v_4 \in \pi$.

13.3. **Orthogonal groups vs. special orthogonal groups.** When W is a quadratic space over F, we consider the associated orthogonal group $G = \mathrm{O}(W)$ and its identity component $G^0 = \mathrm{SO}(W)$. Let π be an irreducible admissible representation of G and π_0 an irreducible constituent of $\pi|_{G^0}$. Fix an element $\mathbf{s} \in G \smallsetminus G^0$ and put $(\mathbf{s}\pi_0)(g) = \pi_0(\mathbf{s}^{-1}g\mathbf{s})$ for $g \in G^0$. Then $\mathbf{s}\pi_0 \cong \pi_0$ if and only if $\pi \otimes \det \ncong \pi$. Moreover, we have:

- If $\pi \otimes \det \ncong \pi$, then we have
$$\pi|_{G^0} = \pi_0, \quad \mathrm{Ind}_{G^0}^{G}(\pi_0) = \pi \oplus (\pi \otimes \det).$$
- If $\pi \otimes \det \cong \pi$, then we have
$$\pi|_{G^0} = \pi_0 \oplus \mathbf{s}\pi_0, \quad \mathrm{Ind}_{G^0}^{G}(\pi_0) = \pi.$$

Lemma 13.1. *Assume that π (and hence π_0) is square integrable. Then we have*

$$\deg \pi_0 = \deg \pi \times \begin{cases} 2 & \text{if } \pi \otimes \det \ncong \pi, \\ 1 & \text{if } \pi \otimes \det \cong \pi. \end{cases}$$

Proof. Recall that

$$\int_G f(g)\, dg = \int_{G^0} f(g)\, dg + \int_{G^0} f(g\mathbf{s})\, dg$$

for $f \in L^1(G)$. Let $v \in \pi_0$ and put $\|v\| = \sqrt{(v,v)}$. Then we have

$$\frac{1}{\deg \pi} \cdot \|v\|^4 = \int_G |(\pi(g)v, v)|^2 \, dg = \int_{G^0} |(\pi(g)v, v)|^2 \, dg + \int_{G^0} |(\pi(g\mathbf{s})v, v)|^2 \, dg$$

and

$$\int_{G^0} |(\pi(g)v, v)|^2 \, dg = \frac{1}{\deg \pi_0} \cdot \|v\|^4.$$

If $\pi \otimes \det \not\cong \pi$, then we have $\pi(\mathbf{s})v \in \pi_0$, so that

$$\int_{G^0} |(\pi(g\mathbf{s})v, v)|^2 \, dg = \frac{1}{\deg \pi_0} \cdot \|\pi(\mathbf{s})v\|^2 \cdot \|v\|^2 = \frac{1}{\deg \pi_0} \cdot \|v\|^4.$$

If $\pi \otimes \det \cong \pi$, then we have $\pi(\mathbf{s})v \notin \pi_0$, so that

$$\int_{G^0} |(\pi(g\mathbf{s})v, v)|^2 \, dg = 0$$

by the Schur orthogonality relation. This completes the proof. □

13.4. **Isometry groups vs. similitude groups.** Let $\tilde{G}$ be a connected reductive linear algebraic group over F and G a connected reductive subgroup of $\tilde{G}$ over F. Put $G' = \tilde{G}/A_{\tilde{G}}$, where $A_{\tilde{G}}$ is the split component of the center of $\tilde{G}$. We assume that:

- G contains the derived group of $\tilde{G}$,
- the natural map $f : G \to G'$ is a central isogeny.

Let

$$N = \ker f \quad \text{and} \quad \mathcal{C} = \operatorname{coker}(G(F) \longrightarrow G'(F)).$$

Then we have an exact sequence

$$1 \longrightarrow G(F)/N(F) \longrightarrow G'(F) \longrightarrow \mathcal{C} \longrightarrow 1.$$

Now let $\tilde{\pi}$ be an irreducible discrete series representation of $\tilde{G}(F)$ on the space $V_{\tilde{\pi}}$. We fix an invariant hermitian inner product $(\cdot, \cdot)$ on $V_{\tilde{\pi}}$. Put

$$\mathfrak{X}_{\tilde{\pi}} = \{\chi \in \operatorname{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^\times) \,|\, \tilde{\pi} \otimes \chi \cong \tilde{\pi}\}.$$

For each $\chi \in \mathfrak{X}_{\tilde{\pi}}$, there exists a nonzero element $\mathcal{A}_\chi \in \mathrm{GL}(V_{\tilde{\pi}})$ such that

$$\mathcal{A}_\chi \circ (\tilde{\pi} \otimes \chi)(\tilde{g}) = \tilde{\pi}(\tilde{g}) \circ \mathcal{A}_\chi$$

for all $\tilde{g} \in \tilde{G}$. We may assume that $\mathcal{A}_\chi$ is unitary with respect to $(\cdot, \cdot)$.

Let $\mathfrak{S}_{\tilde{\pi}}$ be the subgroup of $\mathrm{GL}(V_{\tilde{\pi}})$ generated by $\{\mathcal{A}_\chi \,|\, \chi \in \mathfrak{X}_{\tilde{\pi}}\}$ and $\{z \cdot \mathrm{id}_{V_{\tilde{\pi}}} \,|\, z \in \mathbb{C}^1\}$. Then the map $\mathcal{A}_\chi \mapsto \chi$ induces an exact sequence

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathfrak{S}_{\tilde{\pi}} \longrightarrow \mathfrak{X}_{\tilde{\pi}} \longrightarrow 1.$$

Moreover, $\mathfrak{S}_{\tilde{\pi}} \times G(F)$ acts naturally on $V_{\tilde{\pi}}$ and $(\cdot, \cdot)$ is $\mathfrak{S}_{\tilde{\pi}}$-invariant. By [35, Corollary 2.10], we have a decomposition

$$V_{\tilde{\pi}} \cong \bigoplus_{\eta \in \Pi(\mathfrak{S}_{\tilde{\pi}})} \eta \boxtimes \pi_\eta$$

as representations of $\mathfrak{S}_{\tilde{\pi}} \times G(F)$, where $\Pi(\mathfrak{S}_{\tilde{\pi}})$ is the set of equivalence classes of irreducible representations of $\mathfrak{S}_{\tilde{\pi}}$ such that $z \cdot \mathrm{id}_{V_{\tilde{\pi}}}$ with $z \in \mathbb{C}^1$ acts as the scalar z, and π_η is an irreducible discrete series representation of $G(F)$.

Lemma 13.2. *For* $\eta \in \Pi(\mathfrak{S}_{\tilde{\pi}})$, *we have*

$$\deg \pi_\eta = \frac{\# Z_{\widehat{G'}}^{\Gamma}}{\# Z_{\widehat{G}}^{\Gamma}} \cdot \frac{\dim \eta}{\# \mathfrak{X}_{\tilde{\pi}}} \cdot \deg \tilde{\pi},$$

where $\Gamma = \mathrm{Gal}(\bar{F}/F)$.

Proof. Let v be an element in the π_η-isotypic component of $V_{\tilde{\pi}}$. Since the function

$$\frac{1}{\# \mathcal{C}} \sum_{\chi \in \mathrm{Hom}(\mathcal{C}, \mathbb{C}^\times)} \chi$$

on $G'(F)$ is the characteristic function of $G(F)/N(F)$, we have

$$\begin{aligned} \frac{1}{\deg \pi_\eta} \cdot \|v\|^4 &= \int_{G(F)} (\tilde{\pi}(g)v, v) \cdot \overline{(\tilde{\pi}(g)v, v)} \, dg \\ &= \# N(F) \cdot \int_{G(F)/N(F)} (\tilde{\pi}(g)v, v) \cdot \overline{(\tilde{\pi}(g)v, v)} \, dg \\ &= \frac{\# N(F)}{\# \mathcal{C}} \cdot \sum_{\chi \in \mathrm{Hom}(\mathcal{C}, \mathbb{C}^\times)} \int_{G'(F)} ((\tilde{\pi} \otimes \chi)(g')v, v) \cdot \overline{(\tilde{\pi}(g')v, v)} \cdot |\# N(\bar{F})|_F^{-1} \, dg'. \end{aligned}$$

By the Schur orthogonality relation, we have

$$\int_{G'(F)} ((\tilde{\pi} \otimes \chi)(g')v, v) \cdot \overline{(\tilde{\pi}(g')v, v)} \, dg' = 0$$

unless $\chi \in \mathfrak{X}_{\tilde{\pi}}$. Moreover, if $\chi \in \mathfrak{X}_{\tilde{\pi}}$, then we have

$$\begin{aligned} \int_{G'(F)} ((\tilde{\pi} \otimes \chi)(g')v, v) \cdot \overline{(\tilde{\pi}(g')v, v)} \, dg' &= \int_{G'(F)} (\tilde{\pi}(g')\mathcal{A}_\chi v, \mathcal{A}_\chi v) \cdot \overline{(\tilde{\pi}(g')v, v)} \, dg' \\ &= \frac{1}{\deg \tilde{\pi}} \cdot (\mathcal{A}_\chi v, v) \cdot \overline{(\mathcal{A}_\chi v, v)}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \frac{1}{\deg \pi_\eta} \cdot \|v\|^4 &= \frac{\# N(F)}{\# \mathcal{C} \cdot |\# N(\bar{F})|_F} \cdot \frac{1}{\deg \tilde{\pi}} \cdot \sum_{\chi \in \mathfrak{X}_{\tilde{\pi}}} (\mathcal{A}_\chi v, v) \cdot \overline{(\mathcal{A}_\chi v, v)} \\ &= \frac{\# N(F)}{\# \mathcal{C} \cdot |\# N(\bar{F})|_F} \cdot \frac{1}{\deg \tilde{\pi}} \cdot \# \mathfrak{X}_{\tilde{\pi}} \cdot \int_{\mathfrak{S}_{\tilde{\pi}}} (\mathfrak{s}v, v) \cdot \overline{(\mathfrak{s}v, v)} \, d\mathfrak{s} \\ &= \frac{\# N(F)}{\# \mathcal{C} \cdot |\# N(\bar{F})|_F} \cdot \frac{1}{\deg \tilde{\pi}} \cdot \frac{\# \mathfrak{X}_{\tilde{\pi}}}{\dim \eta} \cdot \|v\|^4, \end{aligned}$$

where the Haar measure $d\mathfrak{s}$ on $\mathfrak{S}_{\tilde{\pi}}$ is normalized so that $\mathrm{vol}(\mathfrak{S}_{\tilde{\pi}}) = 1$. Hence, by the local Euler characteristic formula

$$|\# N(\bar{F})|_F = \frac{\# \mathrm{H}^0(F, N) \cdot \# \mathrm{H}^2(F, N)}{\# \mathrm{H}^1(F, N)},$$

we have

$$\deg \pi_\eta = \frac{\# \mathcal{C} \cdot \# \mathrm{H}^2(F, N)}{\# \mathrm{H}^1(F, N)} \cdot \frac{\dim \eta}{\# \mathfrak{X}_{\tilde{\pi}}} \cdot \deg \tilde{\pi}.$$

Since the identity component of the center of G is anisotropic, we have an exact sequence

$$1 \longrightarrow \mathcal{C} \longrightarrow \mathrm{H}^1(F, N) \longrightarrow \mathrm{H}^1(F, G) \longrightarrow \mathrm{H}^1(F, G') \longrightarrow \mathrm{H}^2(F, N) \longrightarrow 1,$$

so that

$$\frac{\#\mathcal{C} \cdot \#\mathrm{H}^2(F,N)}{\#\mathrm{H}^1(F,N)} = \frac{\#\mathrm{H}^1(F,G')}{\#\mathrm{H}^1(F,G)} = \frac{\#Z_{\widehat{G'}}^{\Gamma}}{\#Z_{\widehat{G}}^{\Gamma}}$$

by the Kottwitz isomorphism [42]. This yields the lemma. □

We will apply this lemma later to the case when $\tilde{G}$ is the similitude group of a space W and $G = G(W)$ is the isometry group of W.

14. **The formal degree conjecture**

In this section, we recall the formal degree conjecture [34] and its refinement due to Gross-Reeder [23]. For simplicity, we shall only consider the case when $G = G(W)$ is the isometry group of a space W or the metaplectic group when W is a symplectic space.

14.1. **Local Langlands correspondence.** In order to state the formal degree conjecture, we need the local Langlands correspondence and its refinement due to Vogan [74]. Let

$$\widehat{G} = \begin{cases} \mathrm{GL}_n(\mathbb{C}) & \text{in Case A,} \\ \mathrm{Sp}_{n-1}(\mathbb{C}) \times \boldsymbol{\mu}_2 & \text{in Case B,} \\ \mathrm{Sp}_n(\mathbb{C}) & \text{in Case C',} \\ \mathrm{SO}_{n+1}(\mathbb{C}) & \text{in Case C'',} \\ \mathrm{O}_n(\mathbb{C}) & \text{in Case D,} \end{cases}$$

$${}^L G = \begin{cases} \mathrm{GL}_n(\mathbb{C}) \rtimes W_F & \text{in Case A,} \\ \{(g,\epsilon,w) \in \mathrm{Sp}_{n-1}(\mathbb{C}) \times \boldsymbol{\mu}_2 \times W_F \mid \epsilon = \chi_W(w)\} & \text{in Case B,} \\ \mathrm{Sp}_n(\mathbb{C}) \times W_F & \text{in Case C',} \\ \mathrm{SO}_{n+1}(\mathbb{C}) \times W_F & \text{in Case C'',} \\ \{(g,w) \in \mathrm{O}_n(\mathbb{C}) \times W_F \mid \det g = \chi_W(w)\} & \text{in Case D.} \end{cases}$$

Here, in Case A, the action of $w \in W_F$ on $\mathrm{GL}_n(\mathbb{C})$ is given by

$$g \longmapsto \begin{cases} g & \text{if } w \in W_E, \\ \mathrm{Ad}(\mathcal{J}_n)({}^t g^{-1}) & \text{if } w \notin W_E, \end{cases}$$

where

$$\mathcal{J}_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ (-1)^{n-1} & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{GL}_n.$$

We remark that:

- in Cases A and C'', where G is linear and connected, $\widehat{G}$ and ${}^L G$ are the dual group and the L-group of G respectively.
- in Cases B and D, where G is linear but disconnected, we have followed [1, §3]. Note that ${}^L G$ is isomorphic to the L-group of G^0.
- in Cases C', where G is nonlinear, we have followed [10, §11]. Note that $\widehat{G}$ and ${}^L G$ are the dual group and the L-group of SO_{n+1} respectively.

We define an element $z \in \widehat{G}$ by

$$z = \begin{cases} -1 & \text{in Cases AD,} \\ (-1,1) & \text{in Case B,} \\ 1 & \text{in Case C.} \end{cases}$$

The (conjectural) local Langlands correspondence asserts that there exists a certain bijection between

the set of equivalence classes of irreducible admissible representations of G

and

the set of pairs (ϕ, η),

where

- $\phi : WD_F \to {}^L G$ is a $\widehat{G}$-conjugacy class of L-parameters;
- η is an irreducible character of the component group
$$\mathcal{S}_\phi = \pi_0(\mathrm{Cent}_{\widehat{G}}(\phi(WD_F)))$$
of the centralizer of $\phi(WD_F)$ in $\widehat{G}$ such that
$$\eta(z_\phi) = \epsilon(W),$$
where z_ϕ is the image of z in $\mathcal{S}_\phi$.

This correspondence has been established by Arthur [4] for special orthogonal groups and symplectic groups.

14.2. **The formal degree conjecture.** Let π be an irreducible discrete series representation of $G = G(W)$ and (ϕ, η) the pair (conjecturally) associated to π by the local Langlands correspondence. Then [34, Conjecture 1.4'] states that:

$$\deg \pi = \frac{\dim \eta}{\# \mathcal{S}_\phi} \cdot |\gamma(0, \mathrm{Ad} \circ \phi, \psi)|,$$

where $\deg \pi$ is the formal degree of π with respect to the Haar measure dg_ψ defined in §13.1 and Ad is the adjoint representation of ${}^L G$ on its Lie algebra $\mathrm{Lie}({}^L G)$. Note that

$$\dim \eta = 1$$

since $\mathcal{S}_\phi$ is abelian.

In [23], Gross-Reeder gave a refinement of the formal degree conjecture where the absolute value sign on the right-hand side can be removed. We explicate this refinement when $G = \mathrm{GL}_n$ or $G(W)$.

14.3. **Formal degrees for GL_n.** We first establish the following proposition, which proves the refined formal degree conjecture [23, Conjectures 7.1, 8.3] for GL_n.

Proposition 14.1. *Let π be an irreducible discrete series representation of $\mathrm{GL}_n(F)$. Then we have*

$$\deg \pi = \frac{1}{n} \cdot \omega_\pi(-1)^{n-1} \cdot \gamma(0, \pi, \mathrm{Ad}, \psi),$$

where ω_π is the central character of π and Ad *is the adjoint representation of $\mathrm{GL}_n(\mathbb{C})$ on $\mathfrak{sl}_n(\mathbb{C})$.*

Proof. By [34, Theorem 3.1], we have

$$\deg \pi = \frac{1}{n} \cdot |\gamma(0, \pi, \mathrm{Ad}, \psi)|.$$

Hence it remains to compute the sign of $\gamma(0,\pi,\mathrm{Ad},\psi)$ (where for $z\in\mathbb{C}^\times$, we call $z/|z|$ the sign of z).

We may assume that ψ is of order zero. Let

$$\{\tau|\det|_F^{(e-1)/2},\tau|\det|_F^{(e-3)/2},\dots,\tau|\det|_F^{-(e-1)/2}\}$$

be the supercuspidal support of π, where τ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_d(F)$ with $n=de$. By the multiplicativity of γ-factors, we have

$$\gamma(s,\pi,\mathrm{Ad},\psi)=\frac{\gamma(s,\pi\times\pi^\vee,\psi)}{\gamma(s,\mathbb{1},\psi)}=\frac{\zeta(s)}{\zeta(1-s)}\cdot\prod_{i=1}^{e}\prod_{j=1}^{e}\gamma(s+e+1-i-j,\tau\times\tau^\vee,\psi).$$

By [7] and [37, Proposition 8.1], we have

$$\gamma(s,\tau\times\tau^\vee,\psi)=\omega_\tau(-1)^{d-1}\cdot q^{-f(s-1/2)}\cdot\frac{\zeta(r(1-s))}{\zeta(rs)},$$

where f is the conductor of $\tau\times\tau^\vee$ and r is the torsion number of τ. Hence the sign of $\gamma(0,\pi,\mathrm{Ad},\psi)$ is equal to

$$\omega_\tau(-1)^{(d-1)e^2}\cdot(-1)^{\#\mathfrak{I}+\#\mathfrak{I}'}=\omega_\tau(-1)^{(d-1)e^2},$$

where

$$\begin{aligned}\mathfrak{I}&=\{(i,j)\in\mathbb{Z}^2\,|\,1\le i,j\le e,\ e+1-i-j\notin\{0,1\}\},\\ \mathfrak{I}'&=\{(i,j)\in\mathbb{Z}^2\,|\,1\le i,j\le e,\ e+1-i-j=1\}.\end{aligned}$$

Since $\omega_\pi=\omega_\tau^e$, we have

$$\omega_\tau(-1)^{(d-1)e^2}=\omega_\pi(-1)^{n-1}.$$

This yields the proposition. □

14.4. **Steinberg representations.** We remark that [23, Conjectures 7.1, 8.3] actually relates

$$\frac{\deg\pi}{\deg\mathrm{St}}\quad\text{to}\quad\frac{\gamma(0,\mathrm{Ad}\circ\phi,\psi)}{\gamma(0,\mathrm{Ad}\circ\phi_0,\psi)},$$

where St is the Steinberg representation of G and

$$\phi_0: WD_F=W_F\times\mathrm{SL}_2(\mathbb{C})\longrightarrow {}^LG$$

is the L-parameter such that $\phi_0|_{W_F}$ is trivial and $\phi_0|_{\mathrm{SL}_2(\mathbb{C})}$ corresponds to the regular unipotent orbit in $\widehat{G}$. (When $G=\mathrm{Mp}(W)$ is the metaplectic group, we set $\mathrm{St}=\delta_\psi^+$ with notation as in [14, §14].) Thus, to derive a formula for $\deg\pi$, we need to compute $\deg\mathrm{St}$.

Lemma 14.2. *Let* St *be the Steinberg representation of $G=G(W)$. Then we have*

$$\deg\mathrm{St}=\frac{1}{\#\mathcal{S}_{\phi_0}}\cdot\gamma(0,\mathrm{Ad}\circ\phi_0,\psi)\times\begin{cases}\epsilon(\frac12,\chi_E,\psi)^{-n/2} & \textit{if } n \textit{ is even in Case A},\\ \epsilon(\frac12,\chi_E,\psi)^{-(n+1)/2} & \textit{if } n \textit{ is odd in Case A},\\ \chi_W(-1)^{n/2}\cdot\epsilon(\frac12,\chi_W,\psi) & \textit{in Case D},\\ 1 & \textit{otherwise}.\end{cases}$$

Proof. By [34, §3.3], [14, Corollary 18] and Lemma 13.1, we have

$$\deg\mathrm{St}=\frac{1}{\#\mathcal{S}_{\phi_0}}\cdot|\gamma(0,\mathrm{Ad}\circ\phi_0,\psi)|.$$

Hence it remains to compute the sign of $\gamma(0,\mathrm{Ad}\circ\phi_0,\psi)$.

For a non-negative integer k, let Sym^k denote the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension $k+1$. Note that

$$\mathrm{Sym}^2(\mathrm{Sym}^k) = \bigoplus_{i=0}^{[\frac{k}{2}]} \mathrm{Sym}^{2k-4i}, \quad \bigwedge^2(\mathrm{Sym}^k) = \bigoplus_{i=0}^{[\frac{k-1}{2}]} \mathrm{Sym}^{2k-2-4i}.$$

Let χ be a quadratic character of $F^\times$. Then we have

$$\gamma(s, \chi \boxtimes \mathrm{Sym}^{2k}, \psi) = \prod_{i=0}^{2k} \gamma(s+k-i, \chi, \psi),$$

where we view $\chi \boxtimes \mathrm{Sym}^{2k}$ as a representation of WD_F. Since

$$\begin{aligned} \gamma(s, \chi, \psi) \cdot \gamma(1-s, \chi, \psi) &= \chi(-1), \\ \lim_{s \to 0} \gamma(s, \mathbb{1}, \psi) \cdot \gamma(s+1, \mathbb{1}, \psi) &= -1, \end{aligned}$$

we have

$$\gamma(0, \chi \boxtimes \mathrm{Sym}^{2k}, \psi) = \gamma(-k, \chi, \psi) \times \begin{cases} -1 & \text{if } \chi = \mathbb{1} \text{ and } k \neq 0, \\ \chi(-1)^k & \text{if } \chi \neq \mathbb{1}. \end{cases}$$

Thus, we see that $\gamma(s, \mathrm{Ad} \circ \phi_0, \psi)$ is equal to

$$\begin{cases} \prod_{i=1}^{n} \gamma(s+n+1-2i, \mathbb{1}, \psi) & \\ \quad \times \prod_{j=1}^{\frac{n}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2n-4j}, \psi) \cdot \gamma(s, \chi_E \boxtimes \mathrm{Sym}^{2n-4j}, \psi) & \text{if } n \text{ is even in Case A}, \\ \prod_{i=1}^{n} \gamma(s+n+1-2i, \chi_E, \psi) & \\ \quad \times \prod_{j=1}^{\frac{n-1}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2n-4j}, \psi) \cdot \gamma(s, \chi_E \boxtimes \mathrm{Sym}^{2n-4j}, \psi) & \text{if } n \text{ is odd in Case A}, \\ \prod_{i=1}^{\frac{n-1}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2n-4i}, \psi) & \text{in Case B}, \\ \prod_{i=1}^{\frac{n}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2n+2-4i}, \psi) & \text{in Case C'}, \\ \prod_{i=1}^{\frac{n}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2n+2-4i}, \psi) & \text{in Case C''}, \\ \gamma(s, \chi_W \boxtimes \mathrm{Sym}^{n-2}, \psi) \cdot \prod_{i=1}^{\frac{n}{2}-1} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2n-2-4i}, \psi) & \text{in Case D}, \end{cases}$$

so that $\gamma(0, \mathrm{Ad} \circ \phi_0, \psi)$ simplifies to:

$$\begin{cases} (-1)^{n/2} \cdot \prod_{i=1}^{n} \gamma(-i+1, \chi_E^i, \psi) & \text{if } n \text{ is even in Case A}, \\ (-1)^{(n-1)/2} \cdot \prod_{i=1}^{n} \gamma(-i+1, \chi_E^i, \psi) & \text{if } n \text{ is odd in Case A}, \\ (-1)^{(n-1)/2} \cdot \prod_{i=1}^{\frac{n-1}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \text{in Case B}, \\ (-1)^{n/2} \cdot \prod_{i=1}^{\frac{n}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \text{in Case C'}, \\ (-1)^{n/2} \cdot \prod_{i=1}^{\frac{n}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \text{in Case C''}, \\ (-1)^{n/2} \cdot \gamma(-\frac{n}{2}+1, \mathbb{1}, \psi) \cdot \prod_{i=1}^{\frac{n}{2}-1} \gamma(-2i+1, \mathbb{1}, \psi) & \text{if } \chi_W = \mathbb{1} \text{ in Case D}, \\ (-1)^{n/2-1} \cdot \chi_W(-1)^{n/2-1} & \\ \quad \times \gamma(-\frac{n}{2}+1, \chi_W, \psi) \cdot \prod_{i=1}^{\frac{n}{2}-1} \gamma(-2i+1, \mathbb{1}, \psi) & \text{if } \chi_W \neq \mathbb{1} \text{ in Case D}. \end{cases}$$

Observe that the sign of $\gamma(-k, \chi, \psi)$ (for a non-negative integer k and a quadratic character χ of $F^\times$) is equal to

$$\begin{cases} -1 & \text{if } \chi = \mathbb{1} \text{ and } k \neq 0, \\ \epsilon(\frac{1}{2}, \chi, \psi) & \text{if } \chi \neq \mathbb{1}. \end{cases}$$

Hence the sign of $\gamma(0, \mathrm{Ad} \circ \phi_0, \psi)$ is equal to

$$\begin{cases} \epsilon(\frac{1}{2}, \chi_E, \psi)^{n/2} & \text{if } n \text{ is even in Case A,} \\ \epsilon(\frac{1}{2}, \chi_E, \psi)^{(n+1)/2} & \text{if } n \text{ is odd in Case A,} \\ \chi_W(-1)^{n/2-1} \cdot \epsilon(\frac{1}{2}, \chi_W, \psi) & \text{in Case D,} \\ 1 & \text{otherwise.} \end{cases}$$

This yields the lemma. □

14.5. **A refined formal degree conjecture.** Let π be an irreducible discrete series representation of $G = G(W)$ and $\phi : WD_F \to {}^L G$ the L-parameter (conjecturally) associated to π. Put

$$\zeta_\phi = \begin{cases} \omega_\pi(-1) \cdot \epsilon(\frac{1}{2}, \chi_E, \psi)^{-n/2} & \text{if } n \text{ is even in Case A,} \\ \epsilon(\frac{1}{2}, \chi_E, \psi)^{-(n+1)/2} & \text{if } n \text{ is odd in Case A,} \\ 1 & \text{in Case B,} \\ 1 & \text{in Case C',} \\ \omega_\pi(-1) & \text{in Case C'',} \\ \chi_W(-1)^{n/2} \cdot \epsilon(\frac{1}{2}, \chi_W, \psi) & \text{in Case D.} \end{cases}$$

By [34, Conjecture 1.4'], [23, Conjectures 7.1, 8.3] and Lemma 14.2, we have the following refined conjecture:

$$\deg \pi = \frac{1}{\# \mathcal{S}_\phi} \cdot \zeta_\phi \cdot \gamma(0, \mathrm{Ad} \circ \phi, \psi).$$

15. Formal degrees and local theta correspondence

The main result of this paper is the following theorem.

Theorem 15.1. *Assume that $p \neq 2$ and $l \geq 0$. Let π be an irreducible discrete series representation of $G(W)$ such that its theta lift $\sigma := \theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ to $H(V)$ is nonzero. Then σ is square integrable. Moreover, we have:*

(i) *If $l = 0$, then we have*

$$\frac{\deg \pi}{\deg \sigma} = \begin{cases} 1 & \textit{in Case A,} \\ 2^{-1} & \textit{in Case B,} \\ 2 & \textit{in Case C'.} \end{cases}$$

(ii) *If $l = 1$, then we have*

$$\frac{\deg \pi}{\deg \sigma} = \boldsymbol{\epsilon} \cdot \omega_\sigma(-1) \cdot \gamma(0, \sigma, \chi_W^{-1}, \psi) \times \begin{cases} 2^{-1} \cdot \gamma(0, \chi_E, \psi) & \textit{in Case A,} \\ 1 & \textit{in Case C'',} \\ 2^{-2} & \textit{in Case D,} \end{cases}$$

where

$$\boldsymbol{\epsilon} = \begin{cases} \chi_W(\daleth)^{-m} \cdot \epsilon(\frac{1}{2}, \chi_E, \psi)^{-1} & \textit{in Case A,} \\ \epsilon(\frac{1}{2}, \chi_V, \psi)^{-1} & \textit{in Case C'',} \\ \chi_W(-1)^{m/2} \cdot \epsilon(\frac{1}{2}, \chi_W, \psi)^{-1} & \textit{in Case D.} \end{cases}$$

(iii) *Assume that $l > 1$ and π is supercuspidal. Then there exists a constant $\boldsymbol{\alpha}$ which does not depend on π such that*

$$\frac{\deg \pi}{\deg \sigma} = \boldsymbol{\alpha} \cdot \omega_\sigma(-1) \cdot \gamma(-\tfrac{l-1}{2}, \sigma, \chi_W^{-1}, \psi).$$

Let $\mathbf{c}$ *be a constant defined by* $\boldsymbol{\alpha} = \mathbf{c} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{s}^{-1} \cdot \boldsymbol{\gamma}$, *where*

$$\boldsymbol{\epsilon} = \begin{cases} \chi_W(\daleth)^{-m} \cdot \epsilon(\frac{1}{2}, \chi_E, \psi)^{-l/2} \cdot \epsilon(V) \cdot \epsilon(W) & \textit{if } l \textit{ is even in Case A,} \\ \chi_W(\daleth)^{-m} \cdot \epsilon(\frac{1}{2}, \chi_E, \psi)^{-(l+1)/2} & \textit{if } l \textit{ is odd in Case A,} \\ \chi_\psi^V(-1)^{-1} \cdot \chi_W(-1)^{m/2} \cdot \epsilon(W) & \textit{in Case B,} \\ \epsilon(V) & \textit{in Case C',} \\ \epsilon(\frac{1}{2}, \chi_V, \psi)^{-1} & \textit{in Case C'',} \\ \chi_W(-1)^{m/2} \cdot \epsilon(\frac{1}{2}, \chi_W, \psi)^{-1} & \textit{in Case D,} \end{cases}$$

$$\boldsymbol{s} = \begin{cases} 2 & \textit{in Case A,} \\ 2^2 & \textit{in Case B,} \\ 1 & \textit{in Case C',} \\ 1 & \textit{in Case C'',} \\ 2^2 & \textit{in Case D,} \end{cases}$$

$$\boldsymbol{\gamma} = \begin{cases} \prod_{i=1}^{l} \gamma(-i+1, \chi_E^i, \psi) & \textit{in Case A,} \\ \prod_{i=1}^{\frac{l}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \textit{in Cases BC',} \\ \prod_{i=1}^{\frac{l-1}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \textit{in Cases C''D.} \end{cases}$$

Then we have

$$\mathbf{c} = 1$$

under Hypothesis (UR) in §19.

The assumption $p \neq 2$ is necessary since the Howe duality conjecture is not known for $p = 2$. (It is used in Proposition 16.1.) This assumption is also necessary in §20.1 and Lemma 20.7. However, even when $p = 2$, the above theorem holds up to a scalar if we admit the Howe duality conjecture, and in particular, if π is supercuspidal.

The proof of Theorem 15.1 will be given in the next five sections. For the rest of this section, we shall explain why Theorem 15.1 is consistent with the refined formal degree conjecture and various conjectures.

15.1. **Conjectures of Adams and Mœglin.** We briefly recall some conjectures due to Adams and Mœglin.

We define a homomorphism

$$\Delta : W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$$

by

$$\Delta(w, g) = (w, g, g) \quad \text{for } w \in W_F \text{ and } g \in \mathrm{SL}_2(\mathbb{C}).$$

Mœglin's conjecture ([57, Corollaire 4.2]). *Let* $\psi : WD_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ *be an A-parameter with associated A-packet* Π_ψ. *Let* $\pi \in \Pi_\psi$. *If* π *is nonzero, irreducible and tempered, then we have*

$$\pi \in \Pi_{\psi \circ \Delta},$$

where $\Pi_{\psi\circ\Delta}$ *is the L-packet associated to the L-parameter* $\psi \circ \Delta : WD_F \to {}^L G$.

