

RESTRICTIONS OF REPRESENTATIONS OF CLASSICAL GROUPS: EXAMPLES

WEE TECK GAN, BENEDICT H. GROSS AND DIPENDRA PRASAD

CONTENTS

1. Introduction	1
2. Discrete series parameters	2
3. Depth zero supercuspidals	5
4. Branching laws for $GL_n(\mathbb{F}_q)$	9
5. Branching laws for $U_n(\mathbb{F}_q)$	12
6. Langlands-Vogan packets for small unitary groups	16
7. Theta correspondence	23
8. Endoscopic packets and theta correspondence	26
9. Skew-hermitian case: $U(1) \times U(1)$	32
10. Restriction from $U(2)$ to $U(1)$	33
11. Theta correspondence for $U(2) \times U(2)$	37
12. Trilinear forms for $U(2)$	40
13. Restriction from $U(3)$ to $U(2)$: endoscopic case	46
14. Restriction from $U(3)$ to $U(2)$: stable case	49
References	56

1. INTRODUCTION

This paper is a sequel to [GGP], where we considered several restriction problems in the representation theory of classical groups over local and global fields. Assuming the Langlands-Vogan parameterization of irreducible representations, we formulated precise conjectures for the solutions of these restriction problems. In the local case, our conjectural answer is given in terms of Langlands parameters and certain natural symplectic root numbers associated to them. In the global case, the conjectural answer is expressed in terms of the central critical value or derivative of a global L -function. For the precise statements of the restriction problems and our conjectures, we refer the reader to [GGP].

The conjectures for the case of special orthogonal groups were contained in the earlier papers [GP1] and [GP2] and were suggested by the results of Waldspurger [Wa1,2,3],

Tunnell-Saito [Tu], [Sa], and Prasad [P1, 2, 3] in certain low rank cases. Since then, there have been further results in the orthogonal case, both locally and globally; see, for example [P4], [GR], [GJS2], and [PT]. Most notably, in two recent preprints [Wa4] and [Wa5], Waldspurger has made spectacular progress towards to a full resolution of the local conjectures of [GP1, GP2].

In this paper, we provide some evidence for the conjectures of [GGP] in the unitary case. More precisely, we shall consider the restriction problems in the following cases:

- (i) the depth zero supercuspidal L -packets of DeBacker-Reeder [DR], which are associated to tame regular discrete L -parameters;
- (ii) certain low rank cases, such as $U(1) \times U(1)$, $U(1) \times U(2)$, $U(2) \times U(2)$ and $U(2) \times U(3)$.

In each case, we shall establish [GGP, Conjecture 16.3].

Acknowledgments: W. T. Gan is partially supported by NSF grant DMS-0801071. B. H. Gross is partially supported by NSF grant DMS 0901102. D. Prasad was partially supported by a Clay Math Institute fellowship during the course of this work. We also thank P. Deligne, S. Kudla, G. Lusztig, M. Reeder, D. Rohrlich, and J.-L. Waldspurger for their help.

2. DISCRETE SERIES PARAMETERS

We begin with the computation of the distinguished character in [GGP, Conjecture 16.3]

$$\chi = \chi_N \times \chi_M : A_M \times A_N \rightarrow \langle \pm 1 \rangle,$$

which is defined using local root numbers, for some discrete series parameters of the unitary groups $G = U(W) \times U(W_0)$, associated to the quadratic extension of local fields k/k_0 .

In general, these discrete series parameters have the form

$$\begin{aligned} M &= \bigoplus_i M_i \\ N &= \bigoplus_j N_j \end{aligned}$$

where the M_i are distinct conjugate-symplectic representations and the N_j are distinct conjugate-orthogonal representations of the Weil-Deligne group of k . The dimension of M is even and the dimension of N is odd. In this case, the centralizer $C_M \times C_N$ of the Langlands parameter is finite.

We will only consider the case here where each $M_i = M(\alpha_i)$ and each $N_j = N(\beta_j)$ is one dimensional. Then α_i is a character of $k^\times/\mathbb{N}k^\times$ with $\alpha_i|_{k_0^\times} = \omega_{k/k_0}$, and β_j is a character of k^\times/k_0^\times . In this case, we have the component groups

$$\begin{aligned} A_M &= \bigoplus \mathbb{Z}/2\mathbb{Z} \cdot e_i \\ A_N &= \bigoplus \mathbb{Z}/2\mathbb{Z} \cdot f_j. \end{aligned}$$

These vector spaces have dimension equal to $\dim M$ and $\dim N$ over $\mathbb{Z}/2\mathbb{Z}$, which is as large as possible. We have

$$\begin{aligned} M^{e_i=-1} &= M(\alpha_i) \\ N^{f_j=-1} &= N(\beta_j). \end{aligned}$$

Fix a nontrivial additive character ψ_0 of k which is trivial on k_0 . By the definition of the character χ , we have the formulae

$$\begin{aligned} \chi(e_i) &= \epsilon(M(\alpha_i) \otimes N, \psi_0) \\ \chi(f_j) &= \epsilon(M \otimes N(\beta_j), \psi_0). \end{aligned}$$

Using the additivity of the local epsilon factors, this becomes

$$\begin{aligned} \chi(e_i) &= \prod_k \epsilon(\alpha_i \beta_k, \psi_0) \\ \chi(f_j) &= \prod_k \epsilon(\alpha_k \beta_j, \psi_0). \end{aligned}$$

Since the products $\alpha_i \beta_j$ are all conjugate-symplectic characters of k^\times , we need a formula to compute their root numbers. We will do this in two different cases - when $k/k_0 = \mathbb{C}/\mathbb{R}$, which we take up now, and then when k/k_0 is unramified which we do in the next section.

Proposition 2.1. *Assume that $k_0 = \mathbb{R}$ and choose an isomorphism $z : k \rightarrow \mathbb{C}$. Let*

$$\alpha = z^{-2a} \cdot (z\bar{z})^a = (\bar{z}/z)^a$$

be a conjugate-symplectic character of k^\times , where a is a half integer, and let

$$\psi_0 = e^{2\pi i \text{Tr}(iz)} = e^{2\pi i(\bar{z}-z)}$$

Then

$$\epsilon(\alpha, \psi_0) = \begin{cases} +1 & \text{if } a > 0; \\ -1 & \text{if } a < 0. \end{cases}$$

Proof. Tate [T, (3.2.5)] gives the formula

$$\epsilon(\alpha, \psi) = i^{2a}$$

when $a > 0$ and $\psi = e^{2\pi i \text{Tr}(z)}$. Since $\psi_0(z) = \psi(iz)$, we find

$$\epsilon(\alpha, \psi_0) = i^{2a} \cdot \alpha(i) = +1$$

in this case. When $a < 0$ we must conjugate the isomorphism $z : k \rightarrow \mathbb{C}$ to use Tate's formula. This changes the character ψ_0 , and hence the sign of ϵ . \square