To state Adams' conjecture, we need to introduce more notation. We assume that $l \geq 0$. Let $G = G(W)$ and $H = H(V)$. Let

$$\xi : {}^L H \longrightarrow {}^L G$$

be an L-homomorphism defined as follows:

- In Case A, fix an element $w_c \in W_F \smallsetminus W_E$ and put

$$\begin{aligned}
\xi(h,1) &= \left(\begin{pmatrix} h & 0 \\ 0 & \mathbf{1}_l \end{pmatrix}, 1 \right) && \text{for } h \in \widehat{H}, \\
\xi(1,w) &= \left(\begin{pmatrix} (\chi_V \chi_W^{-1})(w) \cdot \mathbf{1}_m & 0 \\ 0 & \chi_V(w) \cdot \mathbf{1}_l \end{pmatrix}, w \right) && \text{for } w \in W_E, \\
\xi(1,w_c) &= \left(\begin{pmatrix} \mathcal{J}_m & 0 \\ 0 & \mathcal{J}_l \end{pmatrix} \cdot \mathcal{J}_n^{-1}, w_c \right),
\end{aligned}$$

  where

$$\mathcal{J}_k = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ (-1)^{k-1} & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{GL}_k.$$

- In Case B, put

$$\xi(h,w) = \left(\begin{pmatrix} (\chi_V \chi_W^{-1})(w) \cdot h & 0 \\ 0 & \chi_V(w) \cdot \mathbf{1}_l \end{pmatrix}, \chi_W(w), w \right) \quad \text{for } (h,w) \in {}^L H.$$

- In Case C', put

$$\xi(h,\epsilon,w) = \left(\begin{pmatrix} (\chi_V \chi_W^{-1})(w) \cdot h & 0 \\ 0 & \chi_V(w) \cdot \mathbf{1}_l \end{pmatrix}, w \right) \quad \text{for } (h,\epsilon,w) \in {}^L H.$$

- In Cases C" and D, put

$$\xi(h,w) = \left(\begin{pmatrix} (\chi_V \chi_W^{-1})(w) \cdot h & 0 \\ 0 & \chi_V(w) \cdot \mathbf{1}_l \end{pmatrix}, w \right) \quad \text{for } (h,w) \in {}^L H.$$

For an A-parameter $\psi : \mathit{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L H$, we define an A-parameter $\theta(\psi) : \mathit{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ by

$$\begin{aligned}
\theta(\psi)|_{\mathit{WD}_F} &= \xi \circ \psi|_{\mathit{WD}_F}, \\
\theta(\psi)|_{\mathrm{SL}_2(\mathbb{C})} &= \left(\xi \circ \psi|_{\mathrm{SL}_2(\mathbb{C})} \right) \oplus \mathrm{Sym}^{l-1}.
\end{aligned}$$

Here Sym^{l-1} is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension l (which is interpreted to be 0 when $l = 0$).

Adams' conjecture ([1], [27, Conjecture 7.2])**.** *Let σ be an irreducible admissible representation of H such that its theta lift $\theta_{V,W,\boldsymbol{\chi},\psi}(\sigma)$ to G is nonzero. Let $\psi : \mathit{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L H$ be an A-parameter with associated A-packet Π_ψ. If*

$$\sigma \in \Pi_\psi,$$

then we have

$$\theta_{V,W,\boldsymbol{\chi},\psi}(\sigma) \in \Pi_{\theta(\psi)},$$

where $\Pi_{\theta(\psi)}$ is the A-packet associated to the A-parameter $\theta(\psi) : \mathit{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$.

We also refer the reader to [63], [64] for a certain refined conjecture due to D. Prasad.

In [58], Mœglin has verified Adams' conjecture for a large class of A-parameters in Cases C" and D. However, she has also given some counterexamples to Adams' conjecture (see [58, §7.2.1]). In any case, we note the following consequence of the conjectures of Adams and Mœglin.

Consequence. *Let σ be an irreducible tempered representation of H such that its theta lift $\theta_{V,W,\boldsymbol{\chi},\psi}(\sigma)$ to G is nonzero and tempered. Let $\phi : WD_F \to {}^L H$ be the L-parameter associated to σ. Then the L-parameter associated to $\theta_{V,W,\boldsymbol{\chi},\psi}(\sigma)$ is*

$$\theta(\phi \boxtimes \mathbf{1}) \circ \Delta : WD_F \longrightarrow {}^L G.$$

15.2. **Consistency.** Let σ be an irreducible discrete series representation of H such that its theta lift $\pi := \theta_{V,W,\boldsymbol{\chi},\psi}(\sigma)$ to G is nonzero and square integrable. Let ϕ be the L-parameter associated to σ and put $\phi' = \theta(\phi \boxtimes \mathbf{1}) \circ \Delta$. Then we have

$$\begin{aligned} \phi'|_{W_F} &= \xi \circ \phi|_{W_F}, \\ \phi'|_{\mathrm{SL}_2(\mathbb{C})} &= \big(\xi \circ \phi|_{\mathrm{SL}_2(\mathbb{C})}\big) \oplus \mathrm{Sym}^{l-1}. \end{aligned}$$

In view of the formal degree conjecture and the above consequence of the conjectures of Adams and Mœglin, we should have

$$\frac{\deg \pi}{\deg \sigma} = \frac{\# \mathcal{S}_\phi}{\# \mathcal{S}_{\phi'}} \cdot \frac{\zeta_{\phi'}}{\zeta_\phi} \cdot \frac{\gamma(0, \mathrm{Ad} \circ \phi', \psi)}{\gamma(0, \mathrm{Ad} \circ \phi, \psi)}. \tag{15.1}$$

We first assume that $l = 0$. Then we have

$$\begin{aligned} \# \mathcal{S}_{\phi'} &= \# \mathcal{S}_\phi \times \begin{cases} 1 & \text{in Case A,} \\ 2 & \text{in Case B,} \\ 2^{-1} & \text{in Case C',} \end{cases} \\ \zeta_{\phi'} &= \zeta_\phi, \end{aligned}$$

and

$$\gamma(s, \mathrm{Ad} \circ \phi', \psi) = \gamma(s, \mathrm{Ad} \circ \phi, \psi).$$

Thus Theorem 15.1 is consistent with (15.1) when $l = 0$.

We next assume that $l > 0$. Then we have

$$\frac{\# \mathcal{S}_{\phi'}}{\# \mathcal{S}_\phi} = \begin{cases} 2 & \text{in Case A,} \\ 2^2 & \text{in Case B,} \\ 1 & \text{in Case C',} \\ 1 & \text{in Case C'',} \\ 2^2 & \text{in Case D,} \end{cases}$$

$$\frac{\zeta_{\phi'}}{\zeta_\phi} = \begin{cases} \omega_\pi(-1) \cdot \omega_\sigma(-1) \cdot \epsilon(\tfrac{1}{2}, \chi_E, \psi)^{-l/2} & \text{if n and m are even in Case A,} \\ \omega_\pi(-1) \cdot \epsilon(\tfrac{1}{2}, \chi_E, \psi)^{-(l-1)/2} & \text{if n is even and m is odd in Case A,} \\ \omega_\sigma(-1) \cdot \epsilon(\tfrac{1}{2}, \chi_E, \psi)^{-(l+1)/2} & \text{if n is odd and m is even in Case A,} \\ \epsilon(\tfrac{1}{2}, \chi_E, \psi)^{-l/2} & \text{if n and m are odd in Case A,} \\ 1 & \text{in Case B,} \\ 1 & \text{in Case C',} \\ \omega_\pi(-1) \cdot \chi_V(-1)^{m/2} \cdot \epsilon(\tfrac{1}{2}, \chi_V, \psi)^{-1} & \text{in Case C'',} \\ \omega_\sigma(-1) \cdot \chi_W(-1)^{n/2} \cdot \epsilon(\tfrac{1}{2}, \chi_W, \psi) & \text{in Case D,} \end{cases}$$

and

$$\frac{\gamma(s, \mathrm{Ad} \circ \phi', \psi)}{\gamma(s, \mathrm{Ad} \circ \phi, \psi)} = \prod_{i=1}^{l} \gamma(s + \tfrac{l+1}{2} - i, (\mathrm{std} \otimes \chi_W^{-1}) \circ \phi, \psi) \cdot \beth(s),$$

where std is the standard representation of ${}^L H$ and

$$\gimel(s) = \begin{cases} \prod_{i=1}^{l} \gamma(s+l+1-2i, \mathbb{1}, \psi) & \\ \quad \times \prod_{j=1}^{\frac{l}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2l-4j}, \psi) \cdot \gamma(s, \chi_E \boxtimes \mathrm{Sym}^{2l-4j}, \psi) & \text{if } l \text{ is even in Case A,} \\ \prod_{i=1}^{l} \gamma(s+l+1-2i, \chi_E, \psi) & \\ \quad \times \prod_{j=1}^{\frac{l-1}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2l-4j}, \psi) \cdot \gamma(s, \chi_E \boxtimes \mathrm{Sym}^{2l-4j}, \psi) & \text{if } l \text{ is odd in Case A,} \\ \prod_{i=1}^{\frac{l}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2l+2-4i}, \psi) & \text{in Cases BC',} \\ \prod_{i=1}^{\frac{l-1}{2}} \gamma(s, \mathbb{1} \boxtimes \mathrm{Sym}^{2l-4i}, \psi) & \text{in Cases C''D.} \end{cases}$$

It is expected that

$$\gamma(s, (\mathrm{std} \otimes \chi_W^{-1}) \circ \phi, \psi) = \gamma(s, \sigma, \chi_W^{-1}, \psi) \times \begin{cases} \epsilon(\frac{1}{2}, \chi_E, \psi)^m & \text{in Case A,} \\ 1 & \text{otherwise.} \end{cases}$$

Note that when $[E:F]=2$, Langlands' factor $\lambda(E/F, \psi)$ is equal to $\epsilon(\frac{1}{2}, \chi_E, \psi)$. Hence we have

$$\lim_{s \to 0} \prod_{i=1}^{l} \gamma(s + \tfrac{l+1}{2} - i, (\mathrm{std} \otimes \chi_W^{-1}) \circ \phi, \psi)$$

$$\lim_{s \to 0} \prod_{i=1}^{l} \gamma(s + \tfrac{l+1}{2} - i, \sigma, \chi_W^{-1}, \psi) \times \begin{cases} \epsilon(\frac{1}{2}, \chi_E, \psi)^{lm} & \text{in Case A,} \\ 1 & \text{otherwise,} \end{cases}$$

$$= \pm \gamma(-\tfrac{l-1}{2}, \sigma, \chi_W^{-1}, \psi) \times \begin{cases} \chi_E(-1)^m \cdot \gamma(\frac{1}{2}, \sigma, \chi_W^{-1}, \psi) & \text{if } l \text{ is even in Case A,} \\ \epsilon(\frac{1}{2}, \chi_E, \psi)^{lm} & \text{if } l \text{ is odd in Case A,} \\ \gamma(\frac{1}{2}, \sigma, \chi_W^{-1}, \psi) & \text{in Case B,} \\ \gamma(\frac{1}{2}, \sigma, \chi_W^{-1}, \psi) & \text{in Case C',} \\ \chi_V(-1)^{(l-1)/2} & \text{in Case C'',} \\ \chi_W(-1)^{(l-1)/2} & \text{in Case D.} \end{cases}$$

If π is supercuspidal or $l=1$, then we have

$$\pm = \begin{cases} (-1)^{l/2-1} & \text{if } l \text{ is even,} \\ (-1)^{(l-1)/2} & \text{if } l \text{ is odd,} \end{cases}$$

by Corollary 11.4. Finally, we have

$$\gimel(0) = \begin{cases} (-1)^{l/2} \cdot \prod_{i=1}^{l} \gamma(-i+1, \chi_E^i, \psi) & \text{if } l \text{ is even in Case A,} \\ (-1)^{(l-1)/2} \cdot \prod_{i=1}^{l} \gamma(-i+1, \chi_E^i, \psi) & \text{if } l \text{ is odd in Case A,} \\ (-1)^{l/2} \cdot \prod_{i=1}^{\frac{l}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \text{in Cases BC',} \\ (-1)^{(l-1)/2} \cdot \prod_{i=1}^{\frac{l-1}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \text{in Cases C''D.} \end{cases}$$

Now suppose that either π is supercuspidal or $l=1$. Then we have

$$\frac{\zeta_{\phi'}}{\zeta_\phi} \cdot \frac{\gamma(0, \mathrm{Ad} \circ \phi', \psi)}{\gamma(0, \mathrm{Ad} \circ \phi, \psi)} = \boldsymbol{\zeta} \cdot \gamma(-\tfrac{l-1}{2}, \sigma, \chi_W^{-1}, \psi) \times \begin{cases} \prod_{i=1}^{l} \gamma(-i+1, \chi_E^i, \psi) & \text{in Case A,} \\ \prod_{i=1}^{\frac{l}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \text{in Cases BC',} \\ \prod_{i=1}^{\frac{l-1}{2}} \gamma(-2i+1, \mathbb{1}, \psi) & \text{in Cases C''D,} \end{cases}$$

where

$$\boldsymbol{\zeta} = \begin{cases} -\omega_\pi(-1)\cdot\omega_\sigma(-1)\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-l/2}\cdot\gamma(\frac{1}{2},\sigma,\chi_W^{-1},\psi) & \text{if } n \text{ and } m \text{ are even in Case A,} \\ \omega_\pi(-1)\cdot\chi_E(-1)^{n/2}\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-(l+1)/2} & \text{if } n \text{ is even and } m \text{ is odd in Case A,} \\ \omega_\sigma(-1)\cdot\chi_E(-1)^{m/2}\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-(l+1)/2} & \text{if } n \text{ is odd and } m \text{ is even in Case A,} \\ -\chi_E(-1)\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-l/2}\cdot\gamma(\frac{1}{2},\sigma,\chi_W^{-1},\psi) & \text{if } n \text{ and } m \text{ are odd in Case A,} \\ -\gamma(\frac{1}{2},\sigma,\chi_W^{-1},\psi) & \text{in Case B,} \\ -\gamma(\frac{1}{2},\sigma,\chi_W^{-1},\psi) & \text{in Case C',} \\ \omega_\pi(-1)\cdot\chi_V(-1)^{n/2}\cdot\epsilon(\frac{1}{2},\chi_V,\psi)^{-1} & \text{in Case C'',} \\ \omega_\sigma(-1)\cdot\chi_W(-1)^{m/2}\cdot\epsilon(\frac{1}{2},\chi_W,\psi)^{-1} & \text{in Case D.} \end{cases}$$

Moreover, if l is even, then we have

$$-\gamma(\tfrac{1}{2},\sigma,\chi_W^{-1},\psi) = \gamma(\tfrac{1}{2},\pi,\chi_V^{-1},\psi) = \begin{cases} \omega_\pi(-1)\cdot\chi_V(\daleth)^n\cdot\epsilon(V)\cdot\epsilon(W) & \text{in Case A,} \\ \omega_\pi(-1)\cdot\epsilon(W) & \text{in Case B,} \\ \omega_\pi(-1)\cdot\chi_\psi^W(-1)^{-1}\cdot\chi_V(-1)^{n/2}\cdot\epsilon(V) & \text{in Case C',} \end{cases}$$

by Theorem 11.3 and the epsilon dichotomy (Theorem 11.1). Hence we have

$$\boldsymbol{\zeta} = \begin{cases} \omega_\sigma(-1)\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-l/2}\cdot\epsilon(V)\cdot\epsilon(W) & \text{if } n \text{ and } m \text{ are even in Case A,} \\ \omega_\pi(-1)\cdot\chi_E(-1)^{n/2}\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-(l+1)/2} & \text{if } n \text{ is even and } m \text{ is odd in Case A,} \\ \omega_\sigma(-1)\cdot\chi_E(-1)^{m/2}\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-(l+1)/2} & \text{if } n \text{ is odd and } m \text{ is even in Case A,} \\ \omega_\pi(-1)\cdot\chi_V(\daleth)^{-n}\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-l/2}\cdot\epsilon(V)\cdot\epsilon(W) & \text{if } n \text{ and } m \text{ are odd in Case A,} \\ \omega_\pi(-1)\cdot\epsilon(W) & \text{in Case B,} \\ \omega_\pi(-1)\cdot\chi_\psi^W(-1)^{-1}\cdot\chi_V(-1)^{n/2}\cdot\epsilon(V) & \text{in Case C',} \\ \omega_\pi(-1)\cdot\chi_V(-1)^{n/2}\cdot\epsilon(\frac{1}{2},\chi_V,\psi)^{-1} & \text{in Case C'',} \\ \omega_\sigma(-1)\cdot\chi_W(-1)^{m/2}\cdot\epsilon(\frac{1}{2},\chi_W,\psi)^{-1} & \text{in Case D,} \end{cases}$$

$$= \omega_\sigma(-1)\times\begin{cases} \epsilon(\frac{1}{2},\chi_E,\psi)^{-l/2}\cdot\epsilon(V)\cdot\epsilon(W) & \text{if } n \text{ and } m \text{ are even in Case A,} \\ \chi_W(\daleth)^{-m}\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-(l+1)/2} & \text{if } n \text{ is even and } m \text{ is odd in Case A,} \\ \chi_E(-1)^{m/2}\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-(l+1)/2} & \text{if } n \text{ is odd and } m \text{ is even in Case A,} \\ \chi_W(\daleth)^{-m}\cdot\epsilon(\frac{1}{2},\chi_E,\psi)^{-l/2}\cdot\epsilon(V)\cdot\epsilon(W) & \text{if } n \text{ and } m \text{ are odd in Case A,} \\ \chi_\psi^V(-1)^{-1}\cdot\chi_W(-1)^{m/2}\cdot\epsilon(W) & \text{in Case B,} \\ \epsilon(V) & \text{in Case C',} \\ \epsilon(\frac{1}{2},\chi_V,\psi)^{-1} & \text{in Case C'',} \\ \chi_W(-1)^{m/2}\cdot\epsilon(\frac{1}{2},\chi_W,\psi)^{-1} & \text{in Case D,} \end{cases}$$

by §5.2. Thus Theorem 15.1 is consistent with (15.1) when $l > 0$.

16. **An explicit local theta lifting**

We now begin the proof of Theorem 15.1. In this section, we introduce and study an explicit local theta lifting given by an integral of matrix coefficients. This construction is a local analog of the global theta lifting given by integrating cusp forms against theta functions.

16.1. **Weil representations.**

Assume that $l \geq 0$. For convenience, we write

$$G = G(W), \quad H = H(V), \quad \mathbf{G} = \mathbf{G}(\mathbf{W}), \quad \omega = \omega_{V,W,\boldsymbol{\chi},\psi}, \quad \boldsymbol{\omega} = \omega_{V,\mathbf{W},\boldsymbol{\chi},\psi}.$$

Recall that ω can be realized on the space $S(\mathbb{X})$ of Schwartz-Bruhat functions on $\mathbb{X}$, where $\mathbb{X}$ is a maximal isotropic subspace of $\mathbb{W} = V \otimes W$. Let $(\cdot, \cdot)$ denote the invariant hermitian inner product on $S(\mathbb{X})$.

We recall the setup of §4. Consider the "doubled spaces"

$$\mathbf{W} = W \oplus W_- \quad \text{and} \quad V \otimes \mathbf{W} = \mathbb{W} \oplus \mathbb{W}_-.$$

These lead to the see-saw diagram

$$\begin{tikzcd} \mathbf{G} \arrow[d, dash] \arrow[dr, dash] & H \times H \arrow[d, dash] \arrow[dl, dash] . \\ G \times G & \Delta H \end{tikzcd}$$

The space $V \otimes \mathbf{W}$ has a maximal isotropic subspace $\mathbb{X} \oplus \mathbb{X}_-$ and we have the representation $\omega \otimes \bar{\omega}$ of $(G \times H) \times (G \times H)$ on $S(\mathbb{X} \oplus \mathbb{X}_-)$. Also, $V \otimes \mathbf{W}$ has another maximal isotropic subspace $V \otimes W^\nabla$ and we have the representation $\boldsymbol{\omega}$ of $\mathbf{G} \times H$ on $S(V \otimes W^\nabla)$. There exists an isomorphism

$$\delta : S(\mathbb{X} \oplus \mathbb{X}_-) \xrightarrow{\cong} S(V \otimes W^\nabla)$$

such that

$$\delta(\phi_1 \otimes \bar{\phi}_2)(0) = (\phi_1, \phi_2)$$

for $\phi_1, \phi_2 \in S(\mathbb{X})$.

16.2. **The representation** $R(V, \chi_W)$**.** We also recall some facts from §6.3. For $\varphi \in S(V \otimes W^\nabla)$ and $\mathbf{g} \in \mathbf{G}$, put

$$\mathcal{F}_\varphi(\mathbf{g}) = (\boldsymbol{\omega}(\mathbf{g})\varphi)(0).$$

Then $\varphi \mapsto \mathcal{F}_\varphi$ defines a $\mathbf{G}$-equivariant map

$$\boldsymbol{\omega} \longrightarrow I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2}, \chi_V).$$

For convenience, we write

$$R(V, \chi_W) = R_{\mathbf{W}, \boldsymbol{\chi}, \psi}(V, \chi_W)$$

for the image of this map. Composing with δ, we obtain a map

$$\omega \otimes \bar{\omega} \xrightarrow{\delta} \boldsymbol{\omega} \longrightarrow I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2}, \chi_V).$$

To simplify notation, we shall suppress δ and write

$$\mathcal{F}_{\phi_1 \otimes \bar{\phi}_2} = \mathcal{F}_{\delta(\phi_1 \otimes \bar{\phi}_2)}.$$

16.3. **An integral of matrix coefficients.** Let π be an irreducible discrete series representation of G. Assume that either π is supercuspidal or $0 \leq l \leq 1$. Let $f_{v,v'}$ be a matrix coefficient of π given by

$$f_{v,v'}(g) = (\pi(g)v, v')$$

for $v, v' \in \pi$ and $g \in G$. We shall consider a $G \times G \times H \times H$-equivariant map

$$\mathcal{Z} : \omega \otimes \bar{\omega} \otimes \bar{\pi} \otimes \pi \longrightarrow C^\infty(H)$$

given by an integral of matrix coefficients:

$$\mathcal{Z}(\phi, \phi'; v, v')(h) = \int_G (\omega(gh)\phi, \phi') \cdot \overline{f_{v,v'}(g)} \, dg$$

for $\phi, \phi' \in \omega$, $v, v' \in \pi$ and $h \in H$. Such a construction was first considered by J. S. Li [50] in the stable range. When π is supercuspidal, this integral is clearly absolutely convergent. When π is square integrable and $0 \leq l \leq 1$, we shall see that this integral is also absolutely convergent by relating it to the doubling zeta integral.

More precisely, we have

$$\mathcal{F}_{\phi\otimes\bar{\phi}'}(i(g,1)) = \delta(\omega(g)\phi\otimes\bar{\phi}')(0) = (\omega(g)\phi,\phi')$$

for $\phi,\phi'\in\omega$ and $g\in G$, so that

$$\mathcal{Z}(\phi,\phi';v,v')(h) = \int_G \mathcal{F}_{\omega(h)\phi\otimes\bar{\phi}'}(i(g,1))\cdot\overline{f_{v,v'}(g)}\,dg.$$

The right-hand side is the doubling zeta integral associated to π evaluated at $s=-\frac{l}{2}$. By Lemma 9.4(i), it is absolutely convergent when π is square integrable and $0\le l\le 1$. This shows that

$$\mathcal{Z}(\phi,\phi';v,v')(h) = Z(-\tfrac{l}{2},\mathcal{F}_{\omega(h)\phi\otimes\bar{\phi}'},\bar{f}_{v,v'})$$

and justifies the definition of $\mathcal{Z}$.

16.4. **Properties of $\mathcal{Z}$.** For convenience, we write

$$\Theta(\pi)=\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)\quad\text{and}\quad\theta(\pi)=\theta_{V,W,\boldsymbol{\chi},\psi}(\pi).$$

The following proposition summarizes key properties of $\mathcal{Z}$.

Proposition 16.1. (i) *The image of $\mathcal{Z}$ is contained in $L^2(H)$.*

(ii) *The map $\mathcal{Z}$ factors through the quotient*

$$\omega\otimes\bar{\omega}\otimes\bar{\pi}\otimes\pi\longrightarrow\theta(\pi)\otimes\overline{\theta(\pi)}.$$

(iii) *If $\Theta(\pi)\neq 0$, then $\mathcal{Z}$ is nonzero.*

(iv) *If $p\neq 2$ and $\Theta(\pi)\neq 0$, then $\theta(\pi)$ is square integrable.*

Proof. (i) This will be proved in Lemma C.1.

(ii) The map $\mathcal{Z}$ clearly factors through the quotient

$$\omega\otimes\bar{\omega}\otimes\bar{\pi}\otimes\pi\longrightarrow\Theta(\pi)\otimes\overline{\Theta(\pi)},$$

so that the image of $\mathcal{Z}$ is of finite length. Hence, by (i), the image of $\mathcal{Z}$ must be semisimple. This proves (ii).

(iii) If π is supercuspidal and $\Theta(\pi)\neq 0$, then $\pi\boxtimes\Theta(\pi)$ is a direct summand of ω and hence $\mathcal{Z}$ is clearly nonzero. In general, we argue as in the proof of Proposition 11.2.

Assume that $\Theta(\pi)\neq 0$. In the proof of Proposition 11.2, we have shown that the restriction of $Z(-\frac{l}{2},\cdot,\cdot)$ to the submodule

$$R(V,\chi_W)\oplus R(V',\chi_W)$$

of $I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2},\chi_V)$ is nonzero, where V' is the space of dimension m in the other Witt tower not containing V. (In Cases B and D, we interpret $R(V',\chi_W)=R(V,\chi_W)\otimes\det$.) To see that $\mathcal{Z}$ is nonzero, it suffices to note that the restriction of $Z(-\frac{l}{2},\cdot,\cdot)$ to $R(V',\chi_W)$ must be zero. For if not, then one would have

$$\mathrm{Hom}_{G\times G}(R(V',\chi_W)\otimes\pi^\vee\otimes\pi\chi_V^{-1},\mathbb{C})\neq 0,$$

which by Lemma 6.1 and Proposition 6.2 would imply that $\Theta_{V',W,\boldsymbol{\chi},\psi}(\pi)\neq 0$. However, the scenario that both $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ and $\Theta_{V',W,\boldsymbol{\chi},\psi}(\pi)$ are nonzero contradicts Theorem 5.4. This proves (iii).

(iv) This is an immediate consequence of (i), (ii), (iii) and the Howe duality conjecture (which is known for $p\neq 2$). □

16.5. **An explicit local theta lifting.** Suppose that $p \neq 2$ and $\sigma := \theta(\pi) \neq 0$. We fix a nonzero $G \times H$-equivariant map

$$\theta : \omega \otimes \bar{\pi} \longrightarrow \sigma,$$

which is unique up to a scalar. Since σ is unitary, we have a nonzero $G \times G \times H \times H$-equivariant map

$$\omega \otimes \bar{\omega} \otimes \bar{\pi} \otimes \pi \xrightarrow{\theta \otimes \bar{\theta}} \sigma \otimes \bar{\sigma} \longrightarrow C^{\infty}(H).$$

On the other hand, $\mathcal{Z}$ also provides a nonzero $G \times G \times H \times H$-equivariant map

$$\begin{tikzcd}
\omega \otimes \bar{\omega} \otimes \bar{\pi} \otimes \pi \arrow[r, "\mathcal{Z}"] \arrow[d] & C^{\infty}(H) . \\
\sigma \otimes \bar{\sigma} \arrow[ur] &
\end{tikzcd}$$

Hence, by uniqueness, there exists a nonzero invariant pairing

$$(\cdot, \cdot) : \sigma \otimes \bar{\sigma} \longrightarrow \mathbb{C}$$

such that

$$\mathcal{Z}(\phi, \phi'; v, v')(h) = (\sigma(h)\theta(\phi, v), \theta(\phi', v'))$$

for $\phi, \phi' \in \omega$ and $v, v' \in \pi$.

17. A local Siegel-Weil formula

In this section, we examine the explicit local theta lifting of the character χ_W of H to $\mathbf{G}$ in the other branch of the see-saw diagram in the previous section, and establish a local analog of the Siegel-Weil formula.

17.1. **The map $\mathcal{I}$.** Assume that $l \geq 0$. We define a $\Delta\mathbf{G} \times H \times H$-invariant map

$$\mathcal{I} : \boldsymbol{\omega} \otimes \bar{\boldsymbol{\omega}} \otimes \bar{\chi}_W \otimes \chi_W \longrightarrow \mathbb{C}$$

by

$$\mathcal{I}(\varphi, \varphi') = \int_H (\boldsymbol{\omega}(h)\varphi, \varphi') \cdot \overline{\chi_W(\det h)} \, dh$$

for $\varphi, \varphi' \in \boldsymbol{\omega}$. By [50, Theorem 3.2], this integral is absolutely convergent.

17.2. **The map** proj**.** We shall define another $\Delta\mathbf{G} \times H \times H$-invariant map

$$\mathcal{E} : \boldsymbol{\omega} \otimes \bar{\boldsymbol{\omega}} \otimes \bar{\chi}_W \otimes \chi_W \longrightarrow \mathbb{C}.$$

Let ρ be as in §9.3. Consider the degenerate principal series representation $I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times})$. Here, in Case C', $I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times})$ is interpreted to be the pull-back of $\mathrm{Ind}_{\mathbf{P}}^{\mathrm{Sp}(\mathbf{W})}(|\cdot|_E^{\rho})$ to $\mathrm{Mp}(\mathbf{W})$, i.e., it is non-genuine. For $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times})$, put

$$\mathrm{proj}(\mathcal{F}) = \int_G \mathcal{F}(i(g, 1)) \, dg.$$

Note that this integral is the doubling zeta integral associated to the trivial representation of G evaluated at $s = \rho$. By Lemma 9.3, it is absolutely convergent. Moreover, by Theorem 9.1(iii), there exists $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times})$ such that $\mathrm{proj}(\mathcal{F}) \neq 0$. Thus we obtain a nonzero $G \times G$-invariant map

$$\mathrm{proj} : I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times}) \longrightarrow \mathbb{C}.$$

Lemma 17.1. *We have*

$$\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times}), \mathbb{C}) = \mathrm{Hom}_{\mathbf{G}}(I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times}), \mathbb{C}) = \mathbb{C}\cdot \mathrm{proj}\,.$$

In particular, proj *is* **G***-invariant.*

Proof. Since

$$\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times}), \mathbb{C}) \supset \mathrm{Hom}_{\mathbf{G}}(I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times}), \mathbb{C}) \cong \mathbb{C},$$

it suffices to show that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times}), \mathbb{C}) = 1.$$

This follows from [21] in Case A and [47, Theorem 1.1] in Case C. It remains to treat Cases B and D.

Let V_r be the symplectic space over F of dimension $2r$. By Proposition 7.2(ii), we have

$$I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times}) = R(V_{n-1}, \chi_W) + R(V_{n-1}, \chi_W) \otimes \det.$$

Moreover, we see that

$$\mathrm{Hom}_{G\times G}(R(V_{n-1}, \chi_W)\otimes \det, \mathbb{C}) = \{0\}.$$

Indeed, by Lemma 6.1 and Proposition 6.2,

$$\mathrm{Hom}_{G\times G}(R(V_r, \chi_W)\otimes \det, \mathbb{C}) \cong \mathrm{Hom}_{G\times G}(R(V_r, \chi_W), \det)$$

is nonzero if and only if $\Theta_{V_r,W,\boldsymbol{\chi},\psi}(\det) \neq 0$. But by [65, p. 399], we know that $\Theta_{V_r,W,\boldsymbol{\chi},\psi}(\det) = 0$ unless $r \geq n$. Thus, noting that

$$I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times})/(R(V_{n-1}, \chi_W)\otimes \det) \cong \mathbf{1}_{\mathbf{G}},$$

where $\mathbf{1}_{\mathbf{G}}$ is the trivial representation of $\mathbf{G}$, we have shown that

$$\mathrm{Hom}_{G\times G}(I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times}), \mathbb{C}) \xleftarrow{\cong} \mathrm{Hom}_{G\times G}(\mathbf{1}_{\mathbf{G}}, \mathbb{C}) \cong \mathbb{C},$$

as desired. □

Hence, we obtain a nonzero invariant pairing

$$(\cdot,\cdot) : I_{\mathbf{P}}^{\mathbf{G}}(\tfrac{l}{2}, \chi_V) \otimes \overline{I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2}, \chi_V)} \longrightarrow \mathbb{C}$$

defined by

$$(\mathcal{F}, \mathcal{F}') = \mathrm{proj}(\mathcal{F}\cdot \bar{\mathcal{F}}')$$

for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(\frac{l}{2}, \chi_V)$ and $\mathcal{F}' \in I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2}, \chi_V)$. Note that $\overline{I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2}, \chi_V)} \cong I_{\mathbf{P}}^{\mathbf{G}}(\frac{l}{2}, \chi_V)^\vee$.

17.3. **The map $\mathcal{E}$.**

We can now define the map $\mathcal{E}$. Put

$$V^\dagger = V \oplus \mathbb{H}^{2\rho - m}.$$

Recall that

$$R(V, \chi_W) \subset I_{\mathbf{P}}^{\mathbf{G}}(-\tfrac{l}{2}, \chi_V),$$
$$R(V^\dagger, \chi_W) \subset I_{\mathbf{P}}^{\mathbf{G}}(\tfrac{l}{2}, \chi_V).$$

By Proposition 7.2(i), one knows that $R(V, \chi_W)$ is irreducible and unitarizable.

If $l = 0$, we simply put

$$\mathcal{E}(\varphi, \varphi') = (\mathcal{F}_\varphi, \mathcal{F}_{\varphi'})$$

for $\varphi, \varphi' \in \boldsymbol{\omega}$.

Now suppose that $l > 0$. For convenience, we write

$$\mathcal{M} = M_{\psi,-1}^{\mathrm{LR}}(\tfrac{l}{2}, \chi_V).$$

Recall that $\mathcal{M}$ induces a $\mathbf{G}$-equivariant surjective map

$$\mathcal{M} : R(V^\dagger, \chi_W) \twoheadrightarrow R(V, \chi_W).$$

Since $\ker \mathcal{M}$ is irreducible and $\ker \mathcal{M} \not\cong R(V, \chi_W)$, we have

$$(\mathcal{F}, \mathcal{F}') = 0 \quad \text{for } \mathcal{F} \in \ker \mathcal{M} \text{ and } \mathcal{F}' \in R(V, \chi_W).$$

Hence $\mathcal{M}$ and $(\cdot, \cdot)$ induce a nonzero invariant pairing

$$R(V, \chi_W) \otimes \overline{R(V, \chi_W)} \xleftarrow{\cong} (R(V^\dagger, \chi_W)/\ker \mathcal{M}) \otimes \overline{R(V, \chi_W)} \longrightarrow \mathbb{C}.$$

Let $\mathcal{E}$ be the composition of the $\mathbf{G} \times \mathbf{G} \times H \times H$-equivariant map

$$\boldsymbol{\omega} \otimes \bar{\boldsymbol{\omega}} \otimes \bar{\chi}_W \otimes \chi_W \longrightarrow R(V, \chi_W) \otimes \overline{R(V, \chi_W)}$$

with this pairing. More explicitly, we put

$$\mathcal{E}(\varphi, \varphi') = (\mathcal{F}_\varphi^\dagger, \mathcal{F}_{\varphi'})$$

for $\varphi, \varphi' \in \boldsymbol{\omega}$, where we choose $\mathcal{F}_\varphi^\dagger \in R(V^\dagger, \chi_W)$ such that $\mathcal{M}\mathcal{F}_\varphi^\dagger = \mathcal{F}_\varphi$.

17.4. **A local Siegel-Weil formula.** We have constructed two elements:

$$\mathcal{I}, \mathcal{E} \in \mathrm{Hom}_{\Delta \mathbf{G} \times H \times H}(\boldsymbol{\omega} \otimes \bar{\boldsymbol{\omega}} \otimes \bar{\chi}_W \otimes \chi_W, \mathbb{C}).$$

On the other hand, we have

$$\mathrm{Hom}_{\Delta \mathbf{G} \times H \times H}(\boldsymbol{\omega} \otimes \bar{\boldsymbol{\omega}} \otimes \bar{\chi}_W \otimes \chi_W, \mathbb{C}) \cong \mathrm{Hom}_{\Delta \mathbf{G}}(R(V, \chi_W) \otimes \overline{R(V, \chi_W)}, \mathbb{C}),$$

so that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta \mathbf{G} \times H \times H}(\boldsymbol{\omega} \otimes \bar{\boldsymbol{\omega}} \otimes \bar{\chi}_W \otimes \chi_W, \mathbb{C}) = 1$$

since $R(V, \chi_W)$ is irreducible and unitarizable. Thus, we have proved:

Theorem 17.2. *There exists a constant C such that*

$$\mathcal{I} = C \cdot \mathcal{E}.$$

We shall determine the constant C when G and H are unramified in §19 and when $0 \le l \le 1$ in §20.

18. A local Rallis inner product formula

In this section, we prove Theorem 15.1 up to a scalar, by establishing a local analog of the Rallis inner product formula.

We define a map

$$\mathcal{P} : \omega \otimes \bar{\omega} \otimes \bar{\omega} \otimes \omega \otimes \bar{\pi} \otimes \pi \otimes \pi \otimes \bar{\pi} \longrightarrow \mathbb{C}$$

by

$$\mathcal{P}(\phi_1, \phi_2, \phi_3, \phi_4; v_1, v_2, v_3, v_4) = \int_H (\sigma(h)\theta(\phi_1, v_1), \theta(\phi_2, v_2)) \cdot \overline{(\sigma(h)\theta(\phi_3, v_3), \theta(\phi_4, v_4))} \, dh$$

for $\phi_1, \dots, \phi_4 \in \omega$ and $v_1, \dots, v_4 \in \pi$. This is nothing but the inner product of two matrix coefficients of σ. By definition, we have

$$\begin{aligned} \mathcal{P}(\phi_1, \dots, \phi_4; v_1, \dots, v_4) &= \frac{1}{\deg \sigma} \cdot (\theta(\phi_1, v_1), \theta(\phi_3, v_3)) \cdot \overline{(\theta(\phi_2, v_2), \theta(\phi_4, v_4))} \\ &= \frac{1}{\deg \sigma} \cdot Z(-\tfrac{l}{2}, \mathcal{F}_{\phi_1 \otimes \bar{\phi}_3}, \bar{f}_{v_1, v_3}) \cdot \overline{Z(-\tfrac{l}{2}, \mathcal{F}_{\phi_2 \otimes \bar{\phi}_4}, \bar{f}_{v_2, v_4})}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\mathcal{P}(\phi_1,\dots,\phi_4;v_1,\dots,v_4)\\&=\int_H Z(-\tfrac{l}{2},\mathcal{F}_{\omega(h)\phi_1\otimes\bar{\phi}_2},\bar{f}_{v_1,v_2})\cdot\overline{Z(-\tfrac{l}{2},\mathcal{F}_{\omega(h)\phi_3\otimes\bar{\phi}_4},\bar{f}_{v_3,v_4})}\,dh\\&=\int_H\int_G\int_G\mathcal{F}_{\omega(h)\phi_1\otimes\bar{\phi}_2}(i(g,1))\cdot\overline{\mathcal{F}_{\omega(h)\phi_3\otimes\bar{\phi}_4}(i(g',1))}\cdot\overline{f_{v_1,v_2}(g)}\cdot f_{v_3,v_4}(g')\,dg\,dg'\,dh.\end{aligned}$$

In Lemma C.1 below, we will show that this triple integral is absolutely convergent, so that we may interchange the order of integration. We move the integral over H inside. By definition, we have

$$\begin{aligned}\int_H\mathcal{F}_{\omega(h)\phi_1\otimes\bar{\phi}_2}(i(g,1))\cdot\overline{\mathcal{F}_{\omega(h)\phi_3\otimes\bar{\phi}_4}(i(g',1))}\,dh&=\int_H(\omega(gh)\phi_1,\phi_2)\cdot\overline{(\omega(g'h)\phi_3,\phi_4)}\,dh\\&=\mathcal{I}(\omega(g)\phi_1\otimes\overline{\omega(g')\phi_3},\phi_2\otimes\bar{\phi}_4).\end{aligned}$$

Hence, by the local Siegel-Weil formula (Theorem 17.2), we have

$$\begin{aligned}\mathcal{P}(\phi_1,\dots,\phi_4;v_1,\dots,v_4)&=\int_G\int_G\mathcal{I}(\omega(g)\phi_1\otimes\overline{\omega(g')\phi_3},\phi_2\otimes\bar{\phi}_4)\cdot\overline{f_{v_1,v_2}(g)}\cdot f_{v_3,v_4}(g')\,dg\,dg'\\&=C\int_G\int_G\mathcal{E}(\omega(g)\phi_1\otimes\overline{\omega(g')\phi_3},\phi_2\otimes\bar{\phi}_4)\cdot\overline{f_{v_1,v_2}(g)}\cdot f_{v_3,v_4}(g')\,dg\,dg',\end{aligned}$$

where C is the constant given in Theorem 17.2. For convenience, we write

$$\mathcal{F}=\begin{cases}\mathcal{F}_{\phi_1\otimes\bar{\phi}_3} & \text{if } l=0,\\ \mathcal{F}^{\dagger}_{\phi_1\otimes\bar{\phi}_3} & \text{if } l>0,\end{cases}\quad\text{and}\quad\mathcal{F}'=\mathcal{F}_{\phi_2\otimes\bar{\phi}_4}.$$

Here we choose $\mathcal{F}^{\dagger}_{\phi_1\otimes\bar{\phi}_3}\in I_{\mathbf{P}}^{\mathbf{G}}(\frac{l}{2},\chi_V)$ such that $\mathcal{M}\mathcal{F}^{\dagger}_{\phi_1\otimes\bar{\phi}_3}=\mathcal{F}_{\phi_1\otimes\bar{\phi}_3}$ when $l>0$. Then by definition, we have

$$\begin{aligned}\mathcal{E}(\omega(g)\phi_1\otimes\overline{\omega(g')\phi_3},\phi_2\otimes\bar{\phi}_4)&=\int_G\mathcal{F}(i(g''g,g'))\cdot\chi_V(\det g')^{-1}\cdot\overline{\mathcal{F}'(i(g'',1))}\,dg''\\&=\int_G\mathcal{F}(i(g'^{-1}g''g,1))\cdot\overline{\mathcal{F}'(i(g'',1))}\,dg''.\end{aligned}$$

Changing the variable $g''\mapsto g''g^{-1}$, we have

$$\mathcal{E}(\omega(g)\phi_1\otimes\overline{\omega(g')\phi_3},\phi_2\otimes\bar{\phi}_4)=\int_G\mathcal{F}(i(g'^{-1}g'',1))\cdot\overline{\mathcal{F}'(i(g''g^{-1},1))}\,dg''.$$

Hence we have

$$\begin{aligned}&\mathcal{P}(\phi_1,\dots,\phi_4;v_1,\dots,v_4)\\&=C\int_G\int_G\int_G\mathcal{F}(i(g'^{-1}g'',1))\cdot\overline{\mathcal{F}'(i(g''g^{-1},1))}\cdot\overline{f_{v_1,v_2}(g)}\cdot f_{v_3,v_4}(g')\,dg''\,dg\,dg'.\end{aligned}$$

In Lemma C.2 below, we will show that this triple integral is absolutely convergent, so that we may interchange the order of integration. We move the integral over g'' outside. Changing the variables $g\mapsto g^{-1}g''$ and $g'\mapsto g''g'^{-1}$, we have

$$\begin{aligned}&\mathcal{P}(\phi_1,\dots,\phi_4;v_1,\dots,v_4)\\&=C\int_G\int_G\int_G\mathcal{F}(i(g',1))\cdot\overline{\mathcal{F}'(i(g,1))}\cdot\overline{f_{v_1,v_2}(g^{-1}g'')}\cdot f_{v_3,v_4}(g''g'^{-1})\,dg\,dg'\,dg''.\end{aligned}$$

We may interchange the order of integration again and move the integral over g'' inside. We have

$$\begin{aligned}\int_G \overline{f_{v_1,v_2}(g^{-1}g'')} \cdot f_{v_3,v_4}(g''g'^{-1})\, dg'' &= \int_G \overline{f_{v_1,v_2}(g^{-1}g''g')} \cdot f_{v_3,v_4}(g'')\, dg'' \\ &= \int_G \overline{(\pi(g''g')v_1, \pi(g)v_2)} \cdot (\pi(g'')v_3, v_4)\, dg'' \\ &= \frac{1}{\deg \pi} \cdot \overline{(\pi(g')v_1, v_3)} \cdot (\pi(g)v_2, v_4),\end{aligned}$$

so that

$$\begin{aligned}\mathcal{P}(\phi_1, \dots, \phi_4; v_1, \dots, v_4) &= \frac{C}{\deg \pi} \cdot \int_G \int_G \mathcal{F}(i(g',1)) \cdot \overline{\mathcal{F}'(i(g,1))} \cdot \overline{f_{v_1,v_3}(g')} \cdot f_{v_2,v_4}(g)\, dg\, dg' \\ &= \frac{C}{\deg \pi} \cdot Z(\tfrac{l}{2}, \mathcal{F}, \bar{f}_{v_1,v_3}) \cdot \overline{Z(-\tfrac{l}{2}, \mathcal{F}', \bar{f}_{v_2,v_4})}.\end{aligned}$$

Thus we obtain

$$\frac{1}{\deg \sigma} \cdot Z(-\tfrac{l}{2}, \mathcal{F}_{\phi_1 \otimes \bar{\phi}_3}, \bar{f}_{v_1,v_3}) \cdot \overline{Z(-\tfrac{l}{2}, \mathcal{F}_{\phi_2 \otimes \bar{\phi}_4}, \bar{f}_{v_2,v_4})} = \frac{C}{\deg \pi} \cdot Z(\tfrac{l}{2}, \mathcal{F}, \bar{f}_{v_1,v_3}) \cdot \overline{Z(-\tfrac{l}{2}, \mathcal{F}', \bar{f}_{v_2,v_4})}.$$

When $l = 0$, we conclude that

$$\frac{\deg \pi}{\deg \sigma} = C. \tag{18.1}$$

Henceforth, we assume that $l > 0$. By the local functional equation

$$Z(-\tfrac{l}{2}, \mathcal{F}_{\phi_1 \otimes \bar{\phi}_3}, \bar{f}_{v_1,v_3}) = \varepsilon \cdot \omega_\pi(-1) \cdot \chi_V(\det \beta_W) \cdot |\det \beta_W|_E^{l/2} \cdot \operatorname*{Res}_{s=\frac{l+1}{2}} \gamma(s, \pi, \chi_V^c, \psi) \cdot Z(\tfrac{l}{2}, \mathcal{F}, \bar{f}_{v_1,v_3}),$$

we have

$$\frac{\deg \pi}{\deg \sigma} = C \cdot \varepsilon^{-1} \cdot \omega_\pi(-1)^{-1} \cdot \chi_V(\det \beta_W)^{-1} \cdot |\det \beta_W|_E^{-l/2} \cdot \left(\operatorname*{Res}_{s=\frac{l+1}{2}} \gamma(s, \pi, \chi_V^c, \psi) \right)^{-1}.$$

Here

$$\varepsilon = \begin{cases} \epsilon(W)^{n+1} & \text{in Case A,} \\ \epsilon(W) & \text{in Case B,} \\ \chi_\psi^W(-1)^{-1} & \text{in Case C',} \\ 1 & \text{in Case C'',} \\ \epsilon(\tfrac{1}{2}, \chi_W, \psi)^{-1} & \text{in Case D.} \end{cases}$$

Lemma 18.1. *We have*

$$\begin{aligned} &\varepsilon^{-1} \cdot \omega_\pi(-1)^{-1} \cdot \left(\operatorname*{Res}_{s=\frac{l+1}{2}} \gamma(s, \pi, \chi_V^c, \psi) \right)^{-1} \\ &= -\omega_\sigma(-1) \cdot \gamma(-\tfrac{l-1}{2}, \sigma, \chi_W^{-1}, \psi) \cdot \lim_{s \to 0} s^{-1} \gamma(s, \mathbb{1}_{E^\times}, \psi_E) \cdot \prod_{i=1}^{l-1} \gamma(-i, \mathbb{1}_{E^\times}, \psi_E) \\ &\quad \times \begin{cases} \chi_E(-1)^{mn} \cdot (\chi_W^m / \chi_V^n)(\daleth) \cdot \epsilon(W)^{n+1} & \textit{in Case A,} \\ \chi_\psi^V(-1)^{-1} \cdot \chi_W(-1)^{m/2} \cdot \epsilon(W) & \textit{in Case B,} \\ \chi_V(-1)^{n/2} & \textit{in Case C',} \\ \chi_V(-1)^{n/2+1} & \textit{in Case C'',} \\ \chi_W(-1)^{m/2} \cdot \epsilon(\tfrac{1}{2}, \chi_W, \psi)^{-1} & \textit{in Case D.} \end{cases} \end{aligned}$$

Proof. By §5.2, we have

$$\varepsilon^{-1}\cdot\omega_\pi(-1)^{-1} = \omega_\sigma(-1)\times\begin{cases}(\chi_W^m/\chi_V^n)(\daleth)\cdot\epsilon(W)^{n+1} & \text{in Case A,}\\ \chi_\psi^V(-1)^{-1}\cdot\chi_W(-1)^{m/2}\cdot\epsilon(W) & \text{in Case B,}\\ \chi_V(-1)^{n/2} & \text{in Case C',}\\ \chi_V(-1)^{n/2} & \text{in Case C",}\\ \chi_W(-1)^{m/2}\cdot\epsilon(\frac{1}{2},\chi_W,\psi) & \text{in Case D.}\end{cases}$$

On the other hand, by Theorem 11.3, we have

$$\operatorname*{Res}_{s=\frac{l+1}{2}}\gamma(s,\pi,\chi_V^c,\psi) = \gamma(\tfrac{l+1}{2},\sigma,\chi_W^c,\psi)\cdot\operatorname*{Res}_{s=1}\gamma(s,\mathbb{1}_{E^\times},\psi_E)\cdot\prod_{i=2}^{l}\gamma(i,\mathbb{1}_{E^\times},\psi_E).$$

By the Ten Commandments, we have

$$\begin{aligned}\gamma(\tfrac{l+1}{2},\sigma,\chi_W^c,\psi)^{-1} &= \gamma(\tfrac{l+1}{2},\sigma^\vee,\chi_W,\psi)^{-1} = \gamma(-\tfrac{l-1}{2},\sigma,\chi_W^{-1},\bar{\psi})\\ &= \gamma(-\tfrac{l-1}{2},\sigma,\chi_W^{-1},\psi)\times\begin{cases}\chi_E(-1)^{mn} & \text{in Case A,}\\ 1 & \text{in Case C',}\\ \chi_V(-1) & \text{in Case C",}\\ \chi_W(-1) & \text{in Case D,}\end{cases}\end{aligned}$$

and

$$\gamma(\tfrac{l+1}{2},\sigma,\chi_W^c,\psi)^{-1} = \gamma(\tfrac{l+1}{2},\sigma^\vee,\chi_W,\bar{\psi})^{-1} = \gamma(-\tfrac{l-1}{2},\sigma,\chi_W^{-1},\psi)$$

in Case B. Also, we have

$$\left(\operatorname*{Res}_{s=1}\gamma(s,\mathbb{1}_{E^\times},\psi_E)\cdot\prod_{i=2}^{l}\gamma(i,\mathbb{1}_{E^\times},\psi_E)\right)^{-1} = -\lim_{s\to 0}s^{-1}\gamma(s,\mathbb{1}_{E^\times},\psi_E)\cdot\prod_{i=1}^{l-1}\gamma(-i,\mathbb{1}_{E^\times},\psi_E).$$

This completes the proof. □

Hence, when $l>0$, we conclude that

$$\begin{aligned}\frac{\deg\pi}{\deg\sigma} &= -C\cdot\omega_\sigma(-1)\cdot\gamma(-\tfrac{l-1}{2},\sigma,\chi_W^{-1},\psi)\cdot\chi_V(\det\beta_W)^{-1}\cdot|\det\beta_W|_E^{-l/2}\\ &\quad\times\lim_{s\to 0}s^{-1}\gamma(s,\mathbb{1}_{E^\times},\psi_E)\cdot\prod_{i=1}^{l-1}\gamma(-i,\mathbb{1}_{E^\times},\psi_E)\\ &\quad\times\begin{cases}\chi_E(-1)^{mn}\cdot(\chi_W^m/\chi_V^n)(\daleth)\cdot\epsilon(W)^{n+1} & \text{in Case A,}\\ \chi_\psi^V(-1)^{-1}\cdot\chi_W(-1)^{m/2}\cdot\epsilon(W) & \text{in Case B,}\\ \chi_V(-1)^{n/2} & \text{in Case C',}\\ \chi_V(-1)^{n/2+1} & \text{in Case C",}\\ \chi_W(-1)^{m/2}\cdot\epsilon(\frac{1}{2},\chi_W,\psi)^{-1} & \text{in Case D.}\end{cases}\end{aligned}\tag{18.2}$$

Thus, we have proved Theorem 15.1 up to a scalar. In particular, this completes the proof of Theorem 15.1(iii), except for the assertion that $\mathbf{c}=1$ when G and H are unramified. We will prove this assertion in the next section.

19. Determination of the constant C — the unramified case

In this section, we shall determine the constant C in the local Siegel-Weil formula when G and H are unramified. This will complete the proof of Theorem 15.1(iii).

19.1. **Unramified hypothesis.** We shall make the following hypothesis throughout this section:

Hypothesis (UR).

- $p \neq 2$;
- G and H are quasi-split over F and split over F^{ur};
- $\operatorname{disc} V, \operatorname{disc} W \in \mathfrak{o}_E^\times$;
- χ_V and χ_W are unramified;
- ψ is of order zero;
- $\det \beta_W \in \mathfrak{o}_E^\times$.

We shall show:

Proposition 19.1. *Let C be the constant given in Theorem 17.2. Under Hypothesis (UR), we have:*

(i) *If $l = 0$, then we have*

$$C = \begin{cases} 1 & \textit{in Case A,} \\ 2^{-1} & \textit{in Case B,} \\ 2 & \textit{in Case C'.} \end{cases}$$

(ii) *If $l > 0$, then we have*

$$C = \begin{cases} -2^{-1} \cdot \operatorname{Res}_{s=0} \gamma(s, \mathbb{1}, \psi)^{-1} \cdot \prod_{i=1}^{l-1} \gamma(-i, \chi_E^i, \psi)^{-1} & \textit{in Case A,} \\ -2^{-2} \cdot \operatorname{Res}_{s=0} \gamma(s, \mathbb{1}, \psi)^{-1} \cdot \prod_{i=1}^{\frac{l}{2}-1} \gamma(-2i, \mathbb{1}, \psi)^{-1} & \textit{in Case B,} \\ -\operatorname{Res}_{s=0} \gamma(s, \mathbb{1}, \psi)^{-1} \cdot \prod_{i=1}^{\frac{l}{2}-1} \gamma(-2i, \mathbb{1}, \psi)^{-1} & \textit{in Case C',} \\ -\operatorname{Res}_{s=0} \gamma(s, \mathbb{1}, \psi)^{-1} \cdot \prod_{i=1}^{\frac{l-1}{2}} \gamma(-2i, \mathbb{1}, \psi)^{-1} & \textit{in Case C'',} \\ -2^{-2} \cdot \operatorname{Res}_{s=0} \gamma(s, \mathbb{1}, \psi)^{-1} \cdot \prod_{i=1}^{\frac{l-1}{2}} \gamma(-2i, \mathbb{1}, \psi)^{-1} & \textit{in Case D.} \end{cases}$$

Here the products $\prod_{i=1}^{l-1}$ and $\prod_{i=1}^{\frac{l-1}{2}}$ are interpreted to be 1 when $l = 1$.

In view of (18.2), this completes the proof of Theorem 15.1(iii). The rest of this section is devoted to the proof of Proposition 19.1.

19.2. **Volumes.** Let K_G be a hyperspecial maximal compact subgroup of G stabilizing a self-dual lattice $\mathcal{L}_W$ in W. Similarly, let K_H be a hyperspecial maximal compact subgroup of H stabilizing a self-dual lattice $\mathcal{L}_V$ in V. Then we have

$$\mathrm{vol}(K_G) = \begin{cases} \prod_{i=1}^{n} L(i, \chi_E^i)^{-1} & \text{in Case A,} \\ 2 \cdot \prod_{i=1}^{\frac{n-1}{2}} \zeta(2i)^{-1} & \text{in Case B,} \\ \prod_{i=1}^{\frac{n}{2}} \zeta(2i)^{-1} & \text{in Case C',} \\ \prod_{i=1}^{\frac{n}{2}} \zeta(2i)^{-1} & \text{in Case C'',} \\ 2 \cdot L(\frac{n}{2}, \chi_W)^{-1} \cdot \prod_{i=1}^{\frac{n}{2}-1} \zeta(2i)^{-1} & \text{in Case D,} \end{cases}$$

$$\mathrm{vol}(K_H) = \begin{cases} \prod_{i=1}^{m} L(i, \chi_E^i)^{-1} & \text{in Case A,} \\ \prod_{i=1}^{\frac{m}{2}} \zeta(2i)^{-1} & \text{in Case B,} \\ 2 \cdot \prod_{i=1}^{\frac{m-1}{2}} \zeta(2i)^{-1} & \text{in Case C',} \\ 2 \cdot L(\frac{m}{2}, \chi_V)^{-1} \cdot \prod_{i=1}^{\frac{m}{2}-1} \zeta(2i)^{-1} & \text{in Case C'',} \\ \prod_{i=1}^{\frac{m}{2}} \zeta(2i)^{-1} & \text{in Case D.} \end{cases}$$

19.3. **The map $\mathcal{I}$.** Let $\varphi^o \in \boldsymbol{\omega}$ be the characteristic function of the self-dual lattice $\mathcal{L}_V \otimes \mathcal{L}_W$ in $V \otimes W^\nabla$. We first compute $\mathcal{I}(\varphi^o, \varphi^o)$.

Lemma 19.2. *We have*

$$\mathcal{I}(\varphi^o, \varphi^o) = \mathrm{vol}(K_H) \cdot \frac{L(\frac{l+n+1}{2}, \mathbf{1}_H)}{d_{\mathbf{H}}(\frac{l+n}{2})},$$

where

$$L(s, \mathbf{1}_H) = \begin{cases} \prod_{i=1}^{m} \zeta_E(s + \frac{m+1}{2} - i) & \textit{in Case A,} \\ \prod_{i=0}^{m} \zeta(s + \frac{m}{2} - i) & \textit{in Case B,} \\ \prod_{i=1}^{m-1} \zeta(s + \frac{m}{2} - i) & \textit{in Case C',} \\ L(s, \chi_V) \cdot \prod_{i=1}^{m-1} \zeta(s + \frac{m}{2} - i) & \textit{in Case C'',} \\ \prod_{i=0}^{m} \zeta(s + \frac{m}{2} - i) & \textit{in Case D,} \end{cases}$$

$$d_{\mathbf{H}}(s) = \begin{cases} \prod_{i=1}^{m} L(2s + i, \chi_E^{m-i}) & \textit{in Case A,} \\ \zeta(s + \frac{m+1}{2}) \cdot \prod_{i=1}^{\frac{m}{2}} \zeta(2s + 2i - 1) & \textit{in Case B,} \\ \prod_{i=1}^{\frac{m-1}{2}} \zeta(2s + 2i) & \textit{in Case C',} \\ \prod_{i=1}^{\frac{m}{2}} \zeta(2s + 2i - 1) & \textit{in Case C'',} \\ \zeta(s + \frac{m+1}{2}) \cdot \prod_{i=1}^{\frac{m}{2}} \zeta(2s + 2i - 1) & \textit{in Case D.} \end{cases}$$

Proof. Let $\mathbf{V} = V \oplus V_-$. Let $\mathbf{H}$ be the isometry group of $\mathbf{V}$ and $\mathbf{Q}$ the Siegel parabolic subgroup of $\mathbf{H}$ stabilizing V^Δ. Consider the degenerate principal series representation $I_{\mathbf{Q}}^{\mathbf{H}}(\frac{l+n}{2}, \mathbb{1}_{E^\times})$. We define $\Phi_{\mathbf{H}} \in I_{\mathbf{Q}}^{\mathbf{H}}(\frac{l+n}{2}, \mathbb{1}_{E^\times})$ so that $\Phi_{\mathbf{H}}(\mathbf{k}) = 1$ for $\mathbf{k} \in K_{\mathbf{H}}$, where $K_{\mathbf{H}}$ is the hyperspecial maximal compact subgroup of $\mathbf{H}$ stabilizing the self-dual lattice $\mathcal{L}_V \oplus \mathcal{L}_V$ in $\mathbf{V}$. Then we have

$$\mathcal{I}(\varphi^o, \varphi^o) = Z(\tfrac{l+n}{2}, \Phi_{\mathbf{H}}, \mathbf{1}_H),$$

where $Z(\frac{l+n}{2}, \Phi_{\mathbf{H}}, \mathbf{1}_H)$ is the doubling zeta integral associated to the trivial representation of H evaluated at $s = \frac{l+n}{2}$. Hence the desired identity follows from the unramified computation [49, Proposition 3], [51, Theorem 3.1]. □

19.4. **The map $\mathcal{E}$.** We next compute $\mathcal{E}(\varphi^o, \varphi^o)$.

Lemma 19.3. *We have*

$$\mathcal{E}(\varphi^o, \varphi^o) = \mathrm{vol}(K_G) \cdot \frac{L(\rho + \frac{1}{2}, \mathbf{1}_G)}{d_{\mathbf{G}}(\rho)} \times \begin{cases} 1 & \textit{if } l = 0, \\ \dfrac{b(\frac{l}{2}, \chi_V)}{\mathrm{Res}_{s=\frac{l}{2}} b(-s, \chi_V)} & \textit{if } l > 0, \end{cases}$$

where

$$L(s, \mathbf{1}_G) = \begin{cases} \prod_{i=1}^{n} \zeta_E(s + \frac{n+1}{2} - i) & \textit{in Case A,} \\ \prod_{i=1}^{n-1} \zeta(s + \frac{n}{2} - i) & \textit{in Case B,} \\ \prod_{i=0}^{n} \zeta(s + \frac{n}{2} - i) & \textit{in Case C’,} \\ \prod_{i=0}^{n} \zeta(s + \frac{n}{2} - i) & \textit{in Case C”,} \\ L(s, \chi_W) \cdot \prod_{i=1}^{n-1} \zeta(s + \frac{n}{2} - i) & \textit{in Case D,} \end{cases}$$

$$d_{\mathbf{G}}(s) = \begin{cases} \prod_{i=1}^{n} L(2s + i, \chi_E^{n-i}) & \textit{in Case A,} \\ \prod_{i=1}^{\frac{n-1}{2}} \zeta(2s + 2i) & \textit{in Case B,} \\ \zeta(s + \frac{n+1}{2}) \cdot \prod_{i=1}^{\frac{n}{2}} \zeta(2s + 2i - 1) & \textit{in Case C’,} \\ \zeta(s + \frac{n+1}{2}) \cdot \prod_{i=1}^{\frac{n}{2}} \zeta(2s + 2i - 1) & \textit{in Case C”,} \\ \prod_{i=1}^{\frac{n}{2}} \zeta(2s + 2i - 1) & \textit{in Case D,} \end{cases}$$

$$b(s, \chi_V) = \begin{cases} \prod_{i=1}^{n} L(2s + i, \chi_E^{m+n-i}) & \textit{in Case A,} \\ \prod_{i=1}^{\frac{n-1}{2}} \zeta(2s + 2i) & \textit{in Case B} \\ \prod_{i=1}^{\frac{n}{2}} \zeta(2s + 2i) & \textit{in Case C’} \\ L(s + \frac{n+1}{2}, \chi_V) \cdot \prod_{i=1}^{\frac{n}{2}} \zeta(2s + 2i - 1) & \textit{in Case C”,} \\ \prod_{i=1}^{\frac{n}{2}} \zeta(2s + 2i - 1) & \textit{in Case D.} \end{cases}$$

Proof. Consider the degenerate principal series representation $I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times})$. We define $\Phi_{\mathbf{G}} \in I_{\mathbf{P}}^{\mathbf{G}}(\rho, \mathbb{1}_{E^\times})$ so that $\Phi_{\mathbf{G}}(\mathbf{k}) = 1$ for $\mathbf{k} \in K_{\mathbf{G}}$, where $K_{\mathbf{G}}$ is the hyperspecial maximal compact subgroup of $\mathbf{G}$ stabilizing the self-dual lattice $\mathcal{L}_W \oplus \mathcal{L}_W$ in $\mathbf{W}$. If $l = 0$, then we have

$$\mathcal{F}_{\varphi^o} \cdot \bar{\mathcal{F}}_{\varphi^o} = \Phi_{\mathbf{G}}.$$

Now suppose that $l > 0$. We define $\mathcal{F}_s \in I_{\mathbf{P}}^{\mathbf{G}}(s, \chi_V)$ so that $\mathcal{F}_s(\mathbf{k}) = 1$ for $\mathbf{k} \in K_{\mathbf{G}}$. By the Gindikin-Karpelevich formula, we have

$$M(s, \chi_V)\mathcal{F}_s = \frac{a(s, \chi_V)}{b(s, \chi_V)} \cdot \mathcal{F}_{-s},$$

where

$$a(s, \chi_V) = \begin{cases} \prod_{i=1}^{n} L(2s - i + 1, \chi_E^{m+n-i}) & \text{in Case A,} \\ \prod_{i=1}^{\frac{n-1}{2}} \zeta(2s - 2i + 1) & \text{in Case B,} \\ \prod_{i=1}^{\frac{n}{2}} \zeta(2s - 2i + 1) & \text{in Case C’,} \\ L(s - \frac{n-1}{2}, \chi_V) \cdot \prod_{i=1}^{\frac{n}{2}} \zeta(2s - 2i + 2) & \text{in Case C”,} \\ \prod_{i=1}^{\frac{n}{2}} \zeta(2s - 2i + 2) & \text{in Case D.} \end{cases}$$

In Proposition A.2 below, we will see that

$$M_{\psi}^{\mathrm{LR}}(s, \chi_V) = \frac{b(-s, \chi_V)}{a(s, \chi_V)} \cdot M(s, \chi_V).$$

Hence, putting

$$\mathcal{F}^{\dagger}_{\varphi^o} = \frac{b(\frac{l}{2}, \chi_V)}{\operatorname{Res}_{s=\frac{l}{2}} b(-s, \chi_V)} \cdot \mathcal{F}_{l/2},$$

we obtain $\mathcal{M}\mathcal{F}^{\dagger}_{\varphi^o} = \mathcal{F}_{\varphi^o}$ and

$$\mathcal{F}^{\dagger}_{\varphi^o} \cdot \bar{\mathcal{F}}_{\varphi^o} = \frac{b(\frac{l}{2}, \chi_V)}{\operatorname{Res}_{s=\frac{l}{2}} b(-s, \chi_V)} \cdot \Phi_{\mathbf{G}}.$$

Thus, we have shown that

$$\mathcal{E}(\varphi^o, \varphi^o) = Z(\rho, \Phi_{\mathbf{G}}, \mathbf{1}_G) \times \begin{cases} 1 & \text{if } l = 0, \\ \dfrac{b(\frac{l}{2}, \chi_V)}{\operatorname{Res}_{s=\frac{l}{2}} b(-s, \chi_V)} & \text{if } l > 0, \end{cases}$$

where $Z(\rho, \Phi_{\mathbf{G}}, \mathbf{1}_G)$ is the doubling zeta integral associated to the trivial representation of G evaluated at $s = \rho$. Hence the desired identity follows from the unramified computation [49, Proposition 3], [51, Theorem 3.1]. □

19.5. **Proof of Proposition 19.1.** If $l = 0$, then by Lemmas 19.2 and 19.3, we have

$$C = \frac{\operatorname{vol}(K_H)}{\operatorname{vol}(K_G)} \cdot \frac{d_{\mathbf{G}}(\rho)}{d_{\mathbf{H}}(\frac{n}{2})} \cdot \frac{L(\frac{n+1}{2}, \mathbf{1}_H)}{L(\rho + \frac{1}{2}, \mathbf{1}_G)} = \begin{cases} 1 & \text{in Case A,} \\ 2^{-1} & \text{in Case B,} \\ 2 & \text{in Case C',} \end{cases}$$

as desired.