Corollary 2.2. *Assume that $k_0 = \mathbb{R}$, choose an isomorphism $z : k \rightarrow \mathbb{C}$, and let $\psi_0 = e^{2\pi(z-\bar{z})}$. If M is the sum of the distinct symplectic characters $\alpha_i = (\bar{z}/z)^{a_i}$, where each a_i is a half integer, and N is the sum of the distinct orthogonal characters $\beta_j = (\bar{z}/z)^{b_j}$, where each b_j is an integer, then*

$$\begin{aligned}\chi(e_i) &= (-1)^{m_i} \\ \chi(f_j) &= (-1)^{n_j}\end{aligned}$$

where

$$\begin{aligned}m_i &= \#\{r : a_i + b_r < 0\} \\ n_j &= \#\{r : a_r + b_j < 0\}.\end{aligned}$$

Finally, we note that in the case when $k_0 = \mathbb{R}$, we may order the distinct characters α_i and β_j in the parameter φ so that

$$\begin{aligned}a_1 &> a_2 > a_3 \cdots \text{ in } \frac{1}{2}\mathbb{Z} - \mathbb{Z} \\ b_1 &> b_2 > b_3 \cdots \text{ in } \mathbb{Z}.\end{aligned}$$

Corollary 2.3. *For $i < j$, we have*

$$\begin{aligned}\chi(e_i)\chi(e_j) &= (-1)^{m_{ij}} \\ \chi(f_i)\chi(f_j) &= (-1)^{n_{ij}}.\end{aligned}$$

where

$$\begin{aligned}m_{ij} &= \#\{r : a_i + b_r > 0 > a_j + b_r\} \\ n_{ij} &= \#\{r : b_i + a_r > 0 > b_j + a_r\}.\end{aligned}$$

Since we know how to describe the representations in the L -packets of discrete series parameters when $k_0 = \mathbb{R}$ [GR], the calculation of $\chi(e_i)\chi(e_j)$ and $\chi(f_i)\chi(f_j)$ allows us to say something about the representation $\pi = \pi(\varphi, \chi) = \pi_1 \otimes \pi_2$ of $G(\mathbb{R})$ with $d(\pi) = 1$. The irreducible representations π_1 and π_2 are discrete series representations of even and odd dimensional unitary groups, with infinitesimal characters

$$\begin{aligned}a_1 &> a_2 > a_3 > \cdots \\ b_1 &> b_2 > b_3 > \cdots\end{aligned}$$

in $X^* + \rho$ respectively. Moreover, in the chambers defined by their Harish-Chandra parameters, the simple root walls corresponding to

$$\begin{aligned} e_i - e_{i+1} \text{ is compact} &\iff \chi(e_i) \cdot \chi(e_{i+1}) = -1 \\ f_i - f_{i+1} \text{ is compact} &\iff \chi(f_i) \cdot \chi(f_{i+1}) = -1. \end{aligned}$$

More generally, for $i < j$ the positive root

$$\begin{aligned} e_i - e_j \text{ is compact} &\iff \chi(e_i) \cdot \chi(e_j) = (-1)^{i+j} \\ f_i - f_j \text{ is compact} &\iff \chi(f_i) \cdot \chi(f_j) = (-1)^{i+j}. \end{aligned}$$

This determines the signature of the unitary group $G(\mathbb{R})$, and in almost all cases the discrete series representation π .

We end this section with a remark about branching from $U(n, 1)$ to $U(n)$. According to a theorem of Harish-Chandra, an irreducible admissible (\mathfrak{g}, K) -module is determined by the action of $U(\mathfrak{g})^K$, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of G , on a given K -type which appears in the representation space. Further, the action of $K \times U(\mathfrak{g})$ on the corresponding isotypical component is irreducible. By a theorem of Kostant, for

$$G = U(n, 1) \quad \text{and} \quad K = U(n) \times U(1),$$

$U(\mathfrak{g})^K$ is generated by the centers of the universal enveloping algebras of G and K , and thus is, in particular, abelian. This proves that any irreducible representation of $U(n)$ appears with multiplicity at most one in any irreducible representation of $U(n, 1)$.

3. DEPTH ZERO SUPERCUSPIDALS

In this section, we test the restriction conjecture for some tamely ramified discrete parameters φ of unitary groups. We begin by calculating the local root numbers, assuming that k_0 is non-archimedean and k is the unramified quadratic extension of k_0 .

Proposition 3.1. *Assume that k_0 is non-archimedean, and let k be the unramified quadratic field extension of k_0 . Let ψ_0 be an additive character of k which is trivial on both k_0 and the maximal ideal of the ring of integers A_k , but is nontrivial on A_k . Let α be a conjugate-symplectic character of k^\times of conductor $f(\alpha)$. Then*

$$\epsilon(\alpha, \psi_0) = (-1)^{f(\alpha)+1}.$$

Proof. When k/k_0 is unramified, every conjugate-symplectic character α has the form

$$\alpha = \beta \cdot \mu,$$

where $\beta : k^\times/k_0^\times \rightarrow \mathbb{C}^\times$ is a conjugate-orthogonal character and μ is the unramified quadratic character (which is conjugate-symplectic). By [GGP, Section 5] and [FQ, Theorem 3], we have

$$\epsilon(\beta, \psi_0) = +1$$

for any character ψ_0 of k which is trivial on k_0 . Since μ is unramified, we have [T,(3.4.6)]

$$\epsilon(\alpha, \psi_0) = \epsilon(\beta, \psi_0) \cdot \mu(\pi^{f(\beta)+n(\psi_0)}).$$

Since $f(\beta) = f(\alpha)$ and $n(\psi_0) = -1$, this gives the formula in the proposition. \square

Corollary 3.2. *Assume that k_0 is non-archimedean, and let k be the unramified quadratic field extension of k_0 . Let ψ_0 be an additive character of k which is trivial on both k_0 and the maximal ideal of the ring of integers A_k , but is nontrivial on A_k . Let*

$$M = \bigoplus_i \alpha_i \quad \text{and} \quad N = \bigoplus_j \beta_j$$

where the α_i 's are mutually distinct, tamely ramified, conjugate-symplectic characters, and the β_j 's are mutually distinct, tamely ramified, conjugate-orthogonal characters. Order these characters so that

$$\alpha_1 \beta_1 = \alpha_2 \beta_2 = \cdots = \alpha_p \beta_p = \mu,$$

for $p \geq 0$ and no other products $\alpha_i \beta_j = \mu$. Then

$$\chi(e_i) = \begin{cases} -1 & \text{when } i \leq p; \\ +1 & \text{when } i > p. \end{cases}$$

Similarly,

$$\chi(f_j) = \begin{cases} -1 & \text{when } j \leq p; \\ +1 & \text{when } j > p. \end{cases}$$

Finally, $\chi(-1, 1) = \chi(1, -1) = (-1)^p$.