Now suppose that $l > 0$. By Lemmas 19.2 and 19.3, we have

$$C = \frac{\operatorname{vol}(K_H)}{\operatorname{vol}(K_G)} \cdot \frac{d_{\mathbf{G}}(\rho)}{d_H(\frac{l+n}{2})} \cdot \frac{L(\frac{l+n+1}{2}, \mathbf{1}_H)}{L(\rho + \frac{1}{2}, \mathbf{1}_G)} \cdot \frac{\operatorname{Res}_{s=\frac{l}{2}} b(-s, \chi_V)}{b(\frac{l}{2}, \chi_V)}.$$

Since

$$\mathrm{vol}(K_H) \cdot \operatorname*{Res}_{s=\frac{l}{2}} b(-s, \chi_V) = \begin{cases} -2^{-1} \cdot (\log q)^{-1} \cdot \prod_{i=1}^{l-1} L(-i, \chi_E^i) & \text{in Case A,} \\ -2^{-1} \cdot (\log q)^{-1} \cdot \prod_{i=1}^{\frac{l}{2}-1} \zeta(-2i) & \text{in Case B,} \\ -(\log q)^{-1} \cdot \prod_{i=1}^{\frac{l}{2}-1} \zeta(-2i) & \text{in Case C',} \\ -(\log q)^{-1} \cdot \prod_{i=1}^{\frac{l-1}{2}} \zeta(-2i) & \text{in Case C'',} \\ -2^{-1} \cdot (\log q)^{-1} \cdot \prod_{i=1}^{\frac{l-1}{2}} \zeta(-2i) & \text{in Case D,} \end{cases}$$

$$\frac{d_{\mathbf{G}}(\rho)}{\mathrm{vol}(K_G)} = \begin{cases} \prod_{i=1}^{2n} L(i, \chi_E^i) & \text{in Case A,} \\ 2^{-1} \cdot \prod_{i=1}^{n-1} \zeta(2i) & \text{in Case B,} \\ \zeta(n+1) \cdot \prod_{i=1}^{n} \zeta(2i) & \text{in Case C',} \\ \zeta(n+1) \cdot \prod_{i=1}^{n} \zeta(2i) & \text{in Case C'',} \\ 2^{-1} \cdot L(\frac{n}{2}, \chi_W) \cdot \prod_{i=1}^{n-1} \zeta(2i) & \text{in Case D,} \end{cases}$$

$$\frac{1}{d_{\mathbf{H}}(\frac{l+n}{2}) \cdot b(\frac{l}{2}, \chi_V)} = \begin{cases} \prod_{i=l+1}^{2n} L(i, \chi_E^i)^{-1} & \text{in Case A,} \\ \zeta(n)^{-1} \cdot \prod_{i=\frac{l}{2}+1}^{n-1} \zeta(2i)^{-1} & \text{in Case B,} \\ \prod_{i=\frac{l}{2}+1}^{n} \zeta(2i)^{-1} & \text{in Case C',} \\ L(\frac{l+n+1}{2}, \chi_V)^{-1} \cdot \prod_{i=\frac{l+1}{2}}^{n} \zeta(2i)^{-1} & \text{in Case C'',} \\ \zeta(n)^{-1} \cdot \prod_{i=\frac{l+1}{2}}^{n-1} \zeta(2i)^{-1} & \text{in Case D,} \end{cases}$$

$$\frac{L(\frac{l+n+1}{2}, \mathbf{1}_H)}{L(\rho + \frac{1}{2}, \mathbf{1}_G)} = \begin{cases} \prod_{i=1}^{l} \zeta_E(i)^{-1} & \text{in Case A,} \\ \zeta(n) \cdot \prod_{i=1}^{l} \zeta(i)^{-1} & \text{in Case B,} \\ \zeta(n+1)^{-1} \cdot \prod_{i=1}^{l} \zeta(i)^{-1} & \text{in Case C',} \\ L(\frac{l+n+1}{2}, \chi_V) \cdot \zeta(n+1)^{-1} \cdot \prod_{i=1}^{l} \zeta(i)^{-1} & \text{in Case C'',} \\ L(\frac{n}{2}, \chi_W)^{-1} \cdot \zeta(n) \cdot \prod_{i=1}^{l} \zeta(i)^{-1} & \text{in Case D,} \end{cases}$$

we have

$$C = \begin{cases} -2^{-1} \cdot (\log q)^{-1} \cdot \prod_{i=1}^{l-1} L(-i, \chi_E^i) \cdot \prod_{j=0}^{l-1} L(j+1, \chi_E^j)^{-1} & \text{in Case A,} \\ -2^{-2} \cdot (\log q)^{-1} \cdot \prod_{i=1}^{\frac{l}{2}-1} \zeta(-2i) \cdot \prod_{j=0}^{\frac{l}{2}-1} \zeta(2j+1)^{-1} & \text{in Case B,} \\ -(\log q)^{-1} \cdot \prod_{i=1}^{\frac{l}{2}-1} \zeta(-2i) \cdot \prod_{j=0}^{\frac{l}{2}-1} \zeta(2j+1)^{-1} & \text{in Case C',} \\ -(\log q)^{-1} \cdot \prod_{i=1}^{\frac{l-1}{2}} \zeta(-2i) \cdot \prod_{j=0}^{\frac{l-1}{2}} \zeta(2j+1)^{-1} & \text{in Case C'',} \\ -2^{-2} \cdot (\log q)^{-1} \cdot \prod_{i=1}^{\frac{l-1}{2}} \zeta(-2i) \cdot \prod_{j=0}^{\frac{l-1}{2}} \zeta(2j+1)^{-1} & \text{in Case D.} \end{cases}$$

This completes the proof of Proposition 19.1.

20. **Determination of the constant C — the equal rank case**

Recall that in §18, we have already proved Theorem 15.1(i), (ii) up to a scalar. In this section, we finish the proof of Theorem 15.1(i), (ii).

As an immediate consequence of Theorem 15.1(i), (ii), (18.1) and (18.2), we can determine the constant C in the local Siegel-Weil formula when $0 \leq l \leq 1$. More precisely, we have:

Theorem 20.1. *Let C be the constant given in Theorem 17.2.*

(i) *If $l = 0$, then we have*

$$C = \begin{cases} 1 & \text{in Case A,} \\ 2^{-1} & \text{in Case B,} \\ 2 & \text{in Case C'.} \end{cases}$$

(ii) *If $l = 1$, then we have*

$$C = -\chi_V(\det \beta_W) \cdot |\det \beta_W|_E^{1/2} \cdot \operatorname*{Res}_{s=0} \gamma(s, \mathbb{1}, \psi)^{-1}$$
$$\times \begin{cases} 2^{-1} \cdot \chi_V(\daleth)^n \cdot \epsilon(W)^{n+1} & \text{in Case A,} \\ \chi_V(-1)^{n/2} \cdot \epsilon(\frac{1}{2}, \chi_V, \psi) & \text{in Case C'',} \\ 2^{-2} & \text{in Case D.} \end{cases}$$

The rest of this section is devoted to the proof of Theorem 15.1(i), (ii).

20.1. **Cases B and C'.** Assume that $p \neq 2$. In Cases B and C' (where $l = 0$), we shall determine the constant C using [14]. By symmetry, it suffices to treat Case C'. In [14], the Haar measures dg and dh on $G = \mathrm{Mp}(W)$ and $H = \mathrm{O}(V)$ respectively, are normalized so that

$$\mathrm{vol}(\widetilde{I}_{\mathrm{Sp}(W)}, dg) = 1,$$
$$\mathrm{vol}(I_{\mathrm{SO}(V)}, dh) = \begin{cases} 1 & \text{if } \epsilon(V) = 1, \\ 2 \cdot \frac{q+1}{q-1} & \text{if } \epsilon(V) = -1. \end{cases}$$

Here, $I_{\mathrm{Sp}(W)}$ and $I_{\mathrm{SO}(V)}$ are Iwahori subgroups of $\mathrm{Sp}(W)$ and $\mathrm{SO}(V)$ respectively, and $\widetilde{I}_{\mathrm{Sp}(W)}$ is the preimage of $I_{\mathrm{Sp}(W)}$ in $\mathrm{Mp}(W)$. Let dg_ψ and dh_ψ be the Haar measures on G and H respectively, defined in §13.1.

Lemma 20.2. *We have*

$$\frac{\mathrm{vol}(\widetilde{I}_{\mathrm{Sp}(W)}, dg_\psi)}{\mathrm{vol}(\widetilde{I}_{\mathrm{Sp}(W)}, dg)} = \frac{\mathrm{vol}(I_{\mathrm{SO}(V)}, dh_\psi)}{\mathrm{vol}(I_{\mathrm{SO}(V)}, dh)}.$$

Proof. Let $I^+_{\mathrm{Sp}(W)}$ and $I^+_{\mathrm{SO}(V)}$ be the pro-unipotent radicals of $I_{\mathrm{Sp}(W)}$ and $I_{\mathrm{SO}(V)}$ respectively. Then by [20, (4.11)] and [21, §5], we have

$$\mathrm{vol}(I^+_{\mathrm{Sp}(W)}, dg_\psi) = \mathrm{vol}(I^+_{\mathrm{SO}(V)}, dh_\psi)$$

and hence

$$\frac{\mathrm{vol}(\widetilde{I}_{\mathrm{Sp}(W)}, dg_\psi)}{\mathrm{vol}(I_{\mathrm{SO}(V)}, dh_\psi)} = \frac{[I_{\mathrm{Sp}(W)} : I^+_{\mathrm{Sp}(W)}] \cdot \mathrm{vol}(I^+_{\mathrm{Sp}(W)}, dg_\psi)}{[I_{\mathrm{SO}(V)} : I^+_{\mathrm{SO}(V)}] \cdot \mathrm{vol}(I^+_{\mathrm{SO}(V)}, dh_\psi)} = \frac{[I_{\mathrm{Sp}(W)} : I^+_{\mathrm{Sp}(W)}]}{[I_{\mathrm{SO}(V)} : I^+_{\mathrm{SO}(V)}]}.$$

Since

$$I_{\mathrm{Sp}(W)}/I^+_{\mathrm{Sp}(W)} \cong (\mathbb{F}_q^\times)^r,$$
$$I_{\mathrm{SO}(V)}/I^+_{\mathrm{SO}(V)} \cong \begin{cases} (\mathbb{F}_q^\times)^r & \text{if } \epsilon(V) = 1, \\ (\mathbb{F}_q^\times)^{r-1} \times (\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times) \times \boldsymbol{\mu}_2 & \text{if } \epsilon(V) = -1, \end{cases}$$

where $r = \frac{n}{2} = \frac{m-1}{2}$, the assertion follows. ☐

By [14, Corollary 18, §16], there exist irreducible discrete series representations π and σ of G and H respectively, such that $\theta(\pi) = \sigma$ and

$$\frac{\mathrm{vol}(\widetilde{I}_{\mathrm{Sp}(W)}, dg_\psi)}{\mathrm{vol}(\widetilde{I}_{\mathrm{Sp}(W)}, dg)} \cdot \deg \pi = \frac{\mathrm{vol}(I_{\mathrm{SO}(V)}, dh_\psi)}{\mathrm{vol}(I_{\mathrm{SO}(V)}, dh)} \cdot \deg \sigma_0,$$

where $\sigma_0 = \sigma|_{H^0}$. Hence, by Lemmas 13.1 and 20.2, we obtain

$$\frac{\deg \pi}{\deg \sigma} = 2.$$

In view of (18.1), we conclude that

$$C = 2,$$

and this completes the proof of Theorem 15.1(i) in these cases.

For the rest of this section, we shall only consider Cases A, C" and D.

20.2. **Strategy.** We briefly outline the strategy of the proof of Theorem 15.1(i), (ii). Let π and σ be irreducible supercuspidal representations of G and H respectively, such that $\sigma = \theta(\pi)$. Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_k(E)$ and put $\tau_s = \tau |\det|_E^s$ for $s \in \mathbb{C}$. We consider the induced representations

$$I_P^{G'}(\tau_s \chi_V \otimes \pi) \quad \text{and} \quad I_Q^{H'}(\tau_s \chi_W \otimes \sigma)$$

of $G' = G(\mathbb{H}^k \oplus W)$ and $H' = H(\mathbb{H}^k \oplus V)$ respectively.

Let $s_0 > 0$. If $l \neq 0$, $k = 1$ and $\tau = \mathbb{1}_{E^\times}$, we assume that $s_0 \notin \{\pm\frac{l-1}{2}, \pm\frac{l+1}{2}\}$. Then by Corollary 12.2, $I_P^{G'}(\tau_{s_0}\chi_V \otimes \pi)$ is reducible if and only if $I_Q^{H'}(\tau_{s_0}\chi_W \otimes \sigma)$ is reducible. When these induced representations are reducible, they are of length two, and have unique square integrable constituents π' and σ' respectively. By [24], [60], one knows that

$$\sigma' = \theta(\pi').$$

For simplicity, we assume that $l = 1$. By (18.2), there exist constants $\mathcal{C}$ and $\mathcal{C}'$ which do not depend on the representations such that

$$\frac{\deg \pi}{\deg \sigma} = \mathcal{C} \cdot \omega_\sigma(-1) \cdot \gamma(0, \sigma, \chi_W^{-1}, \psi)$$

and

$$\frac{\deg \pi'}{\deg \sigma'} = \mathcal{C}' \cdot \omega_{\sigma'}(-1) \cdot \gamma(0, \sigma', \chi_W^{-1}, \psi).$$

On the other hand, a result of Heiermann [29] relates $\deg \pi'$ to $\deg \pi$ and $\deg \sigma'$ to $\deg \sigma$ respectively. Thus, one can relate $\mathcal{C}'$ to $\mathcal{C}$.

Using this, one can reduce the determination of $\mathcal{C}$ to that for certain low rank groups. For these low rank groups, the formal degree conjecture is largely known and the theta correspondence is completely understood, so that one can determine the constant $\mathcal{C}$.

20.3. **Setup.** We use the following notation.

- Let $W' = \mathbb{H}^k \oplus W$ and $V' = \mathbb{H}^k \oplus V$.
- Let $\mathbb{H}^k = X \oplus Y \subset W'$ and $\mathbb{H}^k = X' \oplus Y' \subset V'$ be complete polarizations.
- Let G' and H' be the isometry groups of W' and V' respectively.
- Let P and Q be the maximal parabolic subgroups of G' and H' stabilizing X and X' respectively.

- Let $M_P = \mathrm{GL}(X) \times G$ and $M_Q = \mathrm{GL}(X') \times H$ be Levi components of P and Q respectively. Fix isomorphisms $\mathrm{GL}(X) \cong \mathrm{GL}_k(E)$ and $\mathrm{GL}(X') \cong \mathrm{GL}_k(E)$.
- Let A_{M_P} and A_{M_Q} be the split components of the centers of M_P and M_Q respectively.
- Let U_P and U_Q be the unipotent radicals of P and Q respectively.
- Let $\bar{U}_P$ and $\bar{U}_Q$ be the unipotent radicals of the parabolic subgroups of G' and H' opposite to P and Q respectively.
- Let $K_{G'}$ and $K_{H'}$ be the maximal compact subgroups of G' and H' as in §D.1 respectively.
- Let $K^0_{G'} = G'^0 \cap K_{G'}$ and $K^0_{H'} = H'^0 \cap K_{H'}$.
- Let $K_{M_P} = M_P \cap K_{G'}$ and $K_{M_Q} = M_Q \cap K_{H'}$.
- Let $K^0_{M_P} = M^0_P \cap K_{M_P}$ and $K^0_{M_Q} = M^0_Q \cap K_{M_Q}$.
- Let $\gamma(G'^0/M^0_P)$ and $\gamma(H'^0/M^0_Q)$ be the constants defined in [76, §I.1].
- Let π and σ be irreducible supercuspidal representations of G and H respectively, such that $\sigma = \theta(\pi)$.
- Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_k(E)$ and put $\tau_s = \tau|\det|^s_E$ for $s \in \mathbb{C}$.
- Let $\mu(\tau_s\chi_V \otimes \pi)$ and $\mu(\tau_s\chi_W \otimes \sigma)$ be the Plancherel measures associated to the induced representations $I^{G'}_P(\tau_s\chi_V \otimes \pi)$ and $I^{H'}_Q(\tau_s\chi_W \otimes \sigma)$ respectively.
- Let $s_0 > 0$. If $l \neq 0$, $k = 1$ and $\tau = \mathbb{1}_{E^\times}$, we assume that $s_0 \notin \{\pm\frac{l-1}{2}, \pm\frac{l+1}{2}\}$.

Assume that $I^{H'}_Q(\tau_{s_0}\chi_W \otimes \sigma)$ is reducible. Then $I^{G'}_P(\tau_{s_0}\chi_V \otimes \pi)$ is also reducible by Corollary 12.2. Let π' and σ' be the unique irreducible discrete series subrepresentations of $I^{G'}_P(\tau_{s_0}\chi_V \otimes \pi)$ and $I^{H'}_Q(\tau_{s_0}\chi_W \otimes \sigma)$ respectively. By [24], [60], we have

$$\sigma' = \theta(\pi').$$

We remark that

$$\omega_{\sigma'} = \omega_\sigma \cdot \omega_\tau \cdot \chi_W^k \tag{20.1}$$

and

$$\gamma(s, \sigma', \chi_W^{-1}, \psi) = \gamma(s, \sigma, \chi_W^{-1}, \psi) \cdot \gamma(s + s_0, \tau, \psi_E) \cdot \gamma(s - s_0, (\tau^c)^\vee, \psi_E). \tag{20.2}$$

Also, in Case D, we have:

Lemma 20.3. *Let π_0 and π'_0 be irreducible constituents of $\pi|_{G^0}$ and $\pi'|_{G'^0}$ respectively. Then we have*

$$\frac{\deg \pi'_0}{\deg \pi'} = \frac{\deg \pi_0}{\deg \pi}.$$

Proof. By Lemma 13.1, it suffices to show that

$$\pi' \otimes \det \cong \pi' \iff \pi \otimes \det \cong \pi.$$

Let π'_P be the normalized Jacquet module of π' with respect to P. Then we have

$$\pi'_P = \tau_{s_0}\chi_V \otimes \pi,$$

so that the implication $\Rightarrow$ follows. Conversely, since

$$I^{G'}_P(\tau_s\chi_V \otimes \pi) \otimes \det \cong I^{G'}_P(\tau_s\chi_V \otimes (\pi \otimes \det)),$$

the implication $\Leftarrow$ is clear. □

20.4. **A result of Heiermann.** We now recall a result of Heiermann [29].

Proposition 20.4. *Using the notation of §20.3, we have:*

$$\begin{aligned}\deg \pi' &= dk \log q \cdot \deg \tau \cdot \deg \pi \cdot \operatorname*{Res}_{s=s_0} \mu(\tau_s \chi_V \otimes \pi) \\ &\quad \times \gamma(G'^0/M_P^0) \cdot \frac{\mathrm{vol}(K_{M_P}^0) \cdot \mathrm{vol}(U_P \cap K_{G'}) \cdot \mathrm{vol}(\bar{U}_P \cap K_{G'})}{\mathrm{vol}(K_{G'}^0) \cdot \mathrm{vol}(A_{M_P} \cap K_{M_P})}, \\ \deg \sigma' &= dk \log q \cdot \deg \tau \cdot \deg \sigma \cdot \operatorname*{Res}_{s=s_0} \mu(\tau_s \chi_W \otimes \sigma) \\ &\quad \times \gamma(H'^0/M_Q^0) \cdot \frac{\mathrm{vol}(K_{M_Q}^0) \cdot \mathrm{vol}(U_Q \cap K_{H'}) \cdot \mathrm{vol}(\bar{U}_Q \cap K_{H'})}{\mathrm{vol}(K_{H'}^0) \cdot \mathrm{vol}(A_{M_Q} \cap K_{M_Q})}.\end{aligned}$$

Here $d = [E : F]$.

Proof. We only compute $\deg \pi'$. Note that [29] only treats connected reductive linear algebraic groups. Thus, we will deduce the desired identity for orthogonal groups from that for special orthogonal groups.

Let π_0 be an irreducible constituent of $\pi|_{G^0}$. Here, $\pi_0 = \pi$ in Cases A and C". By Lemma B.1, we have

$$\mu(\tau_s \chi_V \otimes \pi_0) = \mu(\tau_s \chi_V \otimes \pi).$$

Since $s_0 > 0$ and $I_P^{G'}(\tau_{s_0} \chi_V \otimes \pi)$ is reducible, $I_{P^0}^{G'^0}(\tau_{s_0} \chi_V \otimes \pi_0)$ is also reducible by [6, Proposition 4.3]. (See also Proposition B.6.) Let π_0' be the unique irreducible discrete series subrepresentation of $I_{P^0}^{G'^0}(\tau_{s_0} \chi_V \otimes \pi_0)$. Here, $\pi_0' = \pi'$ in Cases A and C". Then π_0' is an irreducible constituent of $\pi'|_{G'^0}$. By Lemma 20.3, we have

$$\frac{\deg \pi_0'}{\deg \pi_0} = \frac{\deg \pi'}{\deg \pi}.$$

Hence it remains to compute $\deg \pi_0'$. To simplify notation, we shall suppress the superscript and subscript 0.

Let $\mathrm{Rat}(M_P)$ be the group of algebraic characters of M_P defined over F. Let $X(M_P)$ be the group of unramified characters of M_P and $\mathrm{Im}\, X(M_P)$ the subgroup of unitary unramified characters of M_P. Then the map $\mathrm{Rat}(M_P) \otimes_{\mathbb{Z}} \mathbb{C} \to X(M_P)$ given by $\chi \otimes s \mapsto (m \mapsto |\chi(m)|_F^s)$ is surjective. Similarly, we define $\mathrm{Rat}(A_{M_P})$, $X(A_{M_P})$ and $\mathrm{Im}\, X(A_{M_P})$.

Let α be a simple root of A_{M_P} in U_P. When $n \neq 0$, the image of $\mathrm{N}_{E/F} \circ \det$ under the restriction map $\mathrm{Rat}(M_P) \to \mathrm{Rat}(A_{M_P})$ is $dk\alpha$. Let $\tilde{\alpha}$ be as in [29, §3.2] and $\chi_{s\tilde{\alpha}}$ the image of $s\tilde{\alpha}$ in $X(M_P)$. Then we have $\tilde{\alpha} = \frac{dk}{ft} \cdot \alpha$ and

$$\chi_{s\tilde{\alpha}} = |\det|_E^{s/ft} \otimes \mathbf{1}_G,$$

where f is the residual degree of E/F and t is the torsion number of τ. Even when $n = 0$, the above formula for $\chi_{s\tilde{\alpha}}$ still holds.

We take the Haar measure on $\mathrm{Im}\, X(A_{M_P})$ such that $\mathrm{vol}(\mathrm{Im}\, X(A_{M_P})) = 1$. We take the Haar measure on $\mathrm{Im}\, X(M_P)$ such that the restriction map $\mathrm{res} : \mathrm{Im}\, X(M_P) \to \mathrm{Im}\, X(A_{M_P})$ locally preserves the measures, i.e., if $\mathcal{U}$ is an open subset of $\mathrm{Im}\, X(M_P)$ such that $\mathrm{res}|_{\mathcal{U}} : \mathcal{U} \to \mathrm{res}(\mathcal{U})$ is a homeomorphism, then we have

$$\mathrm{vol}(\mathrm{res}(\mathcal{U})) = \mathrm{vol}(\mathcal{U}).$$

Note that $\mathrm{vol}(\mathrm{Im}\, X(M_P)) = dk/f$. For convenience, we write $\rho = \tau \chi_V \otimes \pi$. Let $\mathcal{O}$ be the orbit of ρ under the action of $\mathrm{Im}\, X(M_P)$. We take the Haar measure on $\mathcal{O}$ such that the natural map

$\operatorname{Im} X(M_P) \to \mathcal{O}$ locally preserves the measures. Note that $\operatorname{vol}(\mathcal{O}) = dk/ft$. For $r \gg 0$, we have

$$\int_{\mathcal{O}} \mu(\rho' \otimes \chi_{r\tilde{\alpha}})\, d\rho' = \int_{\mathcal{O}} \mu(\rho')\, d\rho' + \operatorname{vol}(\mathcal{O}) \cdot \frac{\log q}{2\pi} \cdot 2\pi \operatorname*{Res}_{s=fts_0} \mu(\rho \otimes \chi_{s\tilde{\alpha}}).$$

Then a result of Heiermann [29, Remarque 8.6] says that

$$\begin{aligned}\operatorname{vol}(K_{G'}) \cdot \deg \pi' = \gamma(G'/M_P) \cdot \frac{\operatorname{vol}(K_{M_P})}{\operatorname{vol}(A_{M_P} \cap K_{M_P})} \cdot \deg \rho \\ \times \operatorname{vol}(U_P \cap K_{G'}) \cdot \operatorname{vol}(\bar{U}_P \cap K_{G'}) \cdot \operatorname{vol}(\mathcal{O}) \cdot \frac{\log q}{2\pi} \cdot 2\pi \operatorname*{Res}_{s=fts_0} \mu(\rho \otimes \chi_{s\tilde{\alpha}}).\end{aligned}$$

(See also [30].) We remark that in [29], the Haar measures are normalized so that

$$\operatorname{vol}(K_{G'}) = \operatorname{vol}(K_{M_P}) = \operatorname{vol}(A_{M_P} \cap K_{M_P}) = \operatorname{vol}(U_P \cap K_{G'}) = \operatorname{vol}(\bar{U}_P \cap K_{G'}) = 1.$$

This yields the proposition. □

Remark 20.5. We view $\mathrm{GL}_k(E)$ as an algebraic group over F, so that

$$\deg \tau = \frac{1}{2k} \cdot |\gamma(0, \chi_E, \psi) \cdot \gamma(0, \tau, \operatorname{Ad}, \psi_E)|$$

in Case A.

Corollary 20.6. *Using the notation of* §*20.3, we have:*

$$\frac{\deg \pi'}{\deg \sigma'} = \frac{\deg \pi}{\deg \sigma} \cdot \gamma(s_0 - \tfrac{l-1}{2}, \tau, \psi_E) \cdot \gamma(-s_0 - \tfrac{l-1}{2}, \tau^\vee, \bar{\psi}_E).$$

Proof. We may assume that ψ is of order zero. By Proposition 20.4, we have

$$\begin{aligned}\frac{\deg \pi'}{\deg \sigma'} &= \frac{\deg \pi}{\deg \sigma} \cdot \frac{\operatorname{Res}_{s=s_0} \mu(\tau_s \chi_V \otimes \pi)}{\operatorname{Res}_{s=s_0} \mu(\tau_s \chi_W \otimes \sigma)} \\ &\times \frac{\gamma(G'^0/M_P^0)}{\gamma(H'^0/M_Q^0)} \cdot \frac{\operatorname{vol}(K_{H'}^0)}{\operatorname{vol}(K_{G'}^0)} \cdot \frac{\operatorname{vol}(K_{M_P}^0)}{\operatorname{vol}(K_{M_Q}^0)} \cdot \frac{\operatorname{vol}(U_P \cap K_{G'}) \cdot \operatorname{vol}(\bar{U}_P \cap K_{G'})}{\operatorname{vol}(U_Q \cap K_{H'}) \cdot \operatorname{vol}(\bar{U}_Q \cap K_{H'})}.\end{aligned}$$

Hence, by Theorem 12.1 and Lemma D.2, we have

$$\begin{aligned}\frac{\deg \pi'}{\deg \sigma'} &= \frac{\deg \pi}{\deg \sigma} \cdot \gamma(s_0 - \tfrac{l-1}{2}, \tau, \psi_E) \cdot \gamma(-s_0 - \tfrac{l-1}{2}, \tau^\vee, \bar{\psi}_E) \\ &\times q^{-\Lambda} \cdot \frac{\operatorname{vol}(U_P \cap K_{G'}) \cdot \operatorname{vol}(\bar{U}_P \cap K_{G'})}{\operatorname{vol}(U_Q \cap K_{H'}) \cdot \operatorname{vol}(\bar{U}_Q \cap K_{H'})},\end{aligned}$$

where Λ is the integer given in Lemma D.2. Thus the corollary follows from Lemma D.3. □

We retain the notation of §20.3. By (18.1) and (18.2), there exist constants $\mathcal{C}$ and $\mathcal{C}'$ which do not depend on the representations such that

$$\frac{\deg \pi}{\deg \sigma} = \mathcal{C} \times \begin{cases} 1 & \text{if } l = 0, \\ \omega_\sigma(-1) \cdot \gamma(0, \sigma, \chi_W^{-1}, \psi) & \text{if } l = 1, \end{cases}$$

and

$$\frac{\deg \pi'}{\deg \sigma'} = \mathcal{C}' \times \begin{cases} 1 & \text{if } l = 0, \\ \omega_{\sigma'}(-1) \cdot \gamma(0, \sigma', \chi_W^{-1}, \psi) & \text{if } l = 1. \end{cases}$$

If $l = 1$, then by (20.1) and (20.2), we have

$$\begin{aligned}\frac{\omega_{\sigma'}(-1)}{\omega_\sigma(-1)} \cdot \frac{\gamma(0, \sigma', \chi_W^{-1}, \psi)}{\gamma(0, \sigma, \chi_W^{-1}, \psi)} &= \omega_\tau(-1) \cdot \chi_W(-1)^k \cdot \gamma(s_0, \tau, \psi_E) \cdot \gamma(-s_0, (\tau^c)^\vee, \psi_E) \\ &= \omega_\tau(-1) \cdot \chi_W(-1)^k \cdot \gamma(s_0, \tau, \psi_E) \cdot \gamma(-s_0, \tau^\vee, \psi_E^c) \\ &= \omega_\tau(-1) \cdot \chi_W(-1)^k \cdot \gamma(s_0, \tau, \psi_E) \cdot \gamma(-s_0, \tau^\vee, \psi_E) \\ &= \chi_W(-1)^k \cdot \gamma(s_0, \tau, \psi_E) \cdot \gamma(-s_0, \tau^\vee, \bar{\psi}_E).\end{aligned}$$

In view of Corollary 20.6, we conclude that

$$\frac{\mathcal{C}'}{\mathcal{C}} = \begin{cases} 1 & \text{if } l = 0, \\ \chi_W(-1)^k & \text{if } l = 1. \end{cases} \tag{20.3}$$

20.5. **Conjugate self-dual supercuspidal representations.** To exploit (20.3), we need to verify the assumption of §20.3 that $I_Q^{H'}(\tau_{s_0}\chi_W \otimes \sigma)$ is reducible. When $s_0 = \frac{1}{2}$, this will follow from the existence of an irreducible symplectic or conjugate self-dual supercuspidal representation of $\mathrm{GL}_k(E)$.