Proof. Since our characters are all tamely ramified, we find

$$f(\alpha_i \beta_j) = 1,$$

unless $i = j \leq p$, in which case the product is equal to the unramified character μ and $f(\alpha_i \beta_i) = 0$. Taking the product of epsilon factors giving χ gives the desired result. \square

We take the parameter

$$\begin{aligned} M &= \bigoplus M(\alpha_i) \\ N &= \bigoplus N(\beta_j) \end{aligned}$$

given by the sum of distinct conjugate-symplectic and distinct conjugate-orthogonal characters of k^\times . We will assume that all of these characters are tamely ramified:

$$f(\alpha_i) = f(\beta_j) = 1.$$

The L -packet Π_φ of depth zero supercuspidal representations of the pure inner forms $G = \mathrm{U}(W) \times \mathrm{U}(W_0)$ has been constructed by DeBacker and Reeder [DR], and we briefly

summarize their results in this case. Let V be a hermitian space of dimension n over k . A parameter φ of the above type for the unitary group $U(V) = U_n$ gives, by restriction to the units of k^\times , a regular complex character ρ of the anisotropic torus $S = (U_1)^n$. Each embedding

$$f : S \rightarrow U(V)$$

corresponds to a decomposition of the space V into the direct sum of orthogonal lines, stable under the action of S . Write this decomposition, choosing a basis for each line, as

$$V = \bigoplus kv_i.$$

The $U(V)$ -conjugacy class of the embedding f depends only on the signs

$$\epsilon_i = (-1)^{\text{ord}\langle v_i, v_i \rangle},$$

which must satisfy the one relation

$$\prod_i \epsilon_i = (-1)^{\text{ord}(\text{disc}V)}.$$

Since the two hermitian spaces V and V' of dimension n have distinct hermitian discriminants, there are exactly 2^n conjugacy classes of embeddings f of S into $U(V)$ and $U(V')$. These conjugacy classes correspond bijectively to the characters $\chi = \chi_f$ of the group A_φ , where $\chi(e_i) = \epsilon_i$.

For each embedding $f : S \rightarrow U(V)$, there is a unique maximal compact subgroup $K_f \subset U(V)$ which contains the image. This is the subgroup stabilizing the lattice,

$$L_f = \bigoplus A_k v_i,$$

where we normalize the basis vectors of our S -stable lines to satisfy $0 \leq \text{ord}\langle v_i, v_i \rangle \leq 1$. The reduction mod π of K_f has reductive quotient

$$\bar{K}_f \cong U_p \times U_{n-p},$$

where p is the number of v_i with $(-1)^{\text{ord}\langle v_i, v_i \rangle} = -1$. Hence K_f will be hyperspecial if and only if all of the inner products $\langle v_i, v_i \rangle$ have valuations of the same parity.

The torus $S(q)$ embeds in $\bar{K}_f(q)$, and the regular tame character ρ of $S(q)$ allows us to construct an irreducible, supercuspidal representation $R_f(S, \rho)$ of the finite group $\bar{K}_f(q)$, using the method of Deligne and Lusztig. We view this as a representation of the compact group K_f , and define the representation

$$\pi_\chi = \pi_f, \quad \text{of } U(V)$$

as the compact induction of $R_f(S, \rho)$. These are the 2^n depth zero supercuspidal representation in the L -packet Π_φ .

The Vogan bijection between the set Π_φ and the group of homomorphisms from A_φ to $\langle \pm 1 \rangle$ is normalized as follows. Assume that the hermitian space V is split.

The character $\chi = 1$ of A_φ gives rise to the non-degenerate hermitian A -lattice L in V , with an orthogonal basis whose inner products are units in A . Let N_L be the unipotent radical of an Iwahori subgroup of the hyperspecial maximal subgroup $K = \text{Aut}(L)$ in $\text{U}(V)$. The construction of [GGP, §12] over the ring A gives a surjective homomorphism

$$f + f_0 : N_L \rightarrow A^{n-1} + A(-)$$

where $A(-)$ is the eigenspace where $\sigma = -1$ on A , which consists of the elements of trace 0 to A_0 .

By [DR2], the character $\chi = 1$ of A_φ corresponds to the unique representation π_1 of in the L -packet of φ which is induced from a generic, cuspidal representation of the reductive quotient $\text{U}_n(q)$ of $\text{U}(L)$. All of the generic characters of the unipotent radical $N(q)$ of a Borel subgroup of $\text{U}_n(q)$ are conjugate, and we construct one of them in the following manner.

Let ψ_0 be an additive character of k which is trivial on k_0 and the maximal ideal P of A , but is nontrivial on A . Since A is unramified over A_0 , we have

$$A(+) + A(-) = A_0 + 2 \cdot A.$$

Hence, for elements z in $A(-)$, the character

$$z \mapsto \psi_0(z/2)$$

is nontrivial on $A(-)/P(-)$. Let ψ be any additive character that is trivial on P but nontrivial on A . Then the composition

$$\psi(\Sigma f(n)) \cdot \psi_0(f_0(n)/2) : N_L \rightarrow \mathbb{S}^1$$

gives a generic character of N_L whose reduction mod P is a generic character of $N(q)$. Hence the representation π_1 corresponding to the trivial character of A_φ is generic for the character obtained by scaling the additive character ψ_0 used to define the root number character in Proposition 3.1 and Corollary 3.2 by the factor $1/2$, or equivalently by the factor 2. This is the normalization predicted in [GGP, Conjecture 16.3].

Now consider the parameter of $G = \text{U}(W) \times \text{U}(W_0) = \text{U}_n \times \text{U}_m$ which is given by

$$\begin{aligned} M &= \bigoplus M(\alpha_i) \\ N &= \bigoplus N(\beta_j) \end{aligned}$$

From the calculation of the character $\chi = \chi_N \times \chi_M$ of A_φ in the previous section, we conclude that the irreducible representation π_χ of $G = \text{U}(W) \times \text{U}(W_0)$ is compactly induced from a maximal compact subgroup with reduction isomorphic to

$$(\text{U}_p \times \text{U}_{n-p}) \times (\text{U}_p \times \text{U}_{m-p})$$

over \mathbb{F}_q . Here $p \geq 0$ is the number of pairs (α_i, β_i) with $\alpha_i \beta_i = \mu$. The finite dimensional representation that we are inducing has the form

$$(R \otimes R(\alpha)) \otimes (R^\vee \otimes R(\beta))$$

where R is the Deligne-Lusztig representation of $U_p(q)$ associated to the character $(\alpha_1, \alpha_2, \dots, \alpha_p)$ and R^\vee is its dual representation, associated to the character $(\beta_1, \beta_2, \dots, \beta_p)$. The remaining representations $R(\alpha)$ of $U_{n-p}(q)$ and $R(\beta)$ of $U_{m-p}(q)$ are associated to characters whose components α_i and β_j satisfy $\alpha_i \beta_j \neq \mu$ for all i, j .

As support for [GGP, Conjecture 16.3], we will prove:

Theorem 3.3. *Let π_χ be the depth zero supercuspidal representation of $G = U(W) \times U(W_0)$ defined above, which corresponds to the distinguished character in [GGP, conjecture 16.3]. Then π_χ possesses a Bessel model, in the sense that*

$$\dim \text{Hom}_H(\pi_\chi, \nu) = 1$$

where (H, ν) is as defined in [GGP, §12].