Lemma 20.7. *Assume that $p \neq 2$. Let $k \in \mathbb{N}$.*

(i) *When k is even, there exists an irreducible symplectic supercuspidal representation of* $\mathrm{GL}_k(F)$.

(ii) *When $[E : F] = 2$, there exists an irreducible conjugate self-dual supercuspidal representation of* $\mathrm{GL}_k(E)$.

Proof. (i) See [22, §14], [68, §2].

(ii) By [61, Lemma 2.2], it suffices to show that there exist a tamely ramified extension E' of E of degree k and an involution ς on E' such that $\varsigma|_E = c$.

We first assume that E/F is unramified. We write $k = 2^a b$ with $a \geq 0$ and b odd. Let F' be the unramified extension of F of degree $2b$. Note that $E \subset F'$. Let ς be the nontrivial element in $\mathrm{Gal}(F'/F)$ such that $\varsigma^2 = \mathrm{id}$. Since b is odd, we have $\varsigma|_E = c$. Let $E' = F'(\varpi_F^{1/2^a})$ with some 2^a-th root $\varpi_F^{1/2^a}$ of ϖ_F. We can extend ς to an automorphism ς of E' over F such that $\varsigma(\varpi_F^{1/2^a}) = \varpi_F^{1/2^a}$.

We next assume that E/F is ramified. Let F' be the unramified extension of F of degree k and put $E' = EF'$. Then E' is a Galois extension of F of degree $2k$ and

$$\mathrm{Gal}(E'/F) \cong \mathrm{Gal}(E/F) \times \mathrm{Gal}(F'/F).$$

We can take $\varsigma \in \mathrm{Gal}(E'/F)$ such that

$$\varsigma|_E = c, \quad \varsigma|_{F'} = \mathrm{id}.$$

□

20.6. **Reducibility.** To verify the assumption of §20.3 that $I_Q^{H'}(\tau_{s_0}\chi_W \otimes \sigma)$ is reducible, we need to show that $\mu(\tau_s\chi_W \otimes \sigma)$ has a pole at $s = s_0$.

Recall that $H = H(V)$ and $m = \dim V$. For irreducible admissible representations σ and τ of H and $\mathrm{GL}_k(E)$ respectively, let $\mu(\tau_s \otimes \sigma)$ be the Plancherel measure associated to the induced representation $I_Q^{H'}(\tau_s \otimes \sigma)$.

Lemma 20.8. *Assume that H is non-quasi-split and*

$$m = \begin{cases} 2 & \textit{in Case A}, \\ 4 \textit{ or } 6 & \textit{in Case C''}. \end{cases}$$

Let H^* be the quasi-split inner form of H. Let σ and τ be irreducible supercuspidal representations of H and $\mathrm{GL}_k(E)$ respectively. Let σ^* be an irreducible discrete series representation of H^* associated to σ by the local Jacquet-Langlands correspondence [8]. *Then we have*

$$\mu(\tau_s \otimes \sigma) = \mu(\tau_s \otimes \sigma^*).$$

Proof. As in §11.6, §12.5, we globalize the various objects to give a global-to-local argument. We can find:

- a totally imaginary number field $\mathbb{F}$ such that $\mathbb{F}_{v_1} = \mathbb{F}_{v_2} = F$ for some two places v_1 and v_2;
- either $\mathbb{E} = \mathbb{F}$ or a quadratic extension $\mathbb{E}$ of $\mathbb{F}$ such that $\mathbb{E}_{v_1} = \mathbb{E}_{v_2} = E$;
- a space $\mathbb{V}$ over $\mathbb{E}$ such that
  - $\mathbb{V}_{v_1} = \mathbb{V}_{v_2} = V$,
  - $H(\mathbb{V}_v)$ is quasi-split for all places $v \notin \{v_1, v_2\}$;
- an irreducible cuspidal automorphic representation Σ of $H(\mathbb{V})(\mathbb{A})$ such that
  - $\Sigma_{v_1} = \Sigma_{v_2} = \sigma$,
  - Σ_v has nonzero Iwahori-fixed vectors for all finite places $v \notin \{v_1, v_2\}$;
- an irreducible cuspidal automorphic representation $\mathfrak{T}$ of $\mathrm{GL}_k(\mathbb{A}_{\mathbb{E}})$ such that
  - $\mathfrak{T}_{v_1} = \mathfrak{T}_{v_2} = \tau$,
  - $\mathfrak{T}_v$ has nonzero Iwahori-fixed vectors for all finite places $v \notin \{v_1, v_2\}$.

Let $H(\mathbb{V})^*$ be the quasi-split inner form of $H(\mathbb{V})$. Let Σ^* be an irreducible cuspidal automorphic representation of $H(\mathbb{V})^*(\mathbb{A})$ associated to Σ by the global Jacquet-Langlands correspondence [5], [8] such that

- $\Sigma^*_{v_1} = \Sigma^*_{v_2} = \sigma^*$,
- $\Sigma^*_v = \Sigma_v$ for all places $v \notin \{v_1, v_2\}$.

Then we have the functional equations of global intertwining operators (Proposition B.7)

$$\left(\prod_{v \in S} \mu(\mathfrak{T}_{v,s} \otimes \Sigma_v)^{-1}\right) \cdot \frac{L^S(s, \mathfrak{T} \times \Sigma^\vee)}{L^S(1-s, \mathfrak{T}^\vee \times \Sigma)} \cdot \frac{L^S(-s, \mathfrak{T}^\vee \times \Sigma)}{L^S(1+s, \mathfrak{T} \times \Sigma^\vee)} \cdot \frac{L^S(2s, \mathfrak{T}, R)}{L^S(1-2s, \mathfrak{T}^\vee, R)} \cdot \frac{L^S(-2s, \mathfrak{T}^\vee, R)}{L^S(1+2s, \mathfrak{T}, R)} = 1$$

and

$$\left(\prod_{v \in S} \mu(\mathfrak{T}_{v,s} \otimes \Sigma^*_v)^{-1}\right) \cdot \frac{L^S(s, \mathfrak{T} \times \Sigma^\vee)}{L^S(1-s, \mathfrak{T}^\vee \times \Sigma)} \cdot \frac{L^S(-s, \mathfrak{T}^\vee \times \Sigma)}{L^S(1+s, \mathfrak{T} \times \Sigma^\vee)} \cdot \frac{L^S(2s, \mathfrak{T}, R)}{L^S(1-2s, \mathfrak{T}^\vee, R)} \cdot \frac{L^S(-2s, \mathfrak{T}^\vee, R)}{L^S(1+2s, \mathfrak{T}, R)} = 1$$

for a sufficiently large finite set S of places of $\mathbb{F}$. Here

$$R = \begin{cases} \text{Asai} & \text{in Case A,} \\ \bigwedge^2 & \text{in Cases C”D.} \end{cases}$$

Observing that $1 = 1$, we obtain

$$\mu(\tau_s \otimes \sigma)^2 = \mu(\tau_s \otimes \sigma^*)^2.$$

By the non-negativity of Plancherel measures on the imaginary axis, we deduce the desired identity. □

Lemma 20.9. *Assume that*

$$\begin{cases} 0 \le m \le 2 & \text{in Case A,} \\ 0 \le m \le 4, \text{ or } m = 6 \text{ and } \chi_V = \mathbb{1} & \text{in Case C”,} \\ 0 \le m \le 4 & \text{in Case D.} \end{cases}$$

Let σ and τ be irreducible supercuspidal representations of H and $\mathrm{GL}_k(E)$ respectively. Assume further that

$$\begin{cases} \tau \text{ is conjugate orthogonal} & \text{in Case A,} \\ k \text{ is even and } \tau \text{ is symplectic} & \text{in Case C”D.} \end{cases}$$

Then $I_Q^{H'}(\tau_{1/2} \otimes \sigma)$ is reducible.

Proof. By Proposition B.6, it suffices to show that $\mu(\tau_s \otimes \sigma)$ has a pole at $s = \frac{1}{2}$. We first assume that H is quasi-split and $H \neq \mathrm{Sp}_4$. Then σ and τ are generic with respect to some generic characters. Hence, by [69, Corollary 3.6], we have

$$\mu(\tau_s \otimes \sigma) = \gamma^{\mathrm{LS}}(s, \tau \times \sigma^\vee, \psi) \cdot \gamma^{\mathrm{LS}}(-s, \tau^\vee \times \sigma, \bar{\psi}) \cdot \gamma^{\mathrm{LS}}(2s, \tau, R, \psi) \cdot \gamma^{\mathrm{LS}}(-2s, \tau^\vee, R, \bar{\psi}),$$

where γ^{LS} refers to the γ-factors of Langlands-Shahidi [69] and

$$R = \begin{cases} \text{Asai} & \text{in Case A,} \\ \bigwedge^2 & \text{in Cases C”D.} \end{cases}$$

Moreover, it follows from a result of Henniart [33] that

$$\gamma^{\mathrm{LS}}(2s, \tau, R, \psi) = \alpha \cdot \gamma(2s, R \circ \phi_\tau, \psi)$$

with some root of unity α, where ϕ_τ is the L-parameter associated to τ by the local Langlands correspondence [28], [32]. By assumption, $\gamma(s, R \circ \phi_\tau, \psi)$ has a pole at $s = 1$. Thus, we see that $\mu(\tau_s \otimes \sigma)$ has a pole at $s = \frac{1}{2}$.

We next assume that H is non-quasi-split. Let H^* be the quasi-split inner form of H. Let σ^* be an irreducible discrete series representation of H^* associated to σ by the local Jacquet-Langlands correspondence [8]. Then σ^* and τ are generic with respect to some generic characters. Hence, by Lemma 20.8 and [69, Corollary 3.6], we have

$$\mu(\tau_s \otimes \sigma) = \gamma^{\mathrm{LS}}(s, \tau \times (\sigma^*)^\vee, \psi) \cdot \gamma^{\mathrm{LS}}(-s, \tau^\vee \times \sigma^*, \bar{\psi}) \cdot \gamma^{\mathrm{LS}}(2s, \tau, R, \psi) \cdot \gamma^{\mathrm{LS}}(-2s, \tau^\vee, R, \bar{\psi}).$$

Thus the assertion follows similarly.

Finally, if $H = \mathrm{Sp}_4$, then by [15], we have

$$\mu(\tau_s \otimes \sigma) = \gamma(s, \phi_\tau \otimes \phi_\sigma^\vee, \psi) \cdot \gamma(-s, \phi_\tau^\vee \otimes \phi_\sigma, \bar{\psi}) \cdot \gamma(2s, R \circ \phi_\tau, \psi) \cdot \gamma(-2s, R \circ \phi_\tau^\vee, \bar{\psi}),$$

where ϕ_σ and ϕ_τ are the L-parameters associated to σ and τ respectively, by the local Langlands correspondence [15], [28], [32]. Thus the assertion follows similarly. ☐

20.7. **Reduction to the minimal case.** Assume that $p \neq 2$ and $0 \le l \le 1$. Using (20.3), we will ultimately reduce the proof of Theorem 15.1(i), (ii) to the minimal case, i.e., the case when

$$\begin{cases} V \text{ is anisotropic} & \text{in Case A,} \\ V \text{ is anisotropic} & \text{in Case C”,} \\ W \text{ is anisotropic, or } W = \mathbb{H}^2 \text{ and } V = \mathbb{H} & \text{in Case D.} \end{cases}$$

Note that there is no discrete series representation of $\mathrm{O}_{1,1}$. The following table describes $(G, H) = (G(W), H(V))$ in the minimal case.

	(G, H)
Case A	$(\mathrm{U}_0, \mathrm{U}_0)$, $(\mathrm{U}_1, \mathrm{U}_1)$, $(\mathrm{U}_{1,1}, \mathrm{U}_{2,0})$, $(\mathrm{U}_{2,0}, \mathrm{U}_{2,0})$ if $l = 0$ $(\mathrm{U}_1, \mathrm{U}_0)$, $(\mathrm{U}_{1,1}, \mathrm{U}_1)$, $(\mathrm{U}_{2,0}, \mathrm{U}_1)$, $(\mathrm{U}_{2,1}, \mathrm{U}_{2,0})$ if $l = 1$
Case C"	$(\mathrm{Sp}_0, \mathrm{O}_0)$, $(\mathrm{Sp}_2, \mathrm{O}_{2,0})$, $(\mathrm{Sp}_4, \mathrm{O}_{4,0})$
Case D	$(\mathrm{O}_{2,0}, \mathrm{Sp}_0)$, $(\mathrm{O}_{2,2}, \mathrm{Sp}_2)$, $(\mathrm{O}_{4,0}, \mathrm{Sp}_2)$

However, in Cases C" and D, since k is only allowed to be even in Lemma 20.9, we need to consider the following extra case:

$$\begin{cases} n = 4 \text{ and } m = 4 & \text{if } \chi_V \neq \mathbb{1} \text{ in Case C"}, \\ n = 6 \text{ and } m = 6 & \text{if } \chi_V = \mathbb{1} \text{ in Case C"}, \\ n = 4 \text{ and } m = 2 & \text{if } \chi_W \neq \mathbb{1} \text{ in Case D}, \\ n = 6 \text{ and } m = 4 & \text{if } \chi_W = \mathbb{1} \text{ in Case D}. \end{cases}$$

The following table describes (G, H) in the extra case.

	(G, H)
Case C"	$(\mathrm{Sp}_4, \mathrm{O}_{3,1})$, $(\mathrm{Sp}_6, \mathrm{O}_{3,3})$, $(\mathrm{Sp}_6, \mathrm{O}_{5,1})$
Case D	$(\mathrm{O}_{3,1}, \mathrm{Sp}_2)$, $(\mathrm{O}_{3,3}, \mathrm{Sp}_4)$, $(\mathrm{O}_{5,1}, \mathrm{Sp}_4)$

In the minimal case and the extra case, it follows from case-by-case considerations that there exists an irreducible supercuspidal representation σ of H such that its theta lift $\pi := \theta(\sigma)$ to G is nonzero and supercuspidal. Let $k \in \mathbb{N}$. Assume that k is even in Cases C" and D. By Lemma 20.7, there exists an irreducible supercuspidal representation τ of $\mathrm{GL}_k(E)$ such that

$$\begin{cases} \tau\chi_W \text{ is conjugate orthogonal} & \text{in Case A}, \\ \tau\chi_W \text{ is symplectic} & \text{in Case C"D}. \end{cases}$$

Indeed, for an irreducible admissible representation τ of $\mathrm{GL}_k(E)$ and a character χ of $E^\times$ such that $\chi|_{F^\times} = \chi_E$, $\tau\chi$ is conjugate orthogonal if and only if τ is conjugate symplectic. Thus, in view of Lemma 20.9, we may use (20.3) to reduce the proof of Theorem 15.1(i), (ii) to the minimal case in Case A, and to the minimal case and the extra case in Cases C" and D. Observe that one cannot reduce the cases $(\mathrm{Sp}_6, \mathrm{O}_{3,3})$ and $(\mathrm{O}_{3,3}, \mathrm{Sp}_4)$ to the cases $(\mathrm{Sp}_2, \mathrm{O}_{1,1})$ and $(\mathrm{O}_{1,1}, \mathrm{Sp}_0)$ respectively, since there is no discrete series representation of $\mathrm{O}_{1,1}$.

In Cases C" and D, using the following lemma, we will further reduce the extra case to the minimal case.

Lemma 20.10. *Assume that*

$$\begin{cases} n = 2 \text{ and } m = 2 & \text{if } \chi_V \neq \mathbb{1} \text{ in Case C"}, \\ n = 4 \text{ and } m = 4 & \text{if } \chi_V = \mathbb{1} \text{ in Case C"}, \\ n = 2 \text{ and } m = 0 & \text{if } \chi_W \neq \mathbb{1} \text{ in Case D}, \\ n = 4 \text{ and } m = 2 & \text{if } \chi_W = \mathbb{1} \text{ in Case D}, \end{cases}$$

i.e., that

$$(G, H) = \begin{cases} (\mathrm{Sp}_2, \mathrm{O}_{2,0}),\ (\mathrm{Sp}_4, \mathrm{O}_{2,2}),\ (\mathrm{Sp}_4, \mathrm{O}_{4,0}) & \text{in Case C"}, \\ (\mathrm{O}_{2,0}, \mathrm{Sp}_0),\ (\mathrm{O}_{2,2}, \mathrm{Sp}_2),\ (\mathrm{O}_{4,0}, \mathrm{Sp}_2) & \text{in Case D}. \end{cases}$$

Then there exist an irreducible supercuspidal representation σ of H and a nontrivial quadratic character χ of $\mathrm{GL}_1(F)$ such that

- *its theta lift $\pi := \theta(\sigma)$ to G is nonzero and supercuspidal,*
- *$I_Q^{H'}(\chi\chi_W|\cdot|_F \otimes \sigma)$ is reducible.*

Proof. The lemma follows from case-by-case considerations. We only consider the case $(\mathrm{Sp}_4, \mathrm{O}_{2,2})$.

Recall that

$$\mathrm{GSO}_{2,2} \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/\Delta F^\times.$$

Let τ be an irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ with trivial central character and χ a nontrivial quadratic character of $F^\times$ such that $\tau\chi \ncong \tau$. Let σ be an irreducible constituent of

$$\mathrm{Ind}_{\mathrm{GSO}_{2,2}}^{\mathrm{GO}_{2,2}}(\tau \boxtimes \tau\chi)|_{\mathrm{O}_{2,2}}.$$

Then its theta lift $\theta(\sigma)$ to Sp_4 is nonzero and supercuspidal. Moreover, we have

$$\mu(\chi\chi_W|\cdot|_F^s \otimes \sigma) = \gamma(s, \tau \times \tau, \psi) \cdot \gamma(-s, \tau \times \tau, \bar{\psi}),$$

so that $\mu(\chi\chi_W|\cdot|_F^s \otimes \sigma)$ has a pole at $s = 1$. This completes the proof in this case. □

Thus, in view of Lemmas 20.9 and 20.10, we may use (20.3) to reduce the proof of Theorem 15.1(ii) to the minimal case in Cases C" and D. For example, we apply Lemma 20.10 to reduce the extra case $(\mathrm{Sp}_6, \mathrm{O}_{3,3})$ to the case $(\mathrm{Sp}_4, \mathrm{O}_{2,2})$, and then apply Lemma 20.9 with $k = 2$ to reduce it to the minimal case $(\mathrm{Sp}_0, \mathrm{O}_0)$.

20.8. **Proof for the minimal case.** Finally, to finish the proof of Theorem 15.1(i), (ii), it remains to treat the minimal case. In the minimal case, the formal degree conjecture is *largely* (but not completely) known and the theta correspondence is completely understood. Hence the desired identity in Theorem 15.1(i), (ii) can be proved directly. The only nontrivial cases are:

$$(\mathrm{U}_{2,1}, \mathrm{U}_{2,0}) \quad \text{and} \quad (\mathrm{Sp}_4, \mathrm{O}_{4,0}).$$

In fact, as a consequence of Theorem 15.1(i), (ii), we prove the formal degree conjecture for U_3 and Sp_4. Thus, we need to treat these remaining two cases in detail.

- $(\mathrm{U}_{2,1}, \mathrm{U}_{2,0})$. Recall that

  $$\mathrm{GU}_{2,0} \cong (D^\times \times E^\times)/\Delta F^\times,$$

  where D is the quaternion division algebra over F. Let τ be an irreducible admissible representation of $D^\times$ and χ a character of $E^\times$ such that $\omega_\tau \cdot \chi|_{F^\times} = \mathbb{1}$ and

  $$\sigma := (\tau \boxtimes \chi)|_{\mathrm{U}_{2,0}}$$

  is irreducible. Then σ is stable and its theta lift $\pi := \theta(\sigma)$ to $\mathrm{U}_{2,1}$ is nonzero and square integrable. By (18.2), there exists a constant $\mathcal{C}$ which does not depend on the representations such that

  $$\frac{\deg \pi}{\deg \sigma} = \mathcal{C} \cdot \omega_\sigma(-1) \cdot \gamma(0, \sigma, \chi_W^{-1}, \psi).$$

  Also, recall that

  $$\mathrm{GU}_{1,1} \cong (\mathrm{GL}_2 \times E^\times)/\Delta F^\times.$$

  Let τ^* be an irreducible discrete series representation of $\mathrm{GL}_2(F)$ associated to τ by the local Jacquet-Langlands correspondence and put

  $$\sigma^* := (\tau^* \boxtimes \chi)|_{\mathrm{U}_{1,1}}.$$

Then σ^* is irreducible and its theta lift $\pi^* := \theta(\sigma^*)$ to $\mathrm{U}_{2,1}$ is nonzero and square integrable. For the case $(\mathrm{U}_{2,1}, \mathrm{U}_{1,1})$, Theorem 15.1(ii) can be proved by reducing to the minimal case $(\mathrm{U}_1, \mathrm{U}_0)$. Hence we have

$$\frac{\deg \pi^*}{\deg \sigma^*} = 2^{-1} \cdot \chi_W(\daleth)^{-2} \cdot \epsilon(\tfrac{1}{2}, \chi_E, \psi)^{-1} \cdot \gamma(0, \chi_E, \psi) \cdot \omega_{\sigma^*}(-1) \cdot \gamma(0, \sigma^*, \chi_W^{-1}, \psi).$$

On the other hand, a result of Gelbart-Rogawski-Soudry [17] says that the representations

$$\pi \quad \text{and} \quad \pi^*$$

form an endoscopic L-packet of $\mathrm{U}_{2,1}$. Hence, it follows from a result of Rogawski [67] that

$$\deg \pi = \deg \pi^*.$$

Since $\deg \sigma = \deg \sigma^*$, $\omega_\sigma = \omega_{\sigma^*}$ and $\gamma(s, \sigma, \chi_W^{-1}, \psi) = \gamma(s, \sigma^*, \chi_W^{-1}, \psi)$, we obtain

$$\mathcal{C} = 2^{-1} \cdot \chi_W(\daleth)^{-2} \cdot \epsilon(\tfrac{1}{2}, \chi_E, \psi)^{-1} \cdot \gamma(0, \chi_E, \psi).$$

- $(\mathrm{Sp}_4, \mathrm{O}_{4,0})$. Recall that

$$\mathrm{GSO}_{4,0} \cong (D^\times \times D^\times)/\Delta F^\times,$$

where D is the quaternion division algebra over F. Fix a quadratic extension K of F. Let χ_K be the quadratic character of $F^\times$ associated to K/F by class field theory. Put

$$\tilde{\sigma} := \mathrm{Ind}_{\mathrm{GSO}_{4,0}}^{\mathrm{GO}_{4,0}}(\mathbf{1}_{D^\times} \boxtimes (\chi_K \circ \mathrm{N}_{D/F})),$$

where $\mathbf{1}_{D^\times}$ is the trivial representation of $D^\times$ and $\mathrm{N}_{D/F}$ is the reduced norm on D. Let σ be an irreducible constituent of $\tilde{\sigma}|_{\mathrm{O}_{4,0}}$. Then its theta lift $\pi := \theta(\sigma)$ to Sp_4 is nonzero and square integrable. By (18.2), there exists a constant $\mathcal{C}$ which does not depend on the representations such that

$$\frac{\deg \pi}{\deg \sigma} = \mathcal{C} \cdot \omega_\sigma(-1) \cdot \gamma(0, \sigma, \chi_W^{-1}, \psi).$$

Also, recall that

$$\mathrm{GSO}_{2,2} \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/\Delta F^\times.$$

Put

$$\tilde{\sigma}^* := \mathrm{Ind}_{\mathrm{GSO}_{2,2}}^{\mathrm{GO}_{2,2}}(\mathrm{St} \boxtimes (\mathrm{St} \otimes \chi_K)),$$

where St is the Steinberg representation of $\mathrm{GL}_2(F)$. Let σ^* be an irreducible constituent of $\tilde{\sigma}^*|_{\mathrm{O}_{2,2}}$. Then its theta lift $\pi^* := \theta(\sigma^*)$ to Sp_4 is nonzero and square integrable. For the case $(\mathrm{Sp}_4, \mathrm{O}_{2,2})$, Theorem 15.1(ii) can be proved by reducing to the minimal case $(\mathrm{Sp}_0, \mathrm{O}_0)$. Hence we have

$$\frac{\deg \pi^*}{\deg \sigma^*} = \omega_{\sigma^*}(-1) \cdot \gamma(0, \sigma^*, \chi_W^{-1}, \psi),$$

so that

$$\mathcal{C} = \frac{\deg \pi}{\deg \pi^*}.$$

On the other hand, let $\tilde{\pi}$ and $\tilde{\pi}^*$ be the theta lifts of $\tilde{\sigma}$ and $\tilde{\sigma}^*$ to GSp_4 respectively. Then π and π^* are irreducible constituents of $\tilde{\pi}|_{\mathrm{Sp}_4}$ and $\tilde{\pi}^*|_{\mathrm{Sp}_4}$ respectively. Since both $\tilde{\pi}|_{\mathrm{Sp}_4}$ and $\tilde{\pi}^*|_{\mathrm{Sp}_4}$ are multiplicity-free and of length two, we have

$$\frac{\deg \pi}{\deg \tilde{\pi}} = \frac{\deg \pi^*}{\deg \tilde{\pi}^*}$$

by Lemma 13.2, so that

$$\mathcal{C} = \frac{\deg \tilde{\pi}}{\deg \tilde{\pi}^*}.$$

A result of [16] says that the representations

$$\tilde{\pi} \quad \text{and} \quad \tilde{\pi}^*$$

form an endoscopic L-packet of GSp_4, but the equality

$$\deg \tilde{\pi} = \deg \tilde{\pi}^*$$

has not been proved. To prove this, we appeal to an alternative construction of the L-packet $\{\tilde{\pi}, \tilde{\pi}^*\}$.

Let $\mathcal{V}$ be a quadratic space over F of dimension four with $\chi_{\mathcal{V}} = \chi_K$. Recall that

$$\mathrm{GSO}(\mathcal{V}) \cong (\mathrm{GL}_2(K) \times F^\times)/\Delta K^\times.$$

Then the representation

$$\mathcal{S}t \boxtimes \mathbb{1}$$

of $\mathrm{GSO}(\mathcal{V})$ has two extensions $\tilde{\sigma}^+$ and $\tilde{\sigma}^-$ to $\mathrm{GO}(\mathcal{V})$, where $\mathcal{S}t$ is the Steinberg representation of $\mathrm{GL}_2(K)$. Let $\theta(\tilde{\sigma}^\pm)$ be the theta lift of $\tilde{\sigma}^\pm$ to GSp_4^+ and put

$$\tilde{\pi}^\pm := \mathrm{Ind}_{\mathrm{GSp}_4^+}^{\mathrm{GSp}_4}(\theta(\tilde{\sigma}^\pm)).$$

Here

$$\mathrm{GSp}_4^+ = \{g \in \mathrm{GSp}_4 \,|\, \nu(g) \in \mathrm{N}_{K/F}(K^\times)\},$$

where $\nu : \mathrm{GSp}_4 \to \mathrm{GL}_1$ is the similitude map. Then by [12, Theorem A.11(iii)], we have

$$\{\tilde{\pi}^+, \tilde{\pi}^-\} = \{\tilde{\pi}, \tilde{\pi}^*\}.$$

Moreover, by (18.2), we have

$$\frac{\deg \tilde{\pi}^+}{\deg \tilde{\sigma}^+} = \frac{\deg \tilde{\pi}^-}{\deg \tilde{\sigma}^-}.$$

Since $\deg \tilde{\sigma}^+ = \deg \tilde{\sigma}^-$, we have

$$\deg \tilde{\pi}^+ = \deg \tilde{\pi}^-.$$

Thus we obtain

$$\mathcal{C} = 1.$$

This completes the proof of Theorem 15.1(i), (ii).

20.9. **Formal degrees for U_3, Sp_4 and GSp_4.** As a consequence of Theorem 15.1(ii), we deduce:

Theorem 20.11. *The refined formal degree conjecture holds for the groups* U_3*,* Sp_4 *and* GSp_4*.*

Proof.

- U_3. The formal degree conjecture was proved in [34, Theorem 8.6'] for stable discrete series representations of U_3. On the other hand, by [17], endoscopic L-packets of U_3 can be obtained as theta lifts from U_2. Since the formal degree conjecture is known for U_2, that for endoscopic discrete series representations of U_3 follows from Theorem 15.1(ii).
- Sp_4. By [16], [15], L-packets of Sp_4 can be obtained as theta lifts from $\mathrm{O}_{2,2}$, $\mathrm{O}_{4,0}$ or $\mathrm{O}_{3,3}$. Note that
$$\mathrm{GSO}_{3,3} \cong (\mathrm{GL}_4 \times \mathrm{GL}_1)/\{(z \cdot \mathbf{1}_4, z^{-2}) \,|\, z \in F^\times\}.$$
Since the formal degree conjecture is known for $\mathrm{O}_{2,2}$, $\mathrm{O}_{4,0}$ and $\mathrm{O}_{3,3}$, that for Sp_4 follows from Theorem 15.1(ii).
- GSp_4. In view of Lemma 13.2, the formal degree conjecture for GSp_4 follows from that for Sp_4.

We remark that one can drop the assumption $p \neq 2$. Indeed, even when $p = 2$, the Howe duality conjecture is known for these low rank groups. This completes the proof. □

Appendix A. Analytic properties of normalized intertwining operators

The purpose of this appendix is to establish Proposition 8.1 which describes the analytic properties of the normalized intertwining operator $M^{\mathrm{LR}}_\psi(s, \chi)$.

A.1. **Normalization of intertwining operators.** In §8.2, we have recalled the normalization of the intertwining operator $M(s, \chi)$ due to Lapid-Rallis [49]. However, we have excluded the troublesome Case B. In this subsection, we briefly recall the normalization in Case B.

In Case B, choose an element

$$\beta \in \mathrm{Hom}(W^\nabla, W^\Delta)^{*=-} \quad \text{of rank } n-1$$

and put $\mathcal{K} = \ker\beta \subset W^\nabla$. Note that $\mathrm{im}\,\beta = \mathcal{K}^\perp \subset W^\Delta$. Choose an anisotropic line $\mathfrak{L}$ in $\mathbf{W}$ such that $\mathrm{pr}_{W^\nabla}(\mathfrak{L}) = \mathcal{K}$ and put $\mathcal{K}' = \mathrm{pr}_{W^\Delta}(\mathfrak{L})$. Let $\mathbf{s} \in \mathbf{G}$ be the orthogonal reflection with respect to $\mathfrak{L}$. Then we have $\mathbf{s}(\mathcal{K}) = \mathcal{K}'$.

Now choose an isomorphism $\tilde{\beta} : W^\nabla \to W^\Delta$ such that

- $\tilde{\beta}(\mathcal{K}) = \mathcal{K}'$,
- $\tilde{\beta}|_{\mathcal{K}} = \mathbf{s}|_{\mathcal{K}}$,
- $\tilde{\beta}$ induces the isomorphism $W^\nabla/\mathcal{K} \xrightarrow{\beta} \mathcal{K}^\perp \cong W^\Delta/\mathcal{K}'$.

By [49, Lemma 11], the map $\bar{u} \mapsto \mathrm{pr}_{W^\nabla} \circ \bar{u}|_{\mathcal{K}^\perp}$ induces an isomorphism

$$\bar{U}_{\mathbf{P}} \cap \mathbf{sPs} \xrightarrow{\cong} \mathrm{Hom}(\mathcal{K}^\perp, \mathcal{K}) \cong (W^\nabla/\mathcal{K}) \otimes \mathcal{K}.$$

Moreover, the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbf{W}$ induces pairings

$$(W^\nabla/\mathcal{K}) \times (W^\nabla/\mathcal{K}) \xrightarrow{\mathrm{id}\times\beta} (W^\nabla/\mathcal{K}) \times \mathcal{K}^\perp \longrightarrow F$$

and

$$\mathcal{K} \times \mathcal{K} \xrightarrow{\mathrm{id}\times\mathbf{s}} \mathcal{K} \times \mathcal{K}' \longrightarrow F.$$

We take the self-dual measure on $\bar{U}_{\mathbf{P}} \cap \mathbf{sPs} \cong (W^\nabla/\mathcal{K}) \otimes \mathcal{K}$ with respect to these pairings and ψ.