To prove the existence of a (unique) Bessel model for π_χ , it is sufficient to establish the existence of a Bessel model for the representation

$$R(\alpha) \otimes R(\beta) \quad \text{of } U_{n-p} \times U_{m-p},$$

as there is clearly a unique $U_p \times U_p$ invariant linear form on $(R \otimes R^\vee)$. We will do this in the following two sections, after first studying the situation for general linear groups.

4. BRANCHING LAWS FOR $\text{GL}_n(\mathbb{F}_q)$

In this section, we calculate the restriction of a representation of $\text{GL}_n(\mathbb{F}_q)$ to $\text{GL}_{n-1}(\mathbb{F}_q)$ where $\text{GL}_{n-1}(\mathbb{F}_q)$ sits inside $\text{GL}_n(\mathbb{F}_q)$ in the natural way as

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

These branching laws are surely known in the literature, such as in the work of Thoma [Th]; however, we have preferred to give a different independent treatment.

We begin by recalling the notion of twisted Jacquet functor. Let $P = M \cdot N$ be any group such that N is a normal subgroup of P and let φ be a character of N whose stabilizer in M is denoted by M_φ . The data (N, φ) defines the twisted Jacquet functor from the category of smooth representations of P to the category of smooth representations of M_φ . It associates to a representation V of P the largest quotient $V_{N, \varphi}$ of V on which N operates via the character φ ; clearly $V_{N, \varphi}$ is a representation space for M_φ . The twisted Jacquet functor is exact.

Now let E_{n-1} be the mirabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of matrices whose last row is equal to $(0, 0, \dots, 0, 1)$ and let N_n be the group of upper triangular unipotent matrices in $\mathrm{GL}_n(\mathbb{F}_q)$. We fix a nontrivial character ψ_0 of \mathbb{F}_q and let ψ_n be the character of N_n , given by

$$\psi_n(u) = \psi_0(u_{1,2} + u_{2,3} + \dots + u_{n-1,n}).$$

For a representation π of $\mathrm{GL}_n(\mathbb{F}_q)$, let

$$\pi^i = \text{the } i\text{-th derivative of } \pi,$$

which is a representation of $\mathrm{GL}_{n-i}(\mathbb{F}_q)$. To recall the definition of π^i , if

$$R_{n-i} = \mathrm{GL}_{n-i}(\mathbb{F}_q) \cdot V_i$$

is the subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of matrices

$$\begin{pmatrix} g & v \\ 0 & z \end{pmatrix}$$

with $g \in \mathrm{GL}_{n-i}(\mathbb{F}_q)$, $v \in M(n-i, i)$, $z \in N_i$, and if the character ψ_i of N_i is extended to V_i by extending it trivially across $M(n-i, i)$, then we have

$$\pi^i = \pi_{V_i, \psi_i}.$$

If π is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, then $\pi^i = \pi$ for $i = 0$, and $\pi^n = 1$, the trivial representation of the trivial group $\mathrm{GL}_0(\mathbb{F}_q)$. All the other derivatives of π are 0.

The following proposition is from Bernstein-Zelevinsky [BZ], where it was established for non-archimedean local fields, but their proof works for finite fields as well. It is known as the Leibnitz rule for derivatives.

Proposition 4.1. *For π_1 a representation of $\mathrm{GL}_{n_1}(\mathbb{F}_q)$ and π_2 of $\mathrm{GL}_{n_2}(\mathbb{F}_q)$, we let $\pi_1 \times \pi_2$ denote the representation of $\mathrm{GL}_{n_1+n_2}(\mathbb{F}_q)$ induced from the corresponding representation of the parabolic subgroup with Levi subgroup $\mathrm{GL}_{n_1}(\mathbb{F}_q) \times \mathrm{GL}_{n_2}(\mathbb{F}_q)$. Then there is a composition series of the k -th derivative $(\pi_1 \times \pi_2)^k$ whose successive quotients are $\pi_1^i \times \pi_2^{k-i}$ for $i = 0, \dots, k$.*

Here is a generality from Bernstein and Zelevinsky [BZ].

Proposition 4.2. *Any representation Σ of E_{n-1} has a natural filtration of $E = E_{n-1}$ modules*

$$0 \subset \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \dots \subset \Sigma_n$$

such that

$$\Sigma_{i+1}/\Sigma_i = \mathrm{ind}_{R_i}^E(\Sigma^{n-i} \otimes \psi_{n-i}) \quad \text{for } i = 0, \dots, n-1,$$

where $R_i = \mathrm{GL}_i(\mathbb{F}_q) \cdot V_{n-i}$ is the subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of

$$\begin{pmatrix} g & v \\ 0 & z \end{pmatrix}$$

with $g \in \mathrm{GL}_i(\mathbb{F}_q)$, $v \in M(i, n-i)$, $z \in N_{n-i}$, and the character ψ_{n-i} on N_{n-i} is extended to V_{n-i} by extending it trivially across $M(i, n-i)$.

As a consequence of the above two propositions, we have the following corollary.

Corollary 4.3. *Let $n = n_1 + \cdots + n_r$ be a sum of positive integers, and let π_i be an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{F}_q)$ for $i = 1, \dots, r$. Let*

$$\Pi = \pi_1 \times \cdots \times \pi_r$$

be the corresponding parabolically induced representation of $\mathrm{GL}_n(\mathbb{F}_q)$. Then the restriction of $\pi_1 \times \cdots \times \pi_r$ to $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ is a sum of the following representations:

$$\pi_{i_1} \times \pi_{i_2} \times \cdots \times \pi_{i_s} \times \Sigma[n-1 - (i_1 + \cdots + i_s)]$$

where $1 \leq i_1 < i_2 < \cdots < i_s \leq r$ (the empty sequence is allowed) and $i_1 + \cdots + i_s < n$. Moreover,

$$\Sigma[m] = \mathrm{ind}_{N_m}^{\mathrm{GL}_m(\mathbb{F}_q)} \psi_m$$

denotes the Gelfand-Graev representation of $\mathrm{GL}_m(\mathbb{F}_q)$, with $\Sigma[1]$ equal to the regular representation of \mathbb{F}_q^\times and $\Sigma[0]$ denoting the trivial representation of the trivial group.

Proof. By Proposition 4.2, the restriction of Π to E_{n-1} is the sum of

$$\Pi_{i+1}/\Pi_i = \mathrm{ind}_{R_i}^{E_{n-1}} (\Pi^{n-i} \otimes \psi_{n-i}).$$

Since $\mathrm{GL}_{n-1}(\mathbb{F}_q) \cdot R_i = E_{n-1}$ for any i , it follows that

$$(\Pi_{i+1}/\Pi_i)|_{\mathrm{GL}_{n-1}(\mathbb{F}_q)} = \Pi^{n-i} \times \Sigma[n-1-i],$$

where $\Sigma[n-1-i]$ is the Gelfand-Graev module of $\mathrm{GL}_{n-1-i}(\mathbb{F}_q)$. It only remains to calculate the derivatives Π^{n-i} of Π , but this follows from Proposition 4.1. \square

As a simple consequence of this corollary, we have the following.

Theorem 4.4. *Let $n = n_1 + \cdots + n_r$ be a sum of positive integers, and let π_i be an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{F}_q)$, for $i = 1, \dots, r$. Let $n-1 = m_1 + \cdots + m_s$ be a sum of positive integers, and let μ_i be an irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{F}_q)$. Assume that the representations μ_1, \dots, μ_s are pairwise distinct, so that the corresponding parabolically induced representation $\mu_1 \times \cdots \times \mu_s$ of $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ is irreducible. Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_{n-1}}(\pi_1 \times \cdots \times \pi_r, \mu_1 \times \cdots \times \mu_s)$$

is equal to

$$\prod_{i=1}^s (1 + m_i) \geq 1,$$

where m_i is the multiplicity with which μ_i appears in the set $\{\pi_1 \cdots, \pi_r\}$. In particular, if the π_i 's are mutually distinct as well, then

$$\dim \mathrm{Hom}_{\mathrm{GL}_{n-1}}(\pi_1 \times \cdots \times \pi_r, \mu_1 \times \cdots \times \mu_s) = 2^d$$

where d is the cardinality of the set

$$\{\pi_1, \cdots, \pi_r\} \cap \{\mu_1, \cdots, \mu_s\}.$$

Corollary 4.5. *The restriction of the representation $\pi_1 \times \cdots \times \pi_r$ of $\mathrm{GL}_n(\mathbb{F}_q)$ contains the representation $\mu_1 \times \cdots \times \mu_s$ of $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ (with μ_i 's mutually distinct) with multiplicity one if and only if the sets $\{\pi_1, \cdots, \pi_r\}$ and $\{\mu_1, \cdots, \mu_s\}$ have no common elements.*

5. BRANCHING LAWS FOR $U_n(\mathbb{F}_q)$

In this section, we use the method of base change, also called Shintani descent, to deduce some conclusions about branching laws for the restriction of a representation of $U_n(\mathbb{F}_q)$ to $U(n-1, \mathbb{F}_q)$ from the corresponding results for general linear groups obtained in the previous section. The result is then applied to give a proof of Theorem 3.3.

We make crucial use of the multiplicity 1 theorem for restriction of representations of unitary groups over p -adic fields, which was recently proved by Aizenbud, Gourevitch, Rallis and Schiffmann in [AGRS]. A simple consequence of their result is:

Proposition 5.1. *Let π_1 be an irreducible cuspidal representation of $U_n(\mathbb{F}_q)$ and let*

$$\pi_2 = I_P(\sigma)$$

be an irreducible generalized principal series representation of $U_{n+1}(\mathbb{F}_q)$, where P is a parabolic subgroup of $U_{n+1}(\mathbb{F}_q)$ and σ is a cuspidal representation of a Levi factor of P . We allow the possibility that $P = U_{n+1}$, in which case $\pi_2 = \sigma$ is cuspidal. Then

$$\dim \mathrm{Hom}_{U_n(\mathbb{F}_q)}(\pi_2, \pi_1) \leq 1.$$

Proof. Let k_0 be a local field with \mathbb{F}_q as its residue field and let k be its unramified quadratic extension. Then one can find quasi-split unitary groups $U(W_0)$ and $U(W)$ with $W_0 \subset W$, such that $U(W_0) \times U(W)$ over k_0 contains a hyperspecial maximal compact subgroup $K_0 \times K$ with reductive quotient $U_n(\mathbb{F}_q) \times U_{n+1}(\mathbb{F}_q)$. Moreover, one may find a maximal parabolic subgroup \tilde{P} of $U(W)$, such that $\tilde{P} \cap K$ maps to the parabolic P in the reductive quotient $U_{n+1}(\mathbb{F}_q)$.

Let $\tilde{\pi}_1$ be a depth zero supercuspidal representations of $U(W_0)$ which is obtained from π_1 by compact induction, so that

$$\tilde{\pi}_1 = \mathrm{ind}_{K_0}^{U(W_0)} \pi_1 = \mathrm{Ind}_{K_0}^{U(W_0)} \pi_1.$$

Similarly, let $\tilde{\sigma}$ be a depth zero supercuspidal representation of the Levi factor of \tilde{P} which contains τ as a type. Then we may consider the generalized principal series representation $I_{\tilde{P}}(\tilde{\sigma})$ of $U(W)$ which is irreducible for a generic choice of $\tilde{\sigma}$. Moreover, if K_1 denotes the kernel of the natural projection map

$$K \rightarrow U_{n+1}(\mathbb{F}_q),$$

then one has

$$I_{\tilde{P}}(\tilde{\sigma})^{K_1} = I_P(\sigma).$$

Now by Frobenius reciprocity, we have

$$\begin{aligned} \dim \operatorname{Hom}_{U(W_0)}(I_{\tilde{P}}(\tilde{\sigma}), \tilde{\pi}_1) &= \dim \operatorname{Hom}_{K_0}(I_{\tilde{P}}(\tilde{\sigma}), \pi_1) \\ &= \dim \operatorname{Hom}_{U_n(\mathbb{F}_q)}(I_{\tilde{P}}(\tilde{\sigma})^{K_{0,1}}, \pi_1) \end{aligned}$$

where $K_{0,1}$ is the kernel of the projection map $K_0 \rightarrow U_n(\mathbb{F}_q)$. Since $K_{0,1} \subset K_1$, we have

$$I_{\tilde{P}}(\tilde{\sigma})^{K_{0,1}} \supset I_{\tilde{P}}(\tilde{\sigma})^{K_1} = I_P(\sigma).$$

Thus we conclude that

$$\dim \operatorname{Hom}_{U(W_0)}(I_{\tilde{P}}(\tilde{\sigma}), \tilde{\pi}_1) \geq \dim \operatorname{Hom}_{U_n(\mathbb{F}_q)}(\pi_2, \pi_1).$$

By [AGRS], the LHS is bounded above by 1 for a generic choice of $\tilde{\sigma}$ (so that $I_{\tilde{P}}(\tilde{\sigma})$ is irreducible), and hence so is the RHS. This proves the proposition. \square

A corollary of the above proposition is the uniqueness of Bessel models for cuspidal representations of unitary groups over finite fields.

Proposition 5.2. *Let π_1 be an irreducible cuspidal representation of $U_n(\mathbb{F}_q)$, and let π_2 be an irreducible cuspidal representation of $U_m(\mathbb{F}_q)$ with $n > m$ but $m \not\equiv n \pmod{2}$.*

(i) *Let P be a maximal parabolic subgroup of $U_{n+1}(\mathbb{F}_q)$ with Levi factor $\operatorname{GL}_r(\mathbb{F}_{q^2}) \times U_m(\mathbb{F}_q)$ (so that $m + 2r = n + 1$) and let τ be a cuspidal representation of $\operatorname{GL}_r(\mathbb{F}_{q^2})$. Consider the generalized principal series representation $I_P(\tau \boxtimes \pi_2)$. Then, with the data (H, ν) defined as in [GGP, §12], we have*

$$\operatorname{Hom}_H(\pi_1 \boxtimes \pi_2, \nu) \cong \operatorname{Hom}_{U_n(\mathbb{F}_q)}(I_P(\tau, \pi_2), \pi_1^\vee)$$

(ii) *We have:*

$$\dim \operatorname{Hom}_H(\pi_1 \boxtimes \pi_2, \nu) \leq 1.$$

Proof. (i) This is the finite field analog of [GGP, Theorem 15.1], with the same proof.