We define another Whittaker functional

$$\ell'_\beta : I^{\mathbf{G}}_{\mathbf{P}}(s, \chi) \longrightarrow \mathbb{C}$$

by

$$\ell'_\beta(\mathcal{F}) = \int_{(\bar{U}_{\mathbf{P}} \cap \mathbf{sPs}) \backslash \bar{U}_{\mathbf{P}}} \mathcal{F}(\mathbf{s}\bar{u}) \cdot \psi_\beta(\bar{u})\, d\bar{u}$$

for $\mathcal{F} \in I^{\mathbf{G}}_{\mathbf{P}}(s, \chi)$. Since the space

$$\{\ell \in \mathrm{Hom}_{\bar{U}_{\mathbf{P}}}(I^{\mathbf{G}}_{\mathbf{P}}(s, \chi), \psi_\beta^{-1}) \mid \ell(\mathcal{F}) = 0 \text{ if } \mathcal{F}|_{\mathbf{G}^0} = 0\}$$

is 1-dimensional for all s, we obtain:

$$\ell'_\beta \circ M(s, \chi) = c_{\beta,\psi}(s, \chi) \cdot \ell_\beta$$

for some rational function $c_{\beta,\psi}(s, \chi)$. Following Lapid-Rallis [49], we define a normalized intertwining operator $M^{\mathrm{LR}}_\psi(s, \chi)$ by

$$M^{\mathrm{LR}}_\psi(s, \chi) = c_{\beta,\psi}(s, \chi)^{-1} \cdot M(s, \chi).$$

A.2. **A different normalization.** The intertwining operator $M(s,\chi)$ has been studied in [48] (Case A), [78] (Cases B and D), [73] (Case C') and [46] (Case C"). However, the normalization of $M(s,\chi)$ in these references differs from that in [49]. For the rest of this appendix, we shall transport their results by comparing the two normalizations.

We first recall that the intertwining operator $M(s,\chi)$ depends on the choice of the Haar measure du. We take the self-dual measure du on $U_{\mathbf{P}}$ with respect to the pairing $(u,v) \mapsto \psi(\mathrm{tr}(\mathbf{b}(u)_W \circ \mathbf{b}(v)_W))$.

We now recall the setup of [46], [48], [73], [78]. Let $\{w_i \,|\, i=1,\dots,n\}$ be a basis of W over E and $\{w_i^* \,|\, i=1,\dots,n\}$ the dual basis of W over E such that $\langle w_i, w_j^* \rangle = \delta_{ij}$. Then we have

$$(w_1,\cdots,w_n) = (w_1^*,\cdots,w_n^*)\cdot \mathcal{Q}^c,$$

where $\mathcal{Q} = (\langle w_i, w_j\rangle) \in \mathrm{GL}_n(E)$. We define a basis $\{\mathbf{w}_i, \mathbf{w}_i^* \,|\, i=1,\dots,n\}$ of $\mathbf{W} = W \oplus W_-$ over E by

$$\mathbf{w}_i = (w_i, w_i), \quad \mathbf{w}_i^* = \frac{1}{2}(w_i^*, -w_i^*).$$

Then we have $\langle \mathbf{w}_i, \mathbf{w}_j\rangle = \langle \mathbf{w}_i^*, \mathbf{w}_j^*\rangle = 0$ and $\langle \mathbf{w}_i, \mathbf{w}_j^*\rangle = \delta_{ij}$. Using this basis, we identify $\mathbf{W}$ with the space E^{2n} of column vectors, which induces isomorphisms

$$\mathbf{G} \cong \left\{ \mathbf{g} \in \mathrm{GL}_{2n}(E) \;\middle|\; {}^t\mathbf{g} \begin{pmatrix} 0 & \mathbf{1}_n \\ \epsilon\cdot\mathbf{1}_n & 0 \end{pmatrix} \mathbf{g}^c = \begin{pmatrix} 0 & \mathbf{1}_n \\ \epsilon\cdot\mathbf{1}_n & 0 \end{pmatrix} \right\}$$

and

$$U_{\mathbf{P}} \cong \left\{ \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} \;\middle|\; x \in \mathbf{X} \right\}.$$

Here

$$\epsilon = \begin{cases} -1 & \text{in Cases AC,} \\ 1 & \text{in Cases BD,} \end{cases}$$

and $\mathbf{X} = \{x \in \mathrm{M}_n(E) \,|\, {}^t x^c = -\epsilon x\}$. Let dx be the self-dual measure on $\mathbf{X}$ with respect to the pairing $(x,y) \mapsto \psi(\mathrm{tr}(xy))$. We define an intertwining operator

$$M^\natural(s,\chi) : I_{\mathbf{P}}^{\mathbf{G}}(s,\chi) \longrightarrow I_{\mathbf{P}}^{\mathbf{G}}(-s,(\chi^c)^{-1})$$

by

$$M^\natural(s,\chi)\mathcal{F}(\mathbf{g}) = \int_{\mathbf{X}} \mathcal{F}\left(\begin{pmatrix} 0 & \mathbf{1}_n \\ \epsilon\cdot\mathbf{1}_n & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} \mathbf{g} \right) dx$$

for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(s,\chi)$ and $\mathbf{g} \in \mathbf{G}$.

Lemma A.1. *We have*

$$M(s,\chi) = \chi(2^n \det \mathcal{Q}^c)^{-1} \cdot |2|_E^{-n(s+\rho)} \cdot |\det \mathcal{Q}|_E^{-s} \cdot M^\natural(s,\chi).$$

Proof. The Haar measure $|\det \mathcal{Q}|_E^{\rho} \cdot dx$ on $\mathbf{X}$ induces the Haar measure du on $U_{\mathbf{P}}$. Since

$$w \cdot (\mathbf{w}_1,\cdots,\mathbf{w}_n) = (\mathbf{w}_1^*,\cdots,\mathbf{w}_n^*)\cdot 2\mathcal{Q}^c,$$

we have

$$w = \begin{pmatrix} (2\mathcal{Q}^c)^{-1} & 0 \\ 0 & 2\,{}^t\mathcal{Q} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1}_n \\ \epsilon\cdot\mathbf{1}_n & 0 \end{pmatrix},$$

so that

$$M(s,\chi) = \chi(2^n \det \mathcal{Q}^c)^{-1} |2^n \det \mathcal{Q}^c|_E^{-s-\rho} \cdot |\det \mathcal{Q}|_E^{\rho} \cdot M^\natural(s,\chi).$$

□

Let $\beta \in \mathrm{Hom}(W^{\nabla}, W^{\Delta})^{*=-}$ be of rank

$$\begin{cases} n & \text{in Cases ACD,} \\ n-1 & \text{in Case B.} \end{cases}$$

We define $\boldsymbol{\beta} \in \mathbf{X}$ by

$$\beta \cdot (\mathbf{w}_1^*, \cdots, \mathbf{w}_n^*) = (\mathbf{w}_1, \cdots, \mathbf{w}_n) \cdot \boldsymbol{\beta}.$$

Note that $\det \beta_W = \det \boldsymbol{\beta} \mathcal{Q}^c$. We consider a Whittaker functional

$$\ell^{\natural}_{\boldsymbol{\beta}} : I^{\mathbf{G}}_{\mathbf{P}}(s, \chi) \longrightarrow \mathbb{C}$$

given by

$$\ell^{\natural}_{\boldsymbol{\beta}}(\mathcal{F}) = \int_{\mathbf{X}} \mathcal{F}\left(\begin{pmatrix} 0 & \mathbf{1}_n \\ \epsilon \cdot \mathbf{1}_n & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & x \\ 0 & \mathbf{1}_n \end{pmatrix} \right) \psi(\mathrm{tr}(x\boldsymbol{\beta}))\, dx.$$

In Case B, choose an anisotropic line $\mathfrak{L}$ in $\mathbf{W}$ such that $\mathrm{pr}_{W^{\nabla}}(\mathfrak{L}) = \mathcal{K}$. We may assume that $\mathcal{K} = F\mathbf{w}_n^*$ and $\mathcal{K}^{\perp} = F\mathbf{w}_1 + \cdots + F\mathbf{w}_{n-1}$, so that

$$\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}' & 0 \\ 0 & 0 \end{pmatrix}$$

with some $\boldsymbol{\beta}' \in \mathrm{M}_{n-1}(F)$. For simplicity, we assume that $\mathfrak{L} = F(\mathbf{w}_n - \mathbf{w}_n^*)$. Then we have $\mathcal{K}' = F\mathbf{w}_n$ and $\mathbf{s}(\mathbf{w}_n^*) = \mathbf{w}_n$. Choose an isomorphism $\tilde{\beta} : W^{\nabla} \to W^{\Delta}$ such that

$$\tilde{\boldsymbol{\beta}} = \begin{pmatrix} \boldsymbol{\beta}' & 0 \\ 0 & 1 \end{pmatrix},$$

where $\tilde{\boldsymbol{\beta}} \in \mathrm{M}_n(F)$ is defined by

$$\tilde{\beta} \cdot (\mathbf{w}_1^*, \cdots, \mathbf{w}_n^*) = (\mathbf{w}_1, \cdots, \mathbf{w}_n) \cdot \tilde{\boldsymbol{\beta}}.$$

We consider another Whittaker functional

$$\ell^{\sharp}_{\boldsymbol{\beta}} : I^{\mathbf{G}}_{\mathbf{P}}(s, \chi) \longrightarrow \mathbb{C}$$

given by

$$\ell^{\sharp}_{\boldsymbol{\beta}}(\mathcal{F}) = \int_{\mathbf{X}'} \mathcal{F}\left(\begin{pmatrix} 0 & 0 & \mathbf{1}_{n-1} & 0 \\ 0 & 1 & 0 & 0 \\ \mathbf{1}_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n-1} & 0 & x' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \psi(\mathrm{tr}(x'\boldsymbol{\beta}'))\, dx'.$$

Here $\mathbf{X}' = \{x' \in \mathrm{M}_{n-1}(F) \mid {}^t x' = -x'\}$ and dx' is the self-dual measure on $\mathbf{X}'$ with respect to the pairing $(x', y') \mapsto \psi(\mathrm{tr}(x'y'))$.

A.3. **Computation of $c_{\beta,\psi}(s, \chi)$.** The following proposition gives an explicit formula for the normalizing factor $c_{\beta,\psi}(s, \chi)$ of Lapid-Rallis [49].

Proposition A.2. *We have*

$$\begin{aligned} c_{\beta,\psi}(s, \chi) = {} & c_{\psi}(s, \chi)^{-1} \cdot \chi(2\epsilon)^{-n} \cdot |2|_E^{-n(s+\rho)} \cdot \chi(\det \beta_W)^{-1} \cdot |\det \beta_W|_E^{-s} \\ & \times \begin{cases} \gamma_F(\daleth^2, \psi)^{(n-1)n/2} \cdot \chi_E(\det \boldsymbol{\beta})^{n-1} & \textit{in Case A,} \\ (-1,-1)_F^{n/2} \cdot \gamma_F(\bar{\psi})^{n^2/2} \cdot \gamma_F(\det \boldsymbol{\beta}, \bar{\psi})^{n+1} \cdot \epsilon(-\boldsymbol{\beta}) & \textit{in Case C',} \\ \gamma_F(\bar{\psi})^{n^2/2} \cdot \gamma_F(\det \boldsymbol{\beta}, \bar{\psi})^{n-1} \cdot \gamma(s + \frac{1}{2}, \chi\chi_{\boldsymbol{\beta}}, \psi) & \textit{in Case C'',} \\ 1 & \textit{in Case D,} \end{cases} \end{aligned}$$

and

$$c_{\beta,\psi}(s, \chi) = c_{\psi}(s, \chi)^{-1} \cdot \chi(2)^{-n} \cdot |2|_F^{-n(s+\rho)+\rho} \cdot \chi(\det \tilde{\beta}_W)^{-1} \cdot |\det \tilde{\beta}_W|_F^{-s}$$

in Case B. Here

$$c_\psi(s,\chi) = \begin{cases} \prod_{i=1}^{n} \gamma(2s-i+1, \chi|_{F^\times} \cdot \chi_E^{n-i}, \psi) & \textit{in Case A,} \\ \prod_{i=1}^{\frac{n-1}{2}} \gamma(2s-2i+1, \chi^2, \psi_2) & \textit{in Case B,} \\ \prod_{i=1}^{\frac{n}{2}} \gamma(2s-2i+1, \chi^2, \psi_2) & \textit{in Case C',} \\ \gamma(s-\frac{n-1}{2}, \chi, \psi) \cdot \prod_{i=1}^{\frac{n}{2}} \gamma(2s-2i+2, \chi^2, \psi_2) & \textit{in Case C'',} \\ \prod_{i=1}^{\frac{n}{2}} \gamma(2s-2i+2, \chi^2, \psi_2) & \textit{in Case D,} \end{cases}$$

where $\psi_2(x) = \psi(2x)$ for $x \in F$, and in Case C, $\chi_{\boldsymbol{\beta}}$ is the quadratic character of $F^\times$ associated to $F(\sqrt{\operatorname{disc} \boldsymbol{\beta}})/F$ and $\epsilon(\boldsymbol{\beta})$ is the Hasse invariant of $\boldsymbol{\beta}$.

Proof. We first exclude Case B. By [48, Propositions 3.1, 3A.6], [72, Proposition 4.8], [73, Proposition 4.1, Theorem 5.1], [78, Remark 3.3], we have a functional equation

$$\ell^\natural_{\boldsymbol{\beta}} \circ M^\natural(s,\chi) = c^\natural_{\boldsymbol{\beta},\psi}(s,\chi) \cdot \ell^\natural_{\boldsymbol{\beta}},$$

where

$$\begin{aligned} c^\natural_{\boldsymbol{\beta},\psi}(s,\chi) &= c_\psi(s,\chi)^{-1} \cdot \chi(\det \boldsymbol{\beta})^{-1} \cdot |\det \boldsymbol{\beta}|_E^{-s} \\ &\quad \times \begin{cases} \gamma_F(\daleth^2, \psi)^{(n-1)n/2} \cdot \chi_E(\det \boldsymbol{\beta})^{n-1} & \text{in Case A,} \\ (-1,-1)_F^{n/2} \cdot \gamma_F(\bar{\psi})^{n^2/2} \cdot \gamma_F(\det \boldsymbol{\beta}, \bar{\psi})^{n+1} \cdot \epsilon(\boldsymbol{\beta}) & \text{in Case C',} \\ \gamma_F(\bar{\psi})^{n^2/2} \cdot \gamma_F(\det \boldsymbol{\beta}, \bar{\psi})^{n-1} \cdot \gamma(s+\frac{1}{2}, \chi\chi_{\boldsymbol{\beta}}, \psi) & \text{in Case C'',} \\ 1 & \text{in Case D.} \end{cases} \end{aligned}$$

We remark that one can drop the assumption $\chi^2 = 1$ of [72] using a result of Sweet [73, Theorem 2.1]:

$$\int_{\mathfrak{p}_F^{-d}} \chi(x)|x|_F^{s-1} \gamma_F(\psi_x) \bar{\psi}(x) \, dx = \frac{\gamma(s+\frac{1}{2}, \chi, \psi)}{\gamma(2s, \chi^2, \psi_2)},$$

where $d \gg 0$, $\mathrm{Re}(s) \gg 0$ and dx is the self-dual measure on F with respect to ψ. Since

$$c_{\beta,\psi}(s,\chi) = \chi(2^n \det \mathcal{Q}^c)^{-1} \cdot |2|_E^{-n(s+\rho)} \cdot |\det \mathcal{Q}|_E^{-s} \cdot c^\natural_{\epsilon\boldsymbol{\beta},\psi}(s,\chi),$$

the assertion follows.

Now consider Case B. The functional equation for Case D implies that

$$\ell^\sharp_{\boldsymbol{\beta}} \circ M^\natural(s,\chi) = c^\natural_{\boldsymbol{\beta},\psi}(s,\chi) \cdot \ell^\natural_{\boldsymbol{\beta}},$$

where

$$c^\natural_{\boldsymbol{\beta},\psi}(s,\chi) = c_\psi(s,\psi)^{-1} \cdot \chi(\det \boldsymbol{\beta}')^{-1} \cdot |\det \boldsymbol{\beta}'|_F^{-s+1/2}.$$

Note that

$$\begin{aligned} \ell^\natural_{\boldsymbol{\beta}}(\mathcal{F}) = \int_{F^{n-1}} \int_{\mathbf{X}'} \mathcal{F} &\left(\begin{pmatrix} \mathbf{1}_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbf{1}_{n-1} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n-1} & 0 & 0 & 0 \\ -{}^t x'' & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n-1} & x'' \\ 0 & 0 & 0 & 1 \end{pmatrix} \right. \\ &\quad \times \left. \begin{pmatrix} 0 & 0 & \mathbf{1}_{n-1} & 0 \\ 0 & 1 & 0 & 0 \\ \mathbf{1}_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n-1} & 0 & x' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \psi(\mathrm{tr}(x' \boldsymbol{\beta}')) \, dx' \, dx'', \end{aligned}$$

where dx'' is the self-dual measure on F^{n-1} with respect to the pairing $(x'', y'') \mapsto \psi(2^t x'' y'')$. Since the Haar measure $|\det \boldsymbol{\beta}'|_F^{1/2} \cdot |2|_F^{-(n-1)/2} \cdot dx''$ induces the Haar measure on $\bar{U}_{\mathbf{P}} \cap \mathbf{sPs}$ defined in §A.1, we have

$$c_{\beta,\psi}(s,\chi) = \chi(2^n \det \mathcal{Q}^c)^{-1} \cdot |2|_F^{-n(s+\rho)} \cdot |\det \mathcal{Q}|_F^{-s} \cdot |\det \boldsymbol{\beta}'|_F^{-1/2} \cdot |2|_F^{(n-1)/2} \cdot c^{\natural}_{\boldsymbol{\beta},\psi}(s,\chi).$$

This completes the proof. □

We now prove Proposition 8.1. Put

$$a(s,\chi) = \begin{cases} \prod_{i=1}^{n} L(2s-i+1, \chi|_{F^\times} \cdot \chi_E^{n-i}) & \text{in Case A,} \\ \prod_{i=1}^{\frac{n-1}{2}} L(2s-2i+1, \chi^2) & \text{in Case B,} \\ \prod_{i=1}^{\frac{n}{2}} L(2s-2i+1, \chi^2) & \text{in Case C',} \\ L(s-\frac{n-1}{2}, \chi) \cdot \prod_{i=1}^{\frac{n}{2}} L(2s-2i+2, \chi^2) & \text{in Case C'',} \\ \prod_{i=1}^{\frac{n}{2}} L(2s-2i+2, \chi^2) & \text{in Case D,} \end{cases}$$

$$b(s,\chi) = \begin{cases} \prod_{i=1}^{n} L(2s+i, \chi|_{F^\times} \cdot \chi_E^{n-i}) & \text{in Case A,} \\ \prod_{i=1}^{\frac{n-1}{2}} L(2s+2i, \chi^2) & \text{in Case B} \\ \prod_{i=1}^{\frac{n}{2}} L(2s+2i, \chi^2) & \text{in Case C'} \\ L(s+\frac{n+1}{2}, \chi) \cdot \prod_{i=1}^{\frac{n}{2}} L(2s+2i-1, \chi^2) & \text{in Case C'',} \\ \prod_{i=1}^{\frac{n}{2}} L(2s+2i-1, \chi^2) & \text{in Case D.} \end{cases}$$

By [46], [48, Proposition 3.2], [73, Theorem 5.2], [78, Proposition 8.10], $a(s,\chi)^{-1} \cdot M^{\natural}(s,\chi)$ is holomorphic and nonzero at any s. Since

$$a(s,\chi)^{-1} \cdot b(-s, (\chi^c)^{-1}) \cdot c_\psi(s,\chi)^{-1}$$

is a nonzero exponential function of s, the set of poles of $M^{\mathrm{LR}}(s,\chi)$ is equal to that of $b(-s, (\chi^c)^{-1})$. This completes the proof of Proposition 8.1.

A.4. **A result of Harris-Kudla-Sweet.** In this subsection, we consider Case A. As a byproduct of the above computation, we can transport a result of Harris-Kudla-Sweet [27]. In [27], they defined a normalized intertwining operator $M_\psi^{\mathrm{HKS}}(s,\chi)$ by

$$M_\psi^{\mathrm{HKS}}(s,\chi) = \gamma_F(\daleth^2, \psi)^{-(n-1)n/2} \cdot c_\psi(s,\chi) \cdot M^{\natural}(s,\chi).$$

By Lemma A.1 and Proposition A.2, we have

$$\begin{aligned} M_\psi^{\mathrm{LR}}(s,\chi) &= c_\psi(s,\chi) \cdot \chi(-2)^n \cdot |2|_E^{n(s+\rho)} \cdot \chi(\det \beta_W) \cdot |\det \beta_W|_E^s \\ &\quad \times \gamma_F(\daleth^2, \psi)^{-(n-1)n/2} \cdot \chi_E(\det \boldsymbol{\beta})^{n-1} \\ &\quad \times \chi(2^n \det \mathcal{Q}^c)^{-1} \cdot |2|_E^{-n(s+\rho)} \cdot |\det \mathcal{Q}|_E^{-s} \cdot M_\psi^{\natural}(s,\chi) \\ &= \chi(\det \beta_W) \cdot |\det \beta_W|_E^s \cdot \chi_E(\det \boldsymbol{\beta})^{n-1} \cdot \chi(\det \mathcal{Q})^{-1} \cdot |\det \mathcal{Q}|_E^{-s} \cdot M_\psi^{\mathrm{HKS}}(s,\chi). \end{aligned}$$

Assume that $l = 0$, so that V is a hermitian space over E of dimension n.

Lemma A.3. *We have*

$$M_\psi^{\mathrm{LR}}(0, \chi_V)|_{R_{\mathbf{W},\boldsymbol{\chi},\psi}(V,\chi_W)} = \chi_V(\det \beta_W) \cdot \chi_V(\daleth)^n \cdot \epsilon(V) \cdot \epsilon(W)^n.$$

Proof. It follows from [27, Proposition 6.8] that

$$M_\psi^{\mathrm{HKS}}(0, \chi_V)|_{R_{\mathbf{W},\boldsymbol{\chi},\psi}(V,\chi_W)} = \chi_E(-1)^n \cdot \chi_E(\det V).$$

We remark that the convention in [27] differs from ours:

- their hermitian form $(\cdot,\cdot)$ on V satisfies
$$(av, bw) = a^c (v, w) b$$
for $a, b \in E$ and $v, w \in V$;
- their symplectic form on $\mathbb{W} = V \otimes_E W$ is
$$\frac{1}{2} \cdot \mathrm{tr}_{E/F}((\cdot,\cdot)^c \otimes \langle \cdot,\cdot \rangle).$$

Hence we have
$$M_\psi^{\mathrm{LR}}(0, \chi_V)|_{R_{\mathbf{W},\boldsymbol{\chi},\psi}(V,\chi_W)} = \chi_V(\det \beta_W) \cdot \chi_E(\det \boldsymbol{\beta})^{n-1} \cdot \chi_V(\det \mathcal{Q})^{-1} \cdot \chi_E(-1)^n \cdot \chi_E(\det V).$$

On the other hand, since we have chosen β so that
$$\det \boldsymbol{\beta} \in (-1)^{n/2} \cdot \mathrm{N}_{E/F}(E^\times)$$
when n is even, we have
$$\chi_E(\det \boldsymbol{\beta})^{n-1} = \begin{cases} \chi_E(-1)^{n/2} & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$
Also, we have
$$\begin{aligned} \chi_V(\det \mathcal{Q}) &= \chi_V((-1)^{(n-1)n/2} \cdot \daleth^n) \cdot \chi_V(\daleth^{-n} \cdot \operatorname{disc} W) \\ &= \begin{cases} \chi_V(\daleth)^n & \text{if } n \text{ is even,} \\ \chi_E(-1)^{(n-1)/2} \cdot \chi_V(\daleth)^n \cdot \epsilon(W) & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$
and
$$\chi_E(\det V) = \begin{cases} \chi_E(-1)^{n/2} \cdot \epsilon(V) & \text{if } n \text{ is even,} \\ \chi_E(-1)^{(n-1)/2} \cdot \epsilon(V) & \text{if } n \text{ is odd.} \end{cases}$$
This yields the lemma. □

Let π be an irreducible tempered representation of $G(W)$. Then we have the local functional equation
$$Z(0, M_\psi^{\mathrm{LR}}(0, \chi_V)\mathcal{F}, f) = \epsilon(W)^{n+1} \cdot \omega_\pi(-1) \cdot \chi_V(\det \beta_W) \cdot \gamma(\tfrac{1}{2}, \pi, \chi_V^c, \psi) \cdot Z(0, \mathcal{F}, f)$$
for $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(0, \chi_V)$ and a matrix coefficient f of $\pi^\vee$. Note that $Z(0, \mathcal{F}, f)$ is absolutely convergent by Lemma 9.4(ii). From this, one can deduce Theorem 11.1.

Appendix B. Properties of Plancherel measures

The purpose of this appendix is to recall some basic properties of Plancherel measures which are used in the body of the paper.

B.1. Plancherel measures.

Suppose that
$$W' = \mathbb{H}^k \oplus W,$$
and let $G' = G(W')$ be the isometry groups of W'. We consider a parabolic subgroup $P = MU$ of G' with Levi component
$$M = \mathrm{GL}_{k_1}(E) \times \cdots \times \mathrm{GL}_{k_r}(E) \times G(W),$$
where $k_1 + \cdots + k_r = k$. Let τ_i and π be irreducible admissible representations of $\mathrm{GL}_{k_i}(E)$ and $G(W)$ respectively. Then one can form induced representations
$$I_P^{G'}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) \quad \text{and} \quad I_{\bar{P}}^{G'}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$$

of G', where $\bar{P} = M\bar{U}$ is the parabolic subgroup of G' opposite to P. Now consider the intertwining operator

$$J_{\bar{P}|P}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) : I_P^{G'}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) \longrightarrow I_{\bar{P}}^{G'}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi),$$

which is rational (see [76, Théorème IV.1.1]). Then the Plancherel measure associated to $I_P^{G'}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ is a rational function $\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ of $(\tau_1, \ldots, \tau_r)$ such that

$$J_{P|\bar{P}}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) \circ J_{\bar{P}|P}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) = \mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)^{-1}.$$

Note that $\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ is independent of the choice of the parabolic subgroup P which has Levi component M (see [76, §IV.3]).

B.2. **Haar measures.** At this point, the Plancherel measure $\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ is only well-defined up to a scalar since it depends on the choice of Haar measures on U and $\bar{U}$. We shall take a Haar measure on $U \times \bar{U}$ defined as follows. By induction in stages, it suffices to treat maximal parabolic subgroups of either $G(W)$ or $\mathrm{GL}_k(E)$.

Suppose that $\mathbb{H}^k = X \oplus Y \subset W'$ is a complete polarization and P is the maximal parabolic subgroup of $G' = G(W')$ stabilizing X. Then the unipotent radical U of P is given by:

$$U = \left\{ u \in \mathrm{GL}(W') \;\middle|\; \begin{array}{lll} u_{XX} = \mathrm{id}, & u_{XW} + u_{WY}^* = 0, & u_{XY} + u_{XY}^* + u_{WY}^* \circ u_{WY} = 0 \\ u_{WX} = 0, & u_{WW} = \mathrm{id} & \\ u_{YX} = 0, & u_{YW} = 0, & u_{YY} = \mathrm{id} \end{array} \right\}.$$

Here $u_{\star\bullet} = \mathrm{pr}_\star \circ u|_\bullet \in \mathrm{Hom}(\bullet, \star)$, where $\mathrm{pr}_\star : W' = X \oplus W \oplus Y \to \star$ is the projection, and

$$* : \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(W, X) \quad \text{and} \quad * : \mathrm{Hom}(Y, X) \longrightarrow \mathrm{Hom}(Y, X)$$

are defined by requiring that

$$\langle f(y), w \rangle = \langle y, f^*(w) \rangle \quad \text{for } f \in \mathrm{Hom}(Y, W),\ y \in Y,\ w \in W,$$
$$\langle g(y), y' \rangle = \langle y, g^*(y') \rangle \quad \text{for } g \in \mathrm{Hom}(Y, X),\ y, y' \in Y.$$

In particular, the map $u \mapsto u_{WY}$ induces an exact sequence

$$1 \longrightarrow \mathrm{Hom}(Y, X)^{*=-} \longrightarrow U \longrightarrow \mathrm{Hom}(Y, W) \longrightarrow 1,$$

where $\mathrm{Hom}(Y, X)^{*=-} = \{g \in \mathrm{Hom}(Y, X) \,|\, g^* = -g\}$. Similarly, the map $\bar{u} \mapsto \bar{u}_{YW}$ induces an exact sequence

$$1 \longrightarrow \mathrm{Hom}(X, Y)^{*=-} \longrightarrow \bar{U} \longrightarrow \mathrm{Hom}(W, Y) \longrightarrow 1.$$

In Case A, we replace $\langle \cdot, \cdot \rangle$ by $\daleth^{-1} \cdot \langle \cdot, \cdot \rangle$ and view G' as the isometry group of a hermitian form on W'. Let $\{w_i \,|\, i = 1, \ldots, k\}$ be a basis of X over E and $\{w_{-i} \,|\, i = 1, \ldots, k\}$ the dual basis of Y over E such that $\langle w_i, w_{-j} \rangle = \delta_{ij}$. Using these bases, we obtain isomorphisms

$$\mathrm{Hom}(Y, X)^{*=-} \cong \mathrm{M}_k(E)^{*=-} \quad \text{and} \quad \mathrm{Hom}(X, Y)^{*=-} \cong \mathrm{M}_k(E)^{*=-}.$$

Here $\mathrm{M}_k(E)^{*=-} = \{x \in \mathrm{M}_k(E) \,|\, {}^t x^c = -\epsilon x\}$, where

$$\epsilon = \begin{cases} 1 & \text{in Cases ABD,} \\ -1 & \text{in Case C.} \end{cases}$$

We take the Haar measure

$$\prod_{1 \le i \le k} dx_{ii} \cdot \prod_{1 \le i < j \le k} dx_{ij}$$

on $\mathrm{M}_k(E)^{*=-}$, where:

- dx_{ii} is the self-dual measure on
$$\begin{cases} E_0 & \text{in Case A,} \\ F & \text{in Case C,} \end{cases}$$
with respect to the pairing $(x,y) \mapsto \psi(xy)$. Here, in Cases B and D, the product $\prod_{1 \le i \le k}$ is interpreted to be 1.
- dx_{ij} is the self-dual measure on E with respect to ψ_E.

Thus we obtain Haar measures du' and $d\bar{u}'$ on $\mathrm{Hom}(Y,X)^{*=-}$ and $\mathrm{Hom}(X,Y)^{*=-}$ respectively. Note that the product $du'\, d\bar{u}'$ is independent of the choice of the basis $\{w_i\}$.

Fix a Haar measure du'' on $\mathrm{Hom}(Y,W)$. Let $d\bar{u}''$ be the Haar measure on $\mathrm{Hom}(W,Y)$ which is dual to du'' with respect to the pairing $(u'',\bar{u}'') \mapsto \psi_E(\mathrm{tr}(u'' \circ \bar{u}''))$. Then the product $du''\, d\bar{u}''$ does not depend on the choice of du''.

We define Haar measures du and $d\bar{u}$ on U and $\bar{U}$ by
$$du = du'\, du'' \quad \text{and} \quad d\bar{u} = d\bar{u}' d\bar{u}''$$
respectively. We shall take the Haar measure
$$\begin{cases} du\, d\bar{u} & \text{in Cases AC"D,} \\ |2|_F^{-k} \cdot du\, d\bar{u} & \text{in Cases BC',} \end{cases}$$
on $U \times \bar{U}$.

Now suppose that X is a finite dimensional vector space over E and P is a maximal parabolic subgroup of $\mathrm{GL}(X)$ stabilizing a subspace X' of X. Let U be the unipotent radical of P. If we write
$$X = X' \oplus X'',$$
then we have isomorphisms
$$U \cong \mathrm{Hom}(X'',X') \quad \text{and} \quad \bar{U} \cong \mathrm{Hom}(X',X'').$$
Fix a Haar measure du on $\mathrm{Hom}(X'',X')$. Let $d\bar{u}$ be the Haar measure on $\mathrm{Hom}(X',X'')$ which is dual to du with respect to the pairing $(u,\bar{u}) \mapsto \psi_E(\mathrm{tr}(u \circ \bar{u}))$. Then the product $du\, d\bar{u}$ does not depend on the choice of du. We shall take the Haar measure
$$du\, d\bar{u}$$
on $U \times \bar{U}$.