(ii) If $n = m + 1$, (ii) is a special case of Proposition 5.1. In the general case when $n > m + 1$, we choose τ in the context of (i) so that the induced representation $I_P(\tau \boxtimes \pi_2)$ is irreducible. Then (ii) follows immediately from (i) and Proposition 5.1. \square

After the above propositions, we shall study the restriction problem from $U_n(\mathbb{F}_q)$ to $U_{n-1}(\mathbb{F}_q)$ using Shintani descent. We begin by giving a brief review of this notion.

Let G be a connected reductive algebraic group over \mathbb{F}_q and let $m \geq 1$ be a fixed integer. The group $G(\mathbb{F}_{q^m})$ comes equipped with its Frobenius automorphism F , whose set of fixed points is $G(\mathbb{F}_q)$. There is a natural map, called the norm mapping,

$$\{F\text{-conjugacy classes in } G(\mathbb{F}_{q^m})\} \longrightarrow \{\text{conjugacy classes in } G(\mathbb{F}_q)\}$$

which is a bijection. The norm mapping thus induces an isomorphism of vector spaces

$$\{\text{class functions on } G(\mathbb{F}_q)\} \longrightarrow \{F\text{-class functions on } G(\mathbb{F}_{q^m})\},$$

which is called the base change map, and whose inverse is called Shintani descent. Furthermore, the base change map is an isometry:

$$\langle \chi_1, \chi_2 \rangle_{G(\mathbb{F}_q)} = \langle \chi'_1, \chi'_2 \rangle_{G(\mathbb{F}_{q^m})},$$

where χ_1 and χ_2 are class functions on $G(\mathbb{F}_q)$ which are Shintani descents of the F -class functions χ'_1 and χ'_2 on $G(\mathbb{F}_{q^m})$. Here, the inner products are normalized so that

$$\langle 1, 1 \rangle_{G(\mathbb{F}_q)} = \langle 1, 1 \rangle_{G(\mathbb{F}_{q^m})} = 1.$$

According to Deligne-Lusztig, given a maximal torus T of G defined over \mathbb{F}_q , and a character

$$\theta : T(\mathbb{F}_q) \rightarrow \mathbb{C}^\times,$$

there is a (virtual) representation of $G(\mathbb{F}_q)$ denoted by $R(T, \theta)$, which is called a Deligne-Lusztig representation. Now given a character θ as above, one has the character

$$\theta_m : T_m = T(\mathbb{F}_{q^m}) \rightarrow \mathbb{C}^\times$$

obtained by composing θ with the norm mapping: $T(\mathbb{F}_{q^m}) \rightarrow T(\mathbb{F}_q)$. Thus one may consider the Deligne-Lusztig representation $R(T_m, \theta_m)$. The following lemma is [DL, 5.16]:

Lemma 5.3. *If G has connected center, then if $R(T, \theta)$ is irreducible, so is $R(T_m, \theta_m)$.*

Henceforth, we assume that G has connected center and $R(T, \theta)$ is irreducible. The irreducible representation $R(T_m, \theta_m)$ is invariant by F and thus can be extended (in two ways) to the semi-direct product $G(\mathbb{F}_{q^m}) \rtimes \langle F \rangle$. For any such extension, the restriction of its character to the coset $G(\mathbb{F}_{q^m}) \cdot F$ is a F -class function, and one may consider its Shintani descent. The following is a basic fact in the theory of Shintani descent:

Proposition 5.4. *There is an extension of $R(T_m, \theta_m)$ whose associated Shintani descent is the representation $R(T, \theta)$.*

Now we can begin our study of the restriction problem for unitary groups over finite fields. Let

$$\begin{cases} \pi_1 = R(T, \theta) \\ \pi_2 = R(T', \theta') \end{cases}$$

be irreducible Deligne-Lusztig representations of $U_n(\mathbb{F}_q)$ and $U_{n-1}(\mathbb{F}_q)$ respectively, and let χ_i be the character of π_i . We shall consider the quadratic base change of π_i . By Proposition 5.4, there are extensions of the representations

$$\begin{cases} R(T_2, \theta_2) \text{ of } \mathrm{GL}_n(\mathbb{F}_{q^2}); \\ R(T'_2, \theta'_2) \text{ of } \mathrm{GL}_{n-1}(\mathbb{F}_{q^2}), \end{cases}$$

whose associated Shintani descents are χ_1 and χ_2 respectively. Fixing such an extension in each case, we denote the corresponding character of this distinguished extension by χ'_i .

It follows that

$$2 \cdot \langle \chi'_1, \chi'_2 \rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2}) \rtimes \langle F \rangle} = \langle \chi'_1, \chi'_2 \rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} + \langle \chi'_1, \chi'_2 \rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2}) \cdot F},$$

and thus

$$2 \cdot \langle \chi'_1, \chi'_2 \rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2}) \rtimes \langle F \rangle} = \langle \chi'_1, \chi'_2 \rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} + \langle \chi_1, \chi_2 \rangle_{U_{n-1}(\mathbb{F}_q)}.$$

Now we observe that:

- (i) the left hand side of this last equality is an *even* integer;
- (ii) the quantity $\langle \chi'_1, \chi'_2 \rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})}$ was computed in Theorem 4.4, under the assumption that $R(T'_2, \theta'_2)$ is an irreducible representation;
- (iii) the quantity $\langle \chi_1, \chi_2 \rangle_{U_{n-1}(\mathbb{F}_q)}$ is equal to 0 or 1 in certain cases, by Proposition 5.1.

Together, these observations allow one to compute $\langle \chi_1, \chi_2 \rangle_{U_{n-1}(\mathbb{F}_q)}$ in certain situations.

Namely, let us assume that π_1 and π_2 are irreducible Deligne-Lusztig representations, and suppose further that π_2 is cuspidal. Then the quadratic base change π'_1 and π'_2 of π_1 and π_2 are irreducible full principal series representations of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ and $\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})$. Thus, Theorem 4.4 implies that

$$\langle \chi'_1, \chi'_2 \rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} = \begin{cases} 1, & \text{if the cuspidal supports of } \pi'_1 \text{ and } \pi'_2 \text{ are disjoint,} \\ \text{an even integer,} & \text{otherwise.} \end{cases}$$

On the other hand, by Proposition 5.1, $\langle \chi_1, \chi_2 \rangle_{U_{n-1}(\mathbb{F}_q)}$ is either 0 or 1. Therefore we get the following theorem as our only option.

Theorem 5.5. *Let π_1 and π_2 be irreducible Deligne-Lusztig representations and suppose that π_2 is cuspidal. Then*

$$\dim \operatorname{Hom}_{\mathbf{U}_{n-1}(\mathbb{F}_q)}(\pi_1, \pi_2) \neq 0$$

if and only if the cuspidal supports of the base change representations π'_1 and π'_2 are disjoint, in which case the Hom space has dimension 1.

In particular, this theorem completes the proof of Theorem 3.3. Indeed, in the setting of Theorem 3.3, we need to show that the distinguished representation $\pi_\chi = \pi_1 \times \pi_2$ of $\mathbf{U}(W) \times \mathbf{U}(W_0)$ satisfies

$$\operatorname{Hom}_H(\pi_\chi, \nu) \neq 0.$$

By the argument in the proof of Proposition 5.1, it is sufficient to show that the representation

$$R(\alpha) \otimes R(\beta) \quad \text{of } \mathbf{U}_{n-p}(\mathbb{F}_q) \times \mathbf{U}_{m-p}(\mathbb{F}_q)$$

satisfies

$$\operatorname{Hom}_{H(\mathbb{F}_q)}(R(\alpha) \otimes R(\beta), \nu) \neq 0.$$

The desired non-vanishing then follows from Proposition 5.2(i) and the above theorem, using the fact that the quadratic base change of $R(\alpha)$ and $R(\beta)$ have disjoint cuspidal support.

6. LANGLANDS-VOGAN PACKETS FOR SMALL UNITARY GROUPS

The rest of this paper is devoted to verifying [GGP, Conjecture 16.3] or its variant [GGP, Conjecture 20.1] in various low rank examples in the unitary and symplectic cases. In this section, we explicate the Langlands-Vogan parameterization of irreducible representations of $\mathbf{U}(V)$ where V is a hermitian (or skew-hermitian) space over k of dimension ≤ 3 .

When $\dim_k V = 1$, the group $\mathbf{U}(V)$ is naturally isomorphic to the subgroup k^\times of norm one elements in k^\times , via its scalar action on V . The map

$$x \mapsto x/x^\sigma$$

gives an isomorphism of k^\times/k_0^\times with $\mathbf{U}(V)$. The only other pure inner form of $\mathbf{U}(V)$ is the group $\mathbf{U}(V')$ where V' is obtained from V by scaling the hermitian form on V by an element in $k_0^\times \setminus \mathbb{N}k^\times$.

In this case, an L -parameter for $\mathbf{U}(V)$ is a 1-dimensional conjugate-orthogonal representation M of $WD(k)$, which corresponds via local class field theory to a character of k^\times/k_0^\times , and hence to characters χ_M of $\mathbf{U}(V)$ and χ'_M of $\mathbf{U}(V')$. The Vogan packet associated to M is then

$$\Pi_M = \{\chi_M, \chi'_M\}.$$

The component group A_M is $\mathbb{Z}/2\mathbb{Z}$ and the trivial character of A_M corresponds to the character χ_M of $\mathbf{U}(V)$.

Now consider the case when $\dim V = 2$. We take V to be the split hermitian space, and denote the other rank 2 hermitian space (which is anisotropic) by V' . In this case, the groups $U(V)$ and $U(V')$ are closely related to the group GL_2 and its inner form D^\times , where D is the unique quaternion division algebra over k_0 .

More precisely, given a quaternion algebra B over k_0 (possibly split), we fix an embedding

$$k \hookrightarrow B$$

of algebras over k_0 and regard B as a 2-dimensional vector space over k via left multiplication. All such embeddings of k into B are conjugate under $\text{Aut}_{k_0}(B)$ by the Skolem-Noether theorem. There is an element $b \in B$ (of trace zero) which normalizes k and whose conjugation action on k is the involution σ ; moreover, all other such elements are of the form $\lambda \cdot b$ for $\lambda \in k$. We thus have a decomposition

$$B = k \cdot 1 + k \cdot b.$$

Define a nondegenerate hermitian form on B by

$$\langle x, y \rangle = \text{projection of } x \cdot \bar{y} \text{ onto } k \cdot 1,$$

where $y \mapsto \bar{y}$ is the canonical involution on B ; let V_B be the associated hermitian space. If B is split, then V_B is the split hermitian space V , whereas if B is the quaternion division algebra D over k_0 , then V_B is the anisotropic hermitian space V' .

The associated unitary similitude group is given by

$$\text{GU}(V_B) \cong (B^\times \times k^\times) / \Delta k_0^\times$$

with an element $(b, t) \in B^\times \times k^\times$ acting on B by

$$(b, t)(x) = txb^{-1}.$$

The similitude character is given by

$$(b, t) \mapsto \text{N}t \cdot \text{N}b^{-1},$$

so that

$$U(V_B) = \{(b, t) \in \text{GU}(V_B) : \text{N}b = \text{N}t\}.$$

Observe that $U(V_B)$ is a subgroup of

$$\text{GU}^+(V_B) = ((B^\times)^+ \times k^\times) / \Delta k_0^\times,$$

where

$$(B^\times)^+ = \{b \in B^\times : \text{N}b \in \text{N}k^\times\}.$$

Indeed, it is easy to see that

$$\text{GU}^+(V_B) = U(V_B) \cdot Z_{\text{GU}(V_B)},$$

where

$$Z_{\text{GU}(V_B)} = (k_0^\times \times k^\times) / \Delta k_0^\times \cong k^\times$$

is the center of $\text{GU}(V_B)$.

For later purposes, we describe here a nondegenerate rank 1 hermitian subspace of V_B . Consider the nondegenerate subspace

$$L_B = k \cdot b \hookrightarrow B$$

and observe that its orthogonal complement $L_B^\perp = k \cdot 1$ is isomorphic to $\langle 1 \rangle$. The pointwise stabilizer of L_B^\perp in $U(B)$ is the diagonal subgroup

$$U(L_B) \cong k^\times/k_0^\times \xrightarrow{\Delta} (B^\times \times k^\times)/\Delta k_0^\times.$$

We now come to the representation theory of $U(V_B)$. Observing that the L -packets of $GU(V_B)$ are all singletons, we take an L -packet of $U(V_B)$ to be the set of irreducible constituents of the restriction of an irreducible representation of $GU(V_B)$ to $U(V_B)$. Since

$$GU^+(V_B) = U(V_B) \cdot Z_{GU(V_B)},$$

we see that the restriction of an irreducible representation of $GU(V_B)$ to $U(V_B)$ is completely determined by its restriction to $GU^+(V_B)$. In other words, from the representation theoretic point of view, we may work with $GU^+(V_B)$ in place of $U(V_B)$.