Thus, we have defined the Haar measure on $U \times \bar{U}$ which depends only on the additive character ψ. Let $\mu_\psi(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ denote the corresponding Plancherel measure. Then we have
$$\mu_{\psi_a}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) = |a|_F^{\dim U} \cdot \mu_\psi(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$$
for $a \in F^\times$, where $\psi_a(x) = \psi(ax)$ for $x \in F$. Since we are fixing ψ, we shall henceforth suppress ψ from the notation.

B.3. **Orthogonal groups vs. special orthogonal groups.** When W is a quadratic space over F, we may consider the induced representation $I_{P^0}^{G'^0}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0)$ of $G'^0 = \mathrm{SO}(W')$, where π_0 is an irreducible constituent of $\pi|_{G^0}$, and the associated Plancherel measure $\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0)$. We note:

Lemma B.1. *We have*
$$\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0) = \mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi).$$

Proof. Observe that

$$\begin{aligned} I_P^{G'}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) &\subset I_P^{G'}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \mathrm{Ind}_{G^0}^{G}(\pi_0)) \\ &\cong \mathrm{Ind}_{G'^0}^{G'}(I_{P^0}^{G'^0}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0)) \end{aligned}$$

and

$$J_{\bar{P}|P}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) = \mathrm{Ind}_{G'^0}^{G'}(J_{\bar{P}^0|P^0}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0))|_{I_P^{G'}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)}.$$

From this, one can deduce the desired identity. □

B.4. **Factorization.** Factorizing the intertwining operator $J_{\bar{P}|P}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ as a product of "rank 1" operators, one can express the Plancherel measure $\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ as a product of Plancherel measures for the corank 1 case (see [76, Lemme V.2.1]).

Lemma B.2. *We have:*

$$\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) = \left(\prod_{1 \le i < j \le r} \mu(\tau_i \otimes \tau_j) \cdot \mu(\tau_i \otimes (\tau_j^c)^\vee) \right) \cdot \prod_{1 \le i \le r} \mu(\tau_i \otimes \pi).$$

Here $\mu(\tau_i \otimes \tau_j)$ *is the Plancherel measure associated to the representation of* $\mathrm{GL}_{k_i+k_j}(E)$ *parabolically induced from the representation* $\tau_i \otimes \tau_j$ *of the Levi component* $\mathrm{GL}_{k_i}(E) \times \mathrm{GL}_{k_j}(E)$.

B.5. **Multiplicativity.** We now come to an inductive property of Plancherel measures, known as the multiplicativity. We consider a parabolic subgroup P' of $G = G(W)$ with Levi component

$$\mathrm{GL}_{k'_1}(E) \times \cdots \times \mathrm{GL}_{k'_s}(E) \times G(W''),$$

where $W = \mathbb{H}^{k'} \oplus W''$ and $k'_1 + \cdots + k'_s = k'$. Suppose that

$$\pi \subset I_{P'}^{G}(\tau'_1 \otimes \cdots \otimes \tau'_s \otimes \pi'),$$

where τ'_j and π' are irreducible admissible representations of $\mathrm{GL}_{k'_j}(E)$ and $G(W'')$ respectively. We shall describe the Plancherel measure $\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ in terms of τ_i, τ'_j and π'.

By Lemma B.2, it suffices to consider the corank 1 case, i.e., to consider the Plancherel measure

$$\mu(\tau \otimes \pi)$$

with $r = 1$ and $\tau = \tau_1$. Also, by induction in stages, it suffices to consider the case when

$$\pi \subset I_{P'}^{G}(\tau' \otimes \pi')$$

with $s = 1$ and $\tau' = \tau'_1$.

Proposition B.3. *We have*

$$\mu(\tau \otimes \pi) = \mu(\tau \otimes \tau') \cdot \mu(\tau \otimes (\tau'^c)^\vee) \cdot \mu(\tau \otimes \pi').$$

Proof. Let P'' be the unique parabolic subgroup of $G' = G(W')$ such that $P'' \subset P$ and $M \cap P'' = \mathrm{GL}_k(E) \times P'$. Note that P'' has Levi component

$$\mathrm{GL}_k(E) \times \mathrm{GL}_{k'}(E) \times G(W'').$$

By induction in stages, for an irreducible admissible representation υ of $\mathrm{GL}_{k'}(E)$, we have

$$\begin{aligned} J_{\bar{P}''|P''}(\tau \otimes \upsilon \otimes \pi') &= J_{\bar{P}|P}(\tau \otimes I_{P'}^{G}(\upsilon \otimes \pi')) \circ \mathrm{Ind}_P^{G'}(\mathrm{id} \otimes J_{\bar{P}'|P'}(\upsilon \otimes \pi')) \\ &= \mathrm{Ind}_{\bar{P}}^{G'}(\mathrm{id} \otimes J_{\bar{P}'|P'}(\upsilon \otimes \pi')) \circ J_{\bar{P}|P}(\tau \otimes I_{P'}^{G}(\upsilon \otimes \pi')). \end{aligned}$$

In particular, we have

$$\mu(\tau \otimes \upsilon \otimes \pi') = \mu(\tau \otimes I_{P'}^{G}(\upsilon \otimes \pi')) \cdot \mu(\upsilon \otimes \pi').$$

On the other hand, we have

$$I_P^{G'}(\tau \otimes \pi) \subset I_P^{G'}(\tau \otimes I_{P'}^G(\tau' \otimes \pi'))$$

and

$$J_{\bar{P}|P}(\tau \otimes \pi) = J_{\bar{P}|P}(\tau \otimes I_{P'}^G(\tau' \otimes \pi'))|_{I_P^{G'}(\tau\otimes\pi)},$$

so that

$$\mu(\tau \otimes \pi) = \mu(\tau \otimes I_{P'}^G(\tau' \otimes \pi')) = \left(\frac{\mu(\tau \otimes \upsilon \otimes \pi')}{\mu(\upsilon \otimes \pi')}\right)\Bigg|_{\upsilon=\tau'}.$$

Hence, by Lemma B.2, we obtain

$$\mu(\tau \otimes \pi) = (\mu(\tau \otimes \upsilon) \cdot \mu(\tau \otimes (\upsilon^c)^\vee) \cdot \mu(\tau \otimes \pi'))|_{\upsilon=\tau'}.$$

Since the line $\{(\tau, \upsilon) \,|\, \upsilon = \tau'\}$ is not a singular hyperplane of $\mu(\tau \otimes \upsilon)$ nor that of $\mu(\tau \otimes (\upsilon^c)^\vee)$, we can simply substitute υ by τ' on the right-hand side. This yields the proposition. □

In general, if

$$\pi \subset I_{P'}^G(\tau_1' \otimes \cdots \otimes \tau_s' \otimes \pi'),$$

then by Lemma B.2 and Proposition B.3, we have

$$\begin{aligned}
&\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi) \\
&= \left(\prod_{1\le i<j\le r} \mu(\tau_i \otimes \tau_j) \cdot \mu(\tau_i \otimes (\tau_j^c)^\vee)\right) \cdot \left(\prod_{\substack{1\le i\le r\\ 1\le j\le s}} \mu(\tau_i \otimes \tau_j') \cdot \mu(\tau_i \otimes (\tau_j'^c)^\vee)\right) \cdot \prod_{1\le i\le r} \mu(\tau_i \otimes \pi') \\
&= \left(\prod_{\substack{1\le i\le r\\ 1\le j\le s}} \mu(\tau_i \otimes \tau_j') \cdot \mu(\tau_i \otimes (\tau_j'^c)^\vee)\right) \cdot \mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi').
\end{aligned}$$

We also consider a parabolic subgroup P' of $\mathrm{GL}_{k_i}(E)$ with Levi component

$$\mathrm{GL}_{k_1'}(E) \times \cdots \times \mathrm{GL}_{k_s'}(E),$$

where $k_1' + \cdots + k_s' = k_i$. Suppose that

$$\tau_i \subset I_{P'}^{\mathrm{GL}_{k_i}(E)}(\tau_1' \otimes \cdots \otimes \tau_s'),$$

where τ_j' is an irreducible admissible representation of $\mathrm{GL}_{k_j'}(E)$. Similarly, we can describe the Plancherel measure $\mu(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi)$ in terms of $\tau_{i'}$ (for $i' \neq i$), τ_j' and π'. By Lemma B.2 and induction in stages again, it suffices to consider the Plancherel measure

$$\mu(\tau \otimes \pi)$$

with $r = 1$ and $\tau = \tau_1$ when

$$\tau \subset I_{P'}^{\mathrm{GL}_k(E)}(\tau_1' \otimes \tau_2')$$

with $s = 2$. By a similar argument as in the proof of Proposition B.3, we have:

Proposition B.4. *We have*

$$\mu(\tau \otimes \pi) = \mu(\tau_1' \otimes (\tau_2'^c)^\vee) \cdot \mu(\tau_1' \otimes \pi) \cdot \mu(\tau_2' \otimes \pi).$$

B.6. **Unramified case.** We consider the Plancherel measure $\mu(\tau \otimes \pi)$ for the corank 1 case. Suppose that $G(W)$ is quasi-split and τ and π are submodules of principal series representations:

$$\tau \subset I_B^{\mathrm{GL}_k(E)}(\chi_1 \otimes \cdots \otimes \chi_k) \quad \text{and} \quad \pi \subset I_{P_0}^G(\mu_1 \otimes \cdots \otimes \mu_r \otimes \mathbf{1}_{G(W_{\mathrm{an}})})$$

for some characters χ_i and μ_j of $\mathrm{GL}_1(E)$. Here B and P_0 are Borel subgroups of $\mathrm{GL}_k(E)$ and $G = G(W)$ respectively. Then by the multiplicativity of Plancherel measures, one can reduce the computation of the Plancherel measure $\mu(\tau \otimes \pi)$ to that for the rank 1 groups SL_2, SO_3, SU_3 and Mp_2. For these groups, the Plancherel measure has been computed explicitly (see [40], [73]), and can be expressed as a product of Tate's γ-factors. Thus, we have:

Proposition B.5. *We have*

$$\mu(\tau \otimes \pi) = \gamma(0, \phi_\tau \otimes \phi_\pi^\vee, \psi_E) \cdot \gamma(0, \phi_\tau^\vee \otimes \phi_\pi, \bar{\psi}_E) \cdot \gamma(0, R \circ \phi_\tau, \psi) \cdot \gamma(0, R \circ \phi_\tau^\vee, \bar{\psi}).$$

Here

$$\phi_\tau = \bigoplus_{i=1}^k \chi_i \quad \textit{and} \quad \phi_\pi = \left(\bigoplus_{j=1}^r \left(\mu_j \oplus (\mu_j^c)^{-1}\right) \right) \oplus \mathbf{1},$$

where $\mathbf{1}$ *is the L-parameter associated to the trivial representation of* $G(W_{\mathrm{an}})$ *(note that* ϕ_τ *and* ϕ_π *are not necessarily the L-parameters of* τ *and* π*) and*

$$R = \begin{cases} \mathrm{Asai} & \textit{in Case A,} \\ \mathrm{Sym}^2 & \textit{in Cases BC',} \\ \bigwedge^2 & \textit{in Cases C''D.} \end{cases}$$

Note that the Haar measure on $U \times \bar{U}$ is defined in §B.2 so that the identity in the above proposition holds. When τ and π are unramified (so that χ_i and μ_j are unramified), the Gindikin-Karpelevich formula is a refinement of Proposition B.5 in the sense that it describes the action of the intertwining operator on spherical vectors rather than that of the composition of the two intertwining operators.

B.7. **Plancherel measures and reducibility.** Now suppose that τ and π are unitary and supercuspidal. Let $\Omega(\tau \otimes \pi)$ be the stabilizer of $\tau \otimes \pi$ in the Weyl group $\Omega := \mathrm{Norm}_{G'}(M)/M$. Then the points of reducibility of the induced representation

$$I_P^{G'}(\tau_s \otimes \pi)$$

are determined by the analytic properties of the Plancherel measure

$$\mu(\tau_s \otimes \pi)$$

(where $\tau_s = \tau |\det|_E^s$ for $s \in \mathbb{C}$) as follows.

Proposition B.6. (i) *We have* $\mu(\tau \otimes \pi) \geq 0$.

(ii) *If* $\Omega(\tau \otimes \pi) = \{1\}$*, then* $I_P^{G'}(\tau \otimes \pi)$ *is irreducible and* $\mu(\tau \otimes \pi) > 0$.

(iii) *If* $\Omega(\tau \otimes \pi) \neq \{1\}$*, then* $I_P^{G'}(\tau \otimes \pi)$ *is irreducible if and only if* $\mu(\tau \otimes \pi) = 0$ *(in which case the order of zero is* 2*).*

(iv) *If* $\mu(\tau \otimes \pi) > 0$*, then* $\mu(\tau_s \otimes \pi)$ *is holomorphic on* $\mathbb{R}$.

(v) *If* $\mu(\tau \otimes \pi) = 0$*, then there exists a unique* $s_0 > 0$ *such that* $\mu(\tau_s \otimes \pi)$ *has a pole at* $s = s_0$ *(in which case the order of pole is* 1*).*

(vi) *For* $s_0 > 0$*,* $I_P^{G'}(\tau_{s_0} \otimes \pi)$ *is irreducible if and only if* $\mu(\tau_s \otimes \pi)$ *is holomorphic at* $s = s_0$.

When $G(W)$ is linear and connected, these results are largely due to Harish-Chandra (see [70], [76]); (v) is due to Silberger [71]. The case when $G(W)$ is the (nonlinear) metaplectic group is largely treated in [13, Appendix A]; (v) is due to Hanzer-Muić [25]. The case when $G(W)$ is the (linear but disconnected) orthogonal group can be inferred from a result of Ban-Jantzen [6].

When $G = \mathrm{O}(W)$ is the orthogonal group, the following example is instructive. Let π_0 be an irreducible constituent of $\pi|_{G^0}$. Assume that either $\dim W = 0$ or $\mathbf{s}\pi_0 \not\cong \pi_0$, where $\mathbf{s} \in G \smallsetminus G^0$. Assume further that k is odd and τ is self-dual. Then by [6, Proposition 4.3], $I_{P^0}^{G'^0}(\tau \otimes \pi_0)$ is irreducible but $I_P^{G'}(\tau \otimes \pi)$ is reducible. This is consistent with the fact that

$$\Omega^0(\tau \otimes \pi_0) = \{1\}, \quad \Omega(\tau \otimes \pi) \neq \{1\}, \quad \mu(\tau \otimes \pi_0) = \mu(\tau \otimes \pi) > 0,$$

where $\Omega^0(\tau \otimes \pi_0)$ is the stabilizer of $\tau \otimes \pi_0$ in the Weyl group $\Omega^0 := \mathrm{Norm}_{G'^0}(M^0)/M^0$. Except for this scenario, a result of Ban-Jantzen [6, Proposition 4.3] says that $I_{P^0}^{G'^0}(\tau_s \otimes \pi_0)$ is irreducible if and only if $I_P^{G'}(\tau_s \otimes \pi)$ is irreducible.

B.8. **Global case.** Suppose that $\mathbb{F}$ is a number field with ring of adeles $\mathbb{A} = \mathbb{A}_{\mathbb{F}}$ and $\mathbb{W}$ is a space over $\mathbb{E}$, where either $\mathbb{E} = \mathbb{F}$ or $\mathbb{E}$ is a quadratic extension of $\mathbb{F}$. We fix a nontrivial additive character Ψ of $\mathbb{F}\backslash\mathbb{A}$. Let $\mathfrak{T} \cong \otimes_v \mathfrak{T}_v$ and $\Pi \cong \otimes_v \Pi_v$ be irreducible cuspidal automorphic representations of $\mathrm{GL}_k(\mathbb{A}_{\mathbb{E}})$ and $G(\mathbb{W})(\mathbb{A})$ respectively. We consider the induced representation

$$I_{\mathbb{P}}^{\mathbb{G}'}(\mathfrak{T}_s \otimes \Pi)$$

of $\mathbb{G}'(\mathbb{A})$, where $\mathbb{P}$ is a maximal parabolic subgroup of $\mathbb{G}' = G(\mathbb{H}^k \oplus \mathbb{W})$ with Levi component $\mathrm{GL}_k(\mathbb{E}) \times G(\mathbb{W})$, and the associated global intertwining operator

$$J_{\bar{\mathbb{P}}|\mathbb{P}}(\mathfrak{T}_s \otimes \Pi) : I_{\mathbb{P}}^{\mathbb{G}'}(\mathfrak{T}_s \otimes \Pi) \longrightarrow I_{\bar{\mathbb{P}}}^{\mathbb{G}'}(\mathfrak{T}_s \otimes \Pi).$$

Then we have the global functional equation

$$J_{\mathbb{P}|\bar{\mathbb{P}}}(\mathfrak{T}_s \otimes \Pi) \circ J_{\bar{\mathbb{P}}|\mathbb{P}}(\mathfrak{T}_s \otimes \Pi) = 1.$$

From this and the Gindikin-Karpelevich formula, one can deduce:

Proposition B.7. *We have*

$$\left(\prod_{v \in S} \mu_{\Psi_v}(\mathfrak{T}_{v,s} \otimes \Pi_v)^{-1}\right) \cdot \frac{L^S(s, \mathfrak{T} \times \Pi^\vee)}{L^S(1-s, \mathfrak{T}^\vee \times \Pi)} \cdot \frac{L^S(-s, \mathfrak{T}^\vee \times \Pi)}{L^S(1+s, \mathfrak{T} \times \Pi^\vee)} \cdot \frac{L^S(2s, \mathfrak{T}, R)}{L^S(1-2s, \mathfrak{T}^\vee, R)} \cdot \frac{L^S(-2s, \mathfrak{T}^\vee, R)}{L^S(1+2s, \mathfrak{T}, R)} = 1$$

for a sufficiently large finite set S of places of $\mathbb{F}$. Here

$$R = \begin{cases} \mathrm{Asai} & \textit{in Case A,} \\ \mathrm{Sym}^2 & \textit{in Cases BC',} \\ \bigwedge^2 & \textit{in Cases C''D.} \end{cases}$$

Appendix C. Convergence of integrals

The purpose of this appendix is to show that certain triple integrals appearing in §18 are absolutely convergent.

Assume that $l \geq 0$. Let r and s be the F-ranks of G and H respectively. Put

$$\rho = \begin{cases} \frac{n}{2} & \text{in Case A,} \\ \frac{n-1}{2} & \text{in Cases BD,} \\ \frac{n+1}{2} & \text{in Case C,} \end{cases} \qquad \varrho = \begin{cases} \frac{m}{2} & \text{in Case A,} \\ \frac{m+1}{2} & \text{in Cases BD,} \\ \frac{m-1}{2} & \text{in Case C.} \end{cases}$$

Then we have $l = -m + 2\rho = n - 2\varrho$. Fix maximal split tori A_0^G and A_0^H of G and H respectively. Fix minimal parabolic subgroups $P_0^G \supset A_0^G$ and $P_0^H \supset A_0^H$ of G and H respectively. Let δ_0^G and δ_0^H

be the modulus characters of P_0^G and P_0^H respectively. We may identify A_0^G and A_0^H with $(F^\times)^r$ and $(F^\times)^s$ respectively, so that

$$\delta_0^G(a) = \prod_{i=1}^r |a_i|_E^{2\rho+1-2i}, \quad \delta_0^H(b) = \prod_{j=1}^s |b_j|_E^{2\varrho+1-2j}$$

for $a = (a_1, \dots, a_r) \in A_0^G \cong (F^\times)^r$, $b = (b_1, \dots, b_s) \in A_0^H \cong (F^\times)^s$ and

$$\begin{aligned} A_0^{G,+} &\cong \{(a_1, \dots, a_r) \in (F^\times)^r \,|\, |a_1|_F \le \dots \le |a_r|_F \le 1\}, \\ A_0^{H,+} &\cong \{(b_1, \dots, b_s) \in (F^\times)^s \,|\, |b_1|_F \le \dots \le |b_s|_F \le 1\}. \end{aligned}$$

Let π be an irreducible discrete series representation of G. Assume that either π is supercuspidal or $0 \le l \le 1$.

Lemma C.1. *Let $\phi_1, \phi_2, \phi_3, \phi_4 \in \omega$. Let f and f' be matrix coefficients of π. Then the triple integral*

$$\int_H \int_G \int_G \mathcal{F}_{\omega(h)\phi_1 \otimes \bar{\phi}_2}(i(g,1)) \cdot \overline{\mathcal{F}_{\omega(h)\phi_3 \otimes \bar{\phi}_4}(i(g',1))} \cdot \overline{f(g)} \cdot f'(g') \, dg \, dg' \, dh$$

is absolutely convergent.

Proof. When π is supercuspidal, it suffices to show that the integral

$$\int_H \mathcal{F}_{\omega(h)\phi_1 \otimes \bar{\phi}_2}(1) \cdot \overline{\mathcal{F}_{\omega(h)\phi_3 \otimes \bar{\phi}_4}(1)} \, dh = \int_H (\boldsymbol{\omega}(h)(\phi_1 \otimes \bar{\phi}_3), \phi_2 \otimes \bar{\phi}_4) \cdot \overline{\chi_W(\det h)} \, dh$$

is absolutely convergent. This follows from [50, Theorem 3.2].

Now suppose that $0 \le l \le 1$. It suffices to show that the integral

$$\int_{A_0^{H,+}} \int_{A_0^{G,+}} \int_{A_0^{G,+}} \mathcal{F}_{\omega(b)\phi_1 \otimes \bar{\phi}_2}(i(a,1)) \overline{\mathcal{F}_{\omega(b)\phi_3 \otimes \bar{\phi}_4}(i(a',1))} \cdot \overline{f(a)} f'(a') \cdot \delta_0^G(a)^{-1} \delta_0^G(a')^{-1} \delta_0^H(b)^{-1} \, da \, da' \, db$$

is absolutely convergent. We define a function Υ on $F^\times$ by

$$\Upsilon(x) = \begin{cases} 1 & \text{if } |x|_F \le 1, \\ |x|_E^{-1} & \text{if } |x|_F \ge 1. \end{cases}$$

Then as in the proof of [12, Lemma 9.1], we have

$$|\mathcal{F}_{\omega(b)\phi \otimes \bar{\phi}'}(i(a,1))| = |(\omega(ab)\phi, \phi')| \ll \prod_{i=1}^r |a_i|_E^{m/2} \cdot \prod_{j=1}^s |b_j|_E^{n/2-r} \cdot \prod_{i=1}^r \prod_{j=1}^s \Upsilon(a_i b_j^{-1})$$

for ϕ, $\phi' \in \omega$, $a \in A_0^G$ and $b \in A_0^H$. Thus it suffices to show that the integral

$$\begin{aligned} &\int_{A_0^{H,+}} \int_{A_0^{G,+}} \int_{A_0^{G,+}} \prod_{i=1}^r |a_i|_E^{m/2} |a_i'|_E^{m/2} \cdot \prod_{j=1}^s |b_j|_E^{n-2r} \cdot \prod_{i=1}^r \prod_{j=1}^s \Upsilon(a_i b_j^{-1}) \Upsilon(a_i' b_j^{-1}) \\ &\qquad \times \delta_0^G(a)^{-1/2} \delta_0^G(a')^{-1/2} \delta_0^H(b)^{-1} (1 + \log \|a\| \|a'\|)^{-d} \, da \, da' \, db \\ &= \int_{A_0^{H,+}} \int_{A_0^{G,+}} \int_{A_0^{G,+}} \prod_{i=1}^r |a_i|_E^{-(l+1)/2+i} |a_i'|_E^{-(l+1)/2+i} \cdot \prod_{j=1}^s |b_j|_E^{l-2r-1+2j} \cdot \prod_{i=1}^r \prod_{j=1}^s \Upsilon(a_i b_j^{-1}) \Upsilon(a_i' b_j^{-1}) \\ &\qquad \times (1 + \log \|a\| \|a'\|)^{-d} \, da \, da' \, db \end{aligned}$$

is absolutely convergent for some $d \gg 0$.

Fix a sequence $0 \le k_1 \le \dots \le k_r \le r$. We write $a_1, \dots, a_{2r}$ for

$$a_1, \dots, a_{k_1}, a_1', a_{k_1+1}, \dots, a_{k_2}, a_2', \dots, a_r', a_{k_r+1}, \dots, a_r.$$

Put

$$(t_1, \dots, t_{2r}) = (1, \dots, k_1, 1, k_1 + 1, \dots, k_2, 2, \dots, r, k_r + 1, \dots, r).$$

Note that $(t_1, \dots, t_{2r})$ is a shuffle of an ordered sequence $(1, 2, \dots, r; 1, 2, \dots, r)$ divided into two ordered subsequences. By breaking up the domain of integration, it suffices to show that the integral of

$$\prod_{i=1}^{2r} |a_i|_E^{-(l+1)/2+t_i} (1 - \log |a_i|_F)^{-d} \cdot \prod_{j=1}^{s} |b_j|_E^{l-2r-1+2j} \cdot \prod_{i=1}^{2r} \prod_{j=1}^{s} \Upsilon(a_i b_j^{-1}) \tag{C.1}$$

over

$$|a_1|_F \le \dots \le |a_{2r}|_F \le 1, \quad |b_1|_F \le \dots \le |b_s|_F \le 1$$

is absolutely convergent for some $d \gg 0$.

Fix a sequence $0 = l_0 \le l_1 \le \dots \le l_s \le l_{s+1} = 2r$. By breaking up the domain of integration again, it suffices to show that the integral of (C.1) over

$$\begin{aligned} &|a_1|_F \le \dots \le |a_{l_1}|_F \le |b_1|_F \le |a_{l_1+1}|_F \le \dots \le |a_{l_2}|_F \le |b_2|_F \\ &\le \dots \le |b_j|_F \le |a_{l_j+1}|_F \le \dots \le |a_{l_{j+1}}|_F \le |b_{j+1}|_F \\ &\le \dots \le |b_s|_F \le |a_{l_s+1}|_F \le \dots \le |a_{2r}|_F \le 1 \end{aligned}$$

is absolutely convergent for some $d \gg 0$. Note that a_i with $l_j + 1 \le i \le l_{j+1}$ is in the $(i + j)$-th position and b_j is in the $(l_j + j)$-th position. We introduce new variables $x_1, \dots, x_{2r+s}$ with

$$|x_1|_F \le 1, \dots, |x_{2r+s}|_F \le 1,$$

and set

$$a_i = x_1 \cdots x_{2r+s+1-(i+j)}, \quad b_j = x_1 \cdots x_{2r+s+1-(l_j+j)}.$$

We can write (C.1) as

$$\prod_{k=1}^{2r+s} |x_k|_E^{e_k} \cdot \prod_{i=1}^{2r} (1 - \log |a_i|_F)^{-d}$$

with some $e_k \in \frac{1}{2}\mathbb{Z}$. Since $\prod_{i=1}^{2r} (1 - \log |a_i|_F)^{-d}$ is majorized by

$$\prod_k (1 - \log |x_i|_F)^{-d'}$$

with some $d \gg d' \gg 0$, where k runs over $1 \le k \le 2r + s$ which is not of the form $k = 2r + s + 1 - (l_j + j)$ with $1 \le j \le s$ such that $l_j = 0$, it suffices to show that

$$e_k \ge 0$$

for all k and that

$$e_k > 0$$

if $k = 2r + s + 1 - (l_j + j)$ with $1 \le j \le s$ such that $l_j = 0$.

We first assume that $k = 2r + s + 1 - (i' + j')$ with $1 \le i' \le 2r$ and $0 \le j' \le s$ such that $l_{j'} + 1 \le i' \le l_{j'+1}$. Then the exponents of

$$a_1, \dots, a_{l_1}, b_1, a_{l_1+1}, \dots, a_{l_2}, b_2, \dots, b_{j'}, a_{l_{j'}+1}, \dots, a_{i'}$$

in (C.1) contribute to e_k. The contribution from $\prod_{i=1}^{2r} |a_i|_E^{-(l+1)/2+t_i}$ is greater than or equal to

$$-i' \cdot \frac{l+1}{2} + 2 \cdot \frac{i'}{4} \left(\frac{i'}{2} + 1 \right) = -\frac{i'l}{2} + \frac{i'^2}{4}$$

if i' is even, and

$$-i' \cdot \frac{l+1}{2} + 2 \cdot \frac{i'-1}{4}\left(\frac{i'-1}{2}+1\right) + \frac{i'+1}{2} = -\frac{i'l}{2} + \frac{i'^2}{4} + \frac{1}{4}$$

if i' is odd. The contribution from $\prod_{j=1}^{s} |b_j|_E^{l-2r-1+2j}$ is equal to

$$j'(l-2r-1) + j'(j'+1) = j'l - 2j'r + j'^2.$$

The contribution from $\prod_{i=1}^{2r} \prod_{j=1}^{s} \Upsilon(a_i b_j^{-1})$ is equal to

$$-\sum_{j=1}^{j'}(i'-l_j) + \sum_{j=1}^{j'}(2r-l_j) = -i'j' + 2j'r.$$

Hence we have

$$\begin{aligned} e_k &\geq -\frac{i'l}{2} + \frac{i'^2}{4} + j'l - 2j'r + j'^2 - i'j' + 2j'r \\ &= -\frac{i'l}{2} + \frac{i'^2}{4} + j'l + j'^2 - i'j' \\ &= \left(\frac{i'}{2} - j' - \frac{l}{2}\right)^2 - \frac{l^2}{4} \\ &\geq 0 \end{aligned}$$

if i' is even, and

$$\begin{aligned} e_k &\geq -\frac{i'l}{2} + \frac{i'^2}{4} + \frac{1}{4} + j'l - 2j'r + j'^2 - i'j' + 2j'r \\ &= -\frac{i'l}{2} + \frac{i'^2}{4} + j'l + j'^2 - i'j' + \frac{1}{4} \\ &= \left(\frac{i'}{2} - j' - \frac{l}{2}\right)^2 - \frac{l^2}{4} + \frac{1}{4} \\ &\geq 0 \end{aligned}$$

if i' is odd.

We next assume that $k = 2r + s + 1 - (l_{j'} + j')$ with $1 \leq j' \leq s$. Then the exponents of

$$a_1, \dots, a_{l_1}, b_1, a_{l_1+1}, \dots, a_{l_2}, b_2, \dots, b_{j'}$$

in (C.1) contribute to e_k. Similarly, we have

$$e_k \geq \left(\frac{l_{j'}}{2} - j' - \frac{l}{2}\right)^2 - \frac{l^2}{4} \geq 0$$

if $l_{j'}$ is even, and

$$e_k \geq \left(\frac{l_{j'}}{2} - j' - \frac{l}{2}\right)^2 - \frac{l^2}{4} + \frac{1}{4} \geq 0$$

if $l_{j'}$ is odd. Moreover, if $l_{j'} = 0$, then we have

$$e_k \geq \left(-j' - \frac{l}{2}\right)^2 - \frac{l^2}{4} = j'^2 + j'l > 0.$$

This completes the proof. □

Lemma C.2. *Let* $\mathcal{F} \in I_{\mathbf{P}}^{\mathbf{G}}(\frac{l}{2}, \chi_V)$ *and* $\mathcal{F}' \in I_{\mathbf{P}}^{\mathbf{G}}(-\frac{l}{2}, \chi_V)$*. Let* f *and* f' *be matrix coefficients of* π*. Then the triple integral*

$$\int_G \int_G \int_G \mathcal{F}(i(g'^{-1}g'', 1)) \cdot \overline{\mathcal{F}'(i(g''g^{-1}, 1))} \cdot \overline{f(g)} \cdot f'(g') \, dg \, dg' \, dg''$$

is absolutely convergent.

Proof. When π is supercuspidal, it suffices to show that the integral

$$\int_G \mathcal{F}(i(g'', 1)) \cdot \overline{\mathcal{F}'(i(g'', 1))} \, dg''$$

is absolutely convergent. This follows from Lemma 9.3.

Now suppose that $0 \leq l \leq 1$. It suffices to show that the integral

$$\int_G \int_G \int_G \mathcal{F}(i(g', 1)) \cdot \overline{\mathcal{F}'(i(g, 1))} \cdot \overline{f(g^{-1}g'')} \cdot f'(g''g'^{-1}) \, dg \, dg' \, dg'' \tag{C.2}$$

is absolutely convergent. For any $d \gg 0$, (C.2) is majorized by

$$\int_G \int_G \int_G |\mathcal{F}(i(g', 1))\mathcal{F}'(i(g, 1))| \cdot \Xi(g^{-1}g'')\Xi(g''g'^{-1}) \cdot (1 + \log \|g\|\|g'\|\|g''\|)^{-d} \, dg \, dg' \, dg''.$$

We may assume that

$$|\mathcal{F}(i(g'k', 1))| = |\mathcal{F}(i(g', 1))|, \quad |\mathcal{F}'(i(kg, 1))| = |\mathcal{F}'(i(g, 1))|$$

for $g, g' \in G$, $k, k' \in K$. Since

$$|\mathcal{F}(i(g', 1))\mathcal{F}'(i(g, 1))| = \int_K \int_K |\mathcal{F}(i(g'k', 1))\mathcal{F}'(i(kg, 1))| \, dk \, dk',$$

(C.2) is majorized by

$$\int_G \int_G \int_G \int_K \int_K |\mathcal{F}(i(g'k', 1))\mathcal{F}'(i(kg, 1))| \cdot \Xi(g^{-1}g'')\Xi(g''g'^{-1}) \cdot (1 + \log \|g\|\|g'\|\|g''\|)^{-d} \, dk \, dk' \, dg \, dg' \, dg''$$

$$\int_G \int_G \int_G \int_K \int_K |\mathcal{F}(i(g', 1))\mathcal{F}'(i(g, 1))| \cdot \Xi(g^{-1}kg'')\Xi(g''k'g'^{-1}) \cdot (1 + \log \|g\|\|g'\|\|g''\|)^{-d} \, dk \, dk' \, dg \, dg' \, dg''$$

$$= \int_G \int_G \int_G |\mathcal{F}(i(g', 1))\mathcal{F}'(i(g, 1))| \cdot \Xi(g)\Xi(g')\Xi(g'')^2 \cdot (1 + \log \|g\|\|g'\|\|g''\|)^{-d} \, dg \, dg' \, dg''.$$

By Lemma 9.3, this integral is absolutely convergent. This completes the proof. □

Appendix D. Computation of volumes

The purpose of this appendix is to compute the volumes of certain compact groups which are used in the proof of Corollary 20.6. We only consider Cases A, C" and D.

D.1. **Lattices.** For the computation, we need to fix a minimal parabolic subgroup and an Iwahori subgroup of $G = G(W)$. Here, in Case A, we replace $\langle \cdot, \cdot \rangle$ by $\daleth^{-1} \cdot \langle \cdot, \cdot \rangle$ and view G as the isometry group of a hermitian form on W.

Let r be the F-rank of G. Then we have

$$W = \mathbb{H}^r \oplus W_{\mathrm{an}},$$

where $\mathbb{H}$ is the hyperbolic plane and W_{an} is the anisotropic kernel of W. Choose a basis $\{w_i \,|\, i = \pm 1, \ldots, \pm r\}$ of $\mathbb{H}^r$ over E such that

$$\langle w_i, w_j \rangle = \langle w_{-i}, w_{-j} \rangle = 0 \quad \text{and} \quad \langle w_i, w_{-j} \rangle = \delta_{ij}$$

for $1 \leq i, j \leq r$. Let P_0^G be the minimal parabolic subgroup of G stabilizing a flag

$$Ew_1 \subset Ew_1 + Ew_2 \subset \cdots \subset Ew_1 + \cdots + Ew_r$$

and δ_0^G the modulus character of P_0^G. Let A_0^G be the maximal split torus of G stabilizing $Ew_1, \ldots, Ew_r$. We identify $(a_1, \ldots, a_r) \in (F^\times)^r$ with an element $a \in A_0^G$ such that $aw_i = a_i w_i$ for $1 \leq i \leq r$. Let

$$A_0^{G,+} = \{(a_1, \ldots, a_r) \in (F^\times)^r \mid |a_1|_F \leq \cdots \leq |a_r|_F \leq 1\}.$$

We use a lattice chain in W to define an Iwahori subgroup of G. For $0 \leq i \leq r$, we define a lattice $\mathcal{L}_i$ in W as follows.

(A)
- If $\dim W_{\mathrm{an}} = 0$, let

$$\mathcal{L}_i = \sum_{j=1}^{i} (\varpi_E \mathfrak{d}_E^{-1} w_j + \mathfrak{o}_E w_{-j}) + \sum_{j=i+1}^{r} (\mathfrak{d}_E^{-1} w_j + \mathfrak{o}_E w_{-j}).$$

- If $\dim W_{\mathrm{an}} = 1$, we may assume that $W_{\mathrm{an}} = E$ and $\langle x, x \rangle = \kappa \mathrm{N}_{E/F}(x)$ for $x \in W_{\mathrm{an}}$, where

$$\begin{cases} \kappa \in \mathfrak{o}_F^\times \text{ or } \kappa \text{ is a uniformizer of } \mathfrak{o}_F & \text{if } E/F \text{ is unramified,} \\ \kappa \in \mathfrak{o}_F^\times & \text{if } E/F \text{ is ramified.} \end{cases}$$

  Let

$$\mathcal{L}_i = \sum_{j=1}^{i} (\kappa \varpi_E \mathfrak{d}_E^{-1} w_j + \mathfrak{o}_E w_{-j}) + \sum_{j=i+1}^{r} (\kappa \mathfrak{d}_E^{-1} w_j + \mathfrak{o}_E w_{-j}) + \mathfrak{o}_E.$$

- If $\dim W_{\mathrm{an}} = 2$, we may assume that $W_{\mathrm{an}} = D$ and $\langle x, x \rangle = \mathrm{N}_{D/F}(x)$ for $x \in W_{\mathrm{an}}$, where D is the quaternion division algebra over F and we fix an embedding $E \hookrightarrow D$. Let

$$\mathcal{L}_i = \sum_{j=1}^{i} (\varpi_E \mathfrak{d}_E^{-1} w_j + \mathfrak{o}_E w_{-j}) + \sum_{j=i+1}^{r} (\mathfrak{d}_E^{-1} w_j + \mathfrak{o}_E w_{-j}) + \mathfrak{o}_D.$$

(C") Let

$$\mathcal{L}_i = \sum_{j=1}^{i} (\mathfrak{p}_F w_j + \mathfrak{o}_F w_{-j}) + \sum_{j=i+1}^{r} (\mathfrak{o}_F w_j + \mathfrak{o}_F w_{-j}).$$

(D)
- If $\dim W_{\mathrm{an}} = 0$, let

$$\mathcal{L}_i = \sum_{j=1}^{i} (\mathfrak{p}_F w_j + \mathfrak{o}_F w_{-j}) + \sum_{j=i+1}^{r} (\mathfrak{o}_F w_j + \mathfrak{o}_F w_{-j}).$$

- If $\dim W_{\mathrm{an}} = 2$, we may assume that $W_{\mathrm{an}} = K$ and $\langle x, x \rangle = 2\kappa \mathrm{N}_{K/F}(x)$ for $x \in W_{\mathrm{an}}$, where K is a quadratic extension of F and

$$\begin{cases} \kappa \in \mathfrak{o}_F^\times \text{ or } \kappa \text{ is a uniformizer of } \mathfrak{o}_F & \text{if } K/F \text{ is unramified,} \\ \kappa \in \mathfrak{o}_F^\times & \text{if } K/F \text{ is ramified.} \end{cases}$$

  Let

$$\mathcal{L}_i = \sum_{j=1}^{i} (\kappa \mathfrak{p}_F w_j + \mathfrak{o}_F w_{-j}) + \sum_{j=i+1}^{r} (\kappa \mathfrak{o}_F w_j + \mathfrak{o}_F w_{-j}) + \mathfrak{o}_K.$$

- If $\dim W_{\mathrm{an}} = 4$, we may assume that $W_{\mathrm{an}} = D$ and $\langle x, x \rangle = 2\mathrm{N}_{D/F}(x)$ for $x \in W_{\mathrm{an}}$, where D is the quaternion division algebra over F. Let

$$\mathcal{L}_i = \sum_{j=1}^{i} (\mathfrak{p}_F w_j + \mathfrak{o}_F w_{-j}) + \sum_{j=i+1}^{r} (\mathfrak{o}_F w_j + \mathfrak{o}_F w_{-j}) + \mathfrak{o}_D.$$

Let K_G be the maximal compact subgroup of G stabilizing the lattice $\mathcal{L}_0$ and I_G the Iwahori subgroup of G stabilizing the lattice chain

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots \supset \mathcal{L}_r.$$

Let I_G^+ be the pro-unipotent radical of I_G.

Similarly, we define P_0^H, δ_0^H, A_0^H, $A_0^{H,+}$, K_H, I_H, I_H^+.

D.2. **Computation.** Assume that ψ is of order zero. For a connected reductive linear algebraic group G over F, let I_{G} be an Iwahori subgroup of G and I_{G}^+ the pro-unipotent radical of I_{G}. Put

$$\mathfrak{v}_{\mathsf{G}} = -\log_q \mathrm{vol}(I_{\mathsf{G}}^+, d\mathsf{g}_\psi),$$

where $d\mathsf{g}_\psi$ is the Haar measure on G defined in §13.1. Let $\mathfrak{f}_\bullet$ be the smallest non-negative integer such that $\chi_\bullet(x) = 1$ for all $x \in \mathfrak{o}_F^\times \cap (1 + \mathfrak{p}_F^{\mathfrak{f}_\bullet})$. Note that $\mathfrak{f}_\bullet = a(\chi_\bullet)$, where $a(\chi_\bullet)$ is the Artin conductor of $\chi_\bullet$.

Lemma D.1. *We have*

$$\mathfrak{v}_{\mathrm{GL}_k} = \frac{k(k+1)}{2}$$

and

$$\mathfrak{v}_{G^0} = \begin{cases} \frac{n(n+1)}{2} & \text{if } E/F \text{ is unramified in Case A,} \\ \frac{n}{2}(\frac{n}{2}+1) + \mathfrak{f}_E \cdot \frac{n(n-1)}{4} & \text{if } E/F \text{ is ramified and } n \text{ is even in Case A,} \\ \frac{n^2-1}{4} + \mathfrak{f}_E \cdot \frac{n(n+1)}{4} & \text{if } E/F \text{ is ramified and } n \text{ is odd in Case A,} \\ \frac{n}{2}(\frac{n}{2}+1) & \text{in Case C”,} \\ \frac{n^2}{4} & \text{if } \chi_W \text{ is unramified in Case D,} \\ \frac{n}{2}(\frac{n}{2}-1) + \mathfrak{f}_W \cdot \frac{n-1}{2} & \text{if } \chi_W \text{ is ramified in Case D.} \end{cases}$$

Proof. For a connected reductive linear algebraic group G over F, let

$$\mathfrak{M}_{\mathsf{G}} = \bigoplus_{d \geq 1} \mathfrak{V}_d(1-d)$$

be the motive of G defined by Gross [20], where $\mathfrak{V}_d$ is the Artin motive given in [20, §1]. By [20, (4.11)] and [21, §5], we have

$$\mathfrak{v}_{\mathsf{G}} = \frac{1}{2} \cdot a(\mathfrak{M}_{\mathsf{G}}) + \sum_{d \geq 1} (d-1) \dim \mathfrak{V}_d^{I_F} + \mathrm{rank}_{F^{\mathrm{ur}}} \mathsf{G},$$

where

$$a(\mathfrak{M}_{\mathsf{G}}) = \sum_{d \geq 1} (2d-1) a(\mathfrak{V}_d)$$

is the Artin conductor of $\mathfrak{M}_{\mathsf{G}}$ given in [21, §4] and I_F is the inertia group of F. Also, we have

$$\mathfrak{M}_{\mathrm{GL}_k} = \bigoplus_{i=1}^{k} \mathbb{Q}(1-i)$$

$$\mathfrak{M}_{G^0} = \begin{cases} \bigoplus_{i=1}^{n} \mathbb{Q}[\chi_E^i](1-i) & \text{in Case A,} \\ \bigoplus_{i=1}^{\frac{n}{2}} \mathbb{Q}(1-2i) & \text{in Case C'',} \\ \left(\bigoplus_{i=1}^{\frac{n}{2}-1} \mathbb{Q}(1-2i)\right) \oplus \mathbb{Q}[\chi_W](1-\frac{n}{2}) & \text{in Case D.} \end{cases}$$

This yields the lemma. □

We retain the notation of §20.3.

Lemma D.2. *We have*

$$\frac{\gamma(G'^0/M_P^0)}{\gamma(H'^0/M_Q^0)} = \frac{\mathrm{vol}(K_{G'}^0)}{\mathrm{vol}(K_{H'}^0)} \cdot \frac{\mathrm{vol}(K_{M_Q}^0)}{\mathrm{vol}(K_{M_P}^0)} \cdot q^{-\Lambda},$$

where Λ *is an integer given by the following table.*

Case A, E/F unramified	$\dim V_{\mathrm{an}} = 0$	$\dim V_{\mathrm{an}} = 1$	$\dim V_{\mathrm{an}} = 2$
$\dim W_{\mathrm{an}} = 0$	0	0	$2k$
$\dim W_{\mathrm{an}} = 1$	0	0	$2k$
$\dim W_{\mathrm{an}} = 2$	$-2k$	$-2k$	0

Case A, E/F ramified	$\dim V_{\mathrm{an}} = 0$	$\dim V_{\mathrm{an}} = 1$	$\dim V_{\mathrm{an}} = 2$
$\dim W_{\mathrm{an}} = 0$	$\mathfrak{f}_E km - \mathfrak{f}_E kn$	$\mathfrak{f}_E km - \mathfrak{f}_E kn + \mathfrak{f}_E k$	$\mathfrak{f}_E km - \mathfrak{f}_E kn + 2k$
$\dim W_{\mathrm{an}} = 1$	$\mathfrak{f}_E km - \mathfrak{f}_E kn - \mathfrak{f}_E k$	$\mathfrak{f}_E km - \mathfrak{f}_E kn$	$\mathfrak{f}_E km - \mathfrak{f}_E kn - \mathfrak{f}_E k + 2k$
$\dim W_{\mathrm{an}} = 2$	$\mathfrak{f}_E km - \mathfrak{f}_E kn - 2k$	$\mathfrak{f}_E km - \mathfrak{f}_E kn + \mathfrak{f}_E k - 2k$	$\mathfrak{f}_E km - \mathfrak{f}_E kn$

Case C”	$\dim V_{\mathrm{an}} = 0$	0
	$\dim V_{\mathrm{an}} = 2$, χ_V *unramified*	0
	$\dim V_{\mathrm{an}} = 2$, χ_V *ramified*	$\mathfrak{f}_V k$
	$\dim V_{\mathrm{an}} = 4$	$2k$
Case D	$\dim W_{\mathrm{an}} = 0$	0
	$\dim W_{\mathrm{an}} = 2$, χ_W *unramified*	0
	$\dim W_{\mathrm{an}} = 2$, χ_W *ramified*	$-\mathfrak{f}_W k$
	$\dim W_{\mathrm{an}} = 4$	$-2k$

Proof. Let $I_{G'}$ and $I_{H'}$ be the Iwahori subgroups of G' and H' as in §D.1 respectively. For a reduced root α of A_{M_P} in P, we have

$$U_\alpha \cap K_{G'} = U_\alpha \cap I_{G'}^+,$$

where U_α is the root subgroup associated to α. Hence, by [76, p. 241], we have

$$\gamma(G'^0/M_P^0) = \left(\frac{\mathrm{vol}(I^+_{G'^0})}{\mathrm{vol}(K^0_{G'})}\right)^{-1} \cdot \frac{\mathrm{vol}(I^+_{M_P^0})}{\mathrm{vol}(K^0_{M_P})} \cdot \prod_{\alpha\in\Sigma_{\mathrm{red}}(\bar{P})} [U_\alpha \cap K_{G'} : U_\alpha \cap I^+_{G'}]^{-1},$$

where $\Sigma_{\mathrm{red}}(\bar{P})$ is the set of reduced roots of A_{M_P} in $\bar{P}$. We remark that in [76], the Haar measures are normalized so that

$$\mathrm{vol}(K^0_{G'}) = \mathrm{vol}(K^0_{M_P}) = 1.$$

Similarly, we have

$$\gamma(H'^0/M_Q^0) = \left(\frac{\mathrm{vol}(I^+_{H'^0})}{\mathrm{vol}(K^0_{H'})}\right)^{-1} \cdot \frac{\mathrm{vol}(I^+_{M_Q^0})}{\mathrm{vol}(K^0_{M_Q})} \cdot \prod_{\alpha\in\Sigma_{\mathrm{red}}(\bar{Q})} [U_\alpha \cap K_{H'} : U_\alpha \cap I^+_{H'}]^{-1}.$$

On the other hand, by Lemma D.1, we have

$$\frac{\mathrm{vol}(I^+_{G'^0})}{\mathrm{vol}(I^+_{M_P^0})} = q^{-\Lambda_P} \quad \text{and} \quad \frac{\mathrm{vol}(I^+_{H'^0})}{\mathrm{vol}(I^+_{M_Q^0})} = q^{-\Lambda_Q},$$

where Λ_P and Λ_Q are integers given by the following table.

		Λ_P
Case A	E/F unramified	$2kn + k^2$
	E/F ramified, n even	$kn + \frac{k(k+1)}{2} + \mathfrak{f}_E kn + \mathfrak{f}_E \cdot \frac{k(k-1)}{2}$
	E/F ramified, n odd	$k(n-1) + \frac{k(k+1)}{2} + \mathfrak{f}_E kn + \mathfrak{f}_E \cdot \frac{k(k+1)}{2}$
Case C”		$kn + \frac{k(k+1)}{2}$
Case D	χ_W unramified	$kn + \frac{k(k-1)}{2}$
	χ_W ramified	$k(n-1) + \frac{k(k-1)}{2} + \mathfrak{f}_W k$

		Λ_Q
Case A	E/F unramified	$2km + k^2$
	E/F ramified, m even	$km + \frac{k(k+1)}{2} + \mathfrak{f}_E km + \mathfrak{f}_E \cdot \frac{k(k-1)}{2}$
	E/F ramified, m odd	$k(m-1) + \frac{k(k+1)}{2} + \mathfrak{f}_E km + \mathfrak{f}_E \cdot \frac{k(k+1)}{2}$
Case C”	χ_V unramified	$km + \frac{k(k-1)}{2}$
	χ_V ramified	$k(m-1) + \frac{k(k-1)}{2} + \mathfrak{f}_V k$
Case D		$km + \frac{k(k+1)}{2}$

Moreover, we have

$$\prod_{\alpha\in\Sigma_{\mathrm{red}}(\bar{P})} [U_\alpha \cap K_{G'} : U_\alpha \cap I^+_{G'}]^{-1} = q^{-\Lambda'_P} \quad \text{and} \quad \prod_{\alpha\in\Sigma_{\mathrm{red}}(\bar{Q})} [U_\alpha \cap K_{H'} : U_\alpha \cap I^+_{H'}]^{-1} = q^{-\Lambda'_Q},$$

where Λ'_P and Λ'_Q are integers given by the following table.

		Λ'_P
Case A	E/F unramified, $\dim W_{\rm an} = 0$	$2kn + k^2$
	E/F unramified, $\dim W_{\rm an} = 1$	$2kn + k^2$
	E/F unramified, $\dim W_{\rm an} = 2$	$2k(n-1) + k^2$
	E/F ramified, $\dim W_{\rm an} = 0$	$kn + \frac{k(k+1)}{2}$
	E/F ramified, $\dim W_{\rm an} = 1$	$k(n-1) + \frac{k(k+1)}{2}$
	E/F ramified, $\dim W_{\rm an} = 2$	$k(n-2) + \frac{k(k+1)}{2}$
Case C”		$kn + \frac{k(k+1)}{2}$
Case D	$\dim W_{\rm an} = 0$	$kn + \frac{k(k-1)}{2}$
	$\dim W_{\rm an} = 2$, χ_W unramified	$kn + \frac{k(k-1)}{2}$
	$\dim W_{\rm an} = 2$, χ_W ramified	$k(n-1) + \frac{k(k-1)}{2}$
	$\dim W_{\rm an} = 4$	$k(n-2) + \frac{k(k-1)}{2}$

		Λ'_Q
Case A	E/F unramified, $\dim V_{\rm an} = 0$	$2km + k^2$
	E/F unramified, $\dim V_{\rm an} = 1$	$2km + k^2$
	E/F unramified, $\dim V_{\rm an} = 2$	$2k(m-1) + k^2$
	E/F ramified, $\dim V_{\rm an} = 0$	$km + \frac{k(k+1)}{2}$
	E/F ramified, $\dim V_{\rm an} = 1$	$k(m-1) + \frac{k(k+1)}{2}$
	E/F ramified, $\dim V_{\rm an} = 2$	$k(m-2) + \frac{k(k+1)}{2}$
Case C”	$\dim V_{\rm an} = 0$	$km + \frac{k(k-1)}{2}$
	$\dim V_{\rm an} = 2$, χ_V unramified	$km + \frac{k(k-1)}{2}$
	$\dim V_{\rm an} = 2$, χ_V ramified	$k(m-1) + \frac{k(k-1)}{2}$
	$\dim V_{\rm an} = 4$	$k(m-2) + \frac{k(k-1)}{2}$
Case D		$km + \frac{k(k+1)}{2}$

Thus we obtain

$$\frac{\gamma(G'^0/M_P^0)}{\gamma(H'^0/M_Q^0)} = \frac{\mathrm{vol}(K_{G'}^0)}{\mathrm{vol}(K_{H'}^0)} \cdot \frac{\mathrm{vol}(K_{M_Q}^0)}{\mathrm{vol}(K_{M_P}^0)} \cdot q^{\Lambda_P - \Lambda_Q - \Lambda'_P + \Lambda'_Q}.$$

This yields the lemma. □

D.3. **Computation (continued).** Assume that ψ is of order zero. As in §B.2, we have an exact sequence

$$1 \longrightarrow \mathrm{Hom}(Y, X)^{*=-} \longrightarrow U_P \longrightarrow \mathrm{Hom}(Y, W) \longrightarrow 1. \tag{D.1}$$

Let $\{w_i\}$ be a basis of $\mathbb{H}^k = X \oplus Y$ over E as in §B.2. Using this basis, we identify $\mathrm{Hom}(Y,X)^{*=-}$ and $\mathrm{Hom}(Y,W)$ with $\mathrm{M}_k(E)^{*=-}$ and W^k respectively. We define a Haar measure du on U_P by

$$du = du'\,du'',$$

where:

- du' is the Haar measure on $\mathrm{M}_k(E)^{*=-}$ defined in §B.2;
- du'' is the self-dual measure on W^k with respect to the pairing $(u'',v'') \mapsto \psi_E(\mathrm{tr}(\langle u'',v''\rangle))$.

Similarly, we have an exact sequence

$$1 \longrightarrow \mathrm{Hom}(X,Y)^{*=-} \longrightarrow \bar{U}_P \longrightarrow \mathrm{Hom}(X,W) \longrightarrow 1.$$

Using the basis $\{w_i\}$, we identify $\mathrm{Hom}(X,Y)^{*=-}$ and $\mathrm{Hom}(X,W)$ with $\mathrm{M}_k(E)^{*=-}$ and W^k respectively. We define a Haar measure $d\bar{u}$ on $\bar{U}_P$ by

$$d\bar{u} = d\bar{u}'\,d\bar{u}'',$$

where $d\bar{u}'$ and $d\bar{u}''$ are the above Haar measures on $\mathrm{M}_k(E)^{*=-}$ and W^k respectively. Since

$$\langle (\bar{u}_{YW} \circ u_{WY})(w_{-i}), w_j\rangle = \langle u_{WY}(w_{-i}), \bar{u}^*_{YW}(w_j)\rangle = -\langle u_{WY}(w_{-i}), \bar{u}_{WX}(w_j)\rangle$$

for $u \in U_P$ and $\bar{u} \in \bar{U}_P$, $d\bar{u}''$ induces the Haar measure on $\mathrm{Hom}(W,Y)$ which is dual to du'' with respect to the pairing

$$\mathrm{Hom}(Y,W) \times \mathrm{Hom}(W,Y) \longrightarrow \mathrm{Hom}(Y,Y) \xrightarrow{\mathrm{tr}} E \xrightarrow{\psi_E} \mathbb{C}^\times.$$

In particular, the Haar measure $du\,d\bar{u}$ on $U_P \times \bar{U}_P$ agrees with that defined in §B.2.

Lemma D.3. *We have*

$$\frac{\mathrm{vol}(U_P \cap K_{G'}) \cdot \mathrm{vol}(\bar{U}_P \cap K_{G'})}{\mathrm{vol}(U_Q \cap K_{H'}) \cdot \mathrm{vol}(\bar{U}_Q \cap K_{H'})} = q^\Lambda,$$

where Λ is the integer given in Lemma D.2.

Proof. Put

$$\mathfrak{a} = \begin{cases} \mathfrak{d}_E^{-1} & \text{if } \dim W_{\mathrm{an}} = 0 \text{ in Case A}, \\ \kappa\mathfrak{d}_E^{-1} & \text{if } \dim W_{\mathrm{an}} = 1 \text{ in Case A}, \\ \mathfrak{d}_E^{-1} & \text{if } \dim W_{\mathrm{an}} = 2 \text{ in Case A}, \\ \mathfrak{o}_F & \text{in Case C''}, \\ \mathfrak{o}_F & \text{if } \dim W_{\mathrm{an}} = 0 \text{ in Case D}, \\ \kappa\mathfrak{o}_F & \text{if } \dim W_{\mathrm{an}} = 2 \text{ in Case D}, \\ \mathfrak{o}_F & \text{if } \dim W_{\mathrm{an}} = 4 \text{ in Case D}. \end{cases}$$

Then (D.1) induces an exact sequence

$$1 \longrightarrow \mathrm{Hom}(Y(\mathfrak{o}_E), X(\mathfrak{a}))^{*=-} \longrightarrow U_P \cap K_{G'} \longrightarrow \mathrm{Hom}(Y(\mathfrak{o}_E), \mathcal{L}_0) \longrightarrow 1.$$

Indeed, for $f \in \mathrm{Hom}(Y(\mathfrak{o}_E), \mathcal{L}_0)$, we have $f^* \in \mathrm{Hom}(\mathcal{L}_0, X(\mathfrak{a}))$ and there exists $g \in \mathrm{Hom}(Y(\mathfrak{o}_E), X(\mathfrak{a}))$ such that $g + g^* + f^* \circ f = 0$. Hence we have

$$\mathrm{vol}(U_P \cap K_{G'}) = \mathrm{vol}(\mathrm{M}_k(\mathfrak{a})^{*=-}) \cdot \mathrm{vol}(\mathcal{L}_0)^k.$$

Similarly, we have

$$\mathrm{vol}(\bar{U}_P \cap K_{G'}) = \mathrm{vol}(\mathrm{M}_k(\mathfrak{a}^{-1})^{*=-}) \cdot \mathrm{vol}(\mathfrak{a}^{-1}\mathcal{L}_0)^k.$$

On the other hand, the values

$$\mathrm{vol}(\mathrm{M}_k(\mathfrak{a})^{*=-}), \quad \mathrm{vol}(\mathrm{M}_k(\mathfrak{a}^{-1})^{*=-}), \quad \mathrm{vol}(\mathcal{L}_0)^k, \quad \mathrm{vol}(\mathfrak{a}^{-1}\mathcal{L}_0)^k$$

are given by the following table.

		$\mathrm{vol}(\mathrm{M}_k(\mathfrak{a})^{*=-})$	$\mathrm{vol}(\mathrm{M}_k(\mathfrak{a}^{-1})^{*=-})$
Case A	$\dim W_{\mathrm{an}} = 0$	$\mathfrak{v}_E^k \cdot q^{\mathfrak{f}_E k(k-1)/4}$	$\mathfrak{v}_E^{-k} \cdot q^{-3\mathfrak{f}_E k(k-1)/4}$
	$\dim W_{\mathrm{an}} = 1$	$\mathfrak{v}_E^k \cdot q^{\mathfrak{f}_E k(k-1)/4} \cdot \lvert\kappa\rvert_F^{k^2}$	$\mathfrak{v}_E^{-k} \cdot q^{-3\mathfrak{f}_E k(k-1)/4} \cdot \lvert\kappa\rvert_F^{-k^2}$
	$\dim W_{\mathrm{an}} = 2$	$\mathfrak{v}_E^k \cdot q^{\mathfrak{f}_E k(k-1)/4}$	$\mathfrak{v}_E^{-k} \cdot q^{-3\mathfrak{f}_E k(k-1)/4}$
Case C”		1	1
Case D	$\dim W_{\mathrm{an}} = 0$	1	1
	$\dim W_{\mathrm{an}} = 2$, χ_W unramified	$\lvert\kappa\rvert_F^{k(k-1)/2}$	$\lvert\kappa\rvert_F^{-k(k-1)/2}$
	$\dim W_{\mathrm{an}} = 2$, χ_W ramified	1	1
	$\dim W_{\mathrm{an}} = 4$	1	1

		$\mathrm{vol}(\mathcal{L}_0)^k$	$\mathrm{vol}(\mathfrak{a}^{-1}\mathcal{L}_0)^k$
Case A	$\dim W_{\mathrm{an}} = 0$	1	$q^{-\mathfrak{f}_E kn}$
	$\dim W_{\mathrm{an}} = 1$	$q^{-\mathfrak{f}_E k/2} \cdot \lvert\kappa\rvert_E^{kn/2}$	$q^{-\mathfrak{f}_E kn - \mathfrak{f}_E k/2} \cdot \lvert\kappa\rvert_E^{-kn/2}$
	$\dim W_{\mathrm{an}} = 2$	q^{-k}	$q^{-\mathfrak{f}_E kn - k}$
Case C”		1	1
Case D	$\dim W_{\mathrm{an}} = 0$	1	1
	$\dim W_{\mathrm{an}} = 2$, χ_W unramified	$\lvert\kappa\rvert_F^{kn/2}$	$\lvert\kappa\rvert_F^{-kn/2}$
	$\dim W_{\mathrm{an}} = 2$, χ_W ramified	$q^{-\mathfrak{f}_W k/2}$	$q^{-\mathfrak{f}_W k/2}$
	$\dim W_{\mathrm{an}} = 4$	q^{-k}	q^{-k}

Here, in Case A, we put $\mathfrak{v}_E = \mathrm{vol}(E_0 \cap \mathfrak{d}_E^{-1}, dx)$, where dx is the self-dual measure on E_0 with respect to the pairing $(x, y) \mapsto \psi(xy)$. Also, since $\mathfrak{d}_E = (\varkappa - \varkappa^c)\mathfrak{o}_E$, where $\{1, \varkappa\}$ is a basis of $\mathfrak{o}_E$ over $\mathfrak{o}_F$, we have

$$E_0 \cap \mathfrak{d}_E = \{x \in E_0 \mid xy \in \mathfrak{o}_F \text{ for all } y \in E_0 \cap \mathfrak{d}_E^{-1}\},$$

so that $\mathrm{vol}(E_0 \cap \mathfrak{d}_E, dx) = \mathfrak{v}_E^{-1}$. Hence we have

$$\mathrm{vol}(U_P \cap K_{G'}) \cdot \mathrm{vol}(\bar{U}_P \cap K_{G'}) = q^{-\lambda_P},$$

where λ_P is an integer given by the following table.

		λ_P
Case A	$\dim W_{\mathrm{an}} = 0$	$\mathfrak{f}_E kn + \mathfrak{f}_E \cdot \frac{k(k-1)}{2}$
	$\dim W_{\mathrm{an}} = 1$	$\mathfrak{f}_E kn + \mathfrak{f}_E \cdot \frac{k(k-1)}{2} + \mathfrak{f}_E k$
	$\dim W_{\mathrm{an}} = 2$	$\mathfrak{f}_E kn + \mathfrak{f}_E \cdot \frac{k(k-1)}{2} + 2k$
Case C”		0
Case D	$\dim W_{\mathrm{an}} = 0$	0
	$\dim W_{\mathrm{an}} = 2$, χ_W unramified	0
	$\dim W_{\mathrm{an}} = 2$, χ_W ramified	$\mathfrak{f}_W k$
	$\dim W_{\mathrm{an}} = 4$	$2k$

Similarly, we have

$$\mathrm{vol}(U_Q \cap K_{H'}) \cdot \mathrm{vol}(\bar{U}_Q \cap K_{H'}) = q^{-\lambda_Q},$$

where λ_Q is an integer given by the following table.

		λ_Q
Case A	$\dim V_{\mathrm{an}} = 0$	$\mathfrak{f}_E km + \mathfrak{f}_E \cdot \frac{k(k-1)}{2}$
	$\dim V_{\mathrm{an}} = 1$	$\mathfrak{f}_E km + \mathfrak{f}_E \cdot \frac{k(k-1)}{2} + \mathfrak{f}_E k$
	$\dim V_{\mathrm{an}} = 2$	$\mathfrak{f}_E km + \mathfrak{f}_E \cdot \frac{k(k-1)}{2} + 2k$
Case C”	$\dim V_{\mathrm{an}} = 0$	0
	$\dim V_{\mathrm{an}} = 2$, χ_V unramified	0
	$\dim V_{\mathrm{an}} = 2$, χ_V ramified	$\mathfrak{f}_V k$
	$\dim V_{\mathrm{an}} = 4$	$2k$
Case D		0

Since

$$-\lambda_P + \lambda_Q = \Lambda,$$

the assertion follows. □

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