More precisely, if $\tau \boxtimes \chi$ is an irreducible representation of

$$GU(V_B) = (B^\times \times k^\times)/\Delta k_0^\times,$$

then its restriction to $GU^+(V_B)$ is equal to

$$\tau|_{(B^\times)^+} \boxtimes \chi,$$

and it is known that $\tau|_{(B^\times)^+}$ is either irreducible or is the sum of two inequivalent summands. Moreover, the latter holds if and only if $\tau \otimes \omega_{k/k_0} \cong \tau$, in which case we say that τ is dihedral with respect to k/k_0 . Then the L -packet of $U(V_B)$ associated to τ is the set

$$\Pi_{B,\tau,\chi} = \{(\tau^+ \boxtimes \chi)|_{U(V_B)} : \tau^+ \text{ is an irreducible summand of } \tau|_{(B^\times)^+}\},$$

which has cardinality 1 or 2. Observe that if μ is any character of k_0^\times , then

$$\Pi_{B,\tau \otimes (\mu^{-1} \circ \det), \chi \cdot (\mu \circ N)} = \Pi_{B,\tau,\chi}.$$

If N is the L -parameter of τ , we also write $\Pi_{B,N,\chi}$ for $\Pi_{B,\tau,\chi}$.

To attach L -parameters to these packets, recall that an L -parameter in this case is a two dimensional conjugate-symplectic representation M of $WD(k)$. Now we note:

Proposition 6.1. *(i) Let $\tau \boxtimes \chi$ be an irreducible representation of $GU(V)$, so that $\omega_\tau \cdot \chi|_{k_0^\times} = 1$. If N is the L -parameter of τ , then the representation*

$$M = N|_{WD(k)} \otimes \chi$$

of $WD(k)$ is conjugate-symplectic.

(ii) Conversely, any 2-dimensional conjugate-symplectic representation M of $WD(k)$ arises in this way from an irreducible representation $\tau \boxtimes \chi$ of $\mathrm{GU}(V)$, which is well-defined up to twisting by $(\mu^{-1} \circ \det) \boxtimes \mu \circ \mathbb{N}$ for some character μ of k_0^\times .

Proof. (i) Let

$$(-, -) : N \otimes N \longrightarrow \wedge^2 N = \det N$$

be the natural skew-symmetric $WD(k_0)$ -equivariant form. Also, for $s \in WD(k_0) \setminus WD(k)$, there is a $WD(k)$ -equivariant isomorphism

$$N^s \longrightarrow N$$

given by the action of s^{-1} on N . By composition, we obtain a $WD(k)$ -equivariant bilinear form

$$N \otimes N^s \xrightarrow{1 \otimes s^{-1}} N \otimes N \longrightarrow \det N.$$

Twisting N with χ and N^s with χ^s gives a conjugate duality

$$B : M \otimes M^s \longrightarrow \det N \cdot \chi \cdot \chi^s = 1,$$

where the last equality follows from the fact that

$$\det N \cdot \chi|_{k_0^\times} = 1 \quad \text{on } \mathbb{N}k^\times.$$

To see that this conjugate duality has sign -1 , we write ρ_M for the action of $WD(k)$ on M and ρ_N for the action of $WD(k_0)$ on N and compute:

$$\begin{aligned} & B(n, \rho_M(s^2)m) \\ &= \chi(s^2) \cdot (n, \rho_N(s)^{-1} \cdot \rho_N(s)^2 m) \\ &= \chi(s^2) \cdot (n, \rho_N(s)m) \\ &= -\chi(s^2) \cdot (\rho_N(s)m, n) \\ &= -\chi(s^2) \cdot \det N(s) \cdot (m, \rho_N(s)^{-1}n) \\ &= -\chi(s^2) \cdot \det N(s) \cdot B(m, n). \end{aligned}$$

But if $t \in k_0^\times \setminus \mathbb{N}k^\times$, then

$$\chi(s^2) \cdot \det N(s) = \chi|_{k_0^\times}(t) \cdot \omega_\tau(t) = 1.$$

This proves (i).

(ii) Conversely, if M is conjugate-symplectic, then $\det M$ is conjugate-orthogonal and thus has the form χ/χ^σ for some character χ of $WD(k)$. Moreover, such a χ is well-determined up to a character of the form $\mu \circ \mathbb{N}$. The representation $M \otimes \chi^{-1}$ is then σ -invariant and hence is the restriction to $WD(k)$ of a representation N of $WD(k_0)$. Such an N is not unique, as one can choose to twist any irreducible summand of N by ω_{k/k_0} . In any case, we have

$$M = N|_{WD(k)} \otimes \chi,$$

and

$$\det N \cdot \chi|_{k_0^\times} = 1 \quad \text{or} \quad \omega_{k/k_0}.$$

We need to show that N can be chosen so that the first possibility holds, and that this choice of N is unique up to twisting by ω_{k/k_0} (for χ fixed). For this, we consider various cases:

- (a) if N is reducible, then by twisting one of its irreducible summand (which is 1-dimensional) by ω_{k/k_0} if necessary, we can ensure that $\det N \cdot \chi|_{k_0^\times} = 1$. For a fixed choice of χ , the only other N for which this holds is $N \otimes \omega_{k/k_0}$.
- (b) if M is irreducible, then Schur's lemma implies that any two conjugate dualities of M are multiples of each other and thus must have sign -1 in our setting. On the other hand, the construction in the proof of (i) gives a conjugate duality on M which has sign

$$\begin{cases} +1, & \text{if } \det N \cdot \chi|_{k_0^\times} = \omega_{k/k_0}; \\ -1, & \text{if } \det N \cdot \chi|_{k_0^\times} = 1. \end{cases}$$

Thus, $\det N \cdot \chi|_{k_0^\times} = 1$ in this case.

- (c) if M is reducible but N is irreducible, then we have

$$M = M_1 + M_2$$

with M_1 and M_2 distinct conjugate-symplectic, and

$$M_1 \cdot M_2 = \chi/\chi^\sigma.$$

Moreover,

$$N = \text{ind}_{WD(k)}^{WD(k_0)} M_1 \chi^{-1}$$

so that

$$\det N = \chi^{-1}|_{k_0^\times}.$$

In particular, $\det N \cdot \chi|_{k_0^\times} = 1$ in this case as well.

Hence (ii) is proved. □

In view of the above proposition, we set the L -parameter associated to the packet $\Pi_{B,\tau,\chi}$ to be the conjugate-symplectic representation

$$M = N|_{WD(k)} \otimes \chi,$$

with N the L -parameter of τ . Given a conjugate-symplectic M , with associated pair (τ, χ) as in Proposition 6.1(ii), the associated Vogan packet is

$$\Pi_M = \bigcup_B \Pi_{B,N,\chi},$$

where the union is taken over the two quaternion algebras over k_0 .

Remark: It has been shown by Konno-Konno [KK] that the above construction of L -parameters agrees with the one supplied by the theory of twisted endoscopy (i.e. base change to $\mathrm{GL}(2)$ over k), which has been achieved by Rogawski [Ro] using the stable trace formula.

The following table lists the various possibilities of M , Π_M and the component group A_M , depending on the type of τ 's.

τ	M	Π_M	A_M
non-dihedral principal series (with respect to k/k_0)	$P + {}^\sigma P^\vee$, $P \not\cong {}^\sigma P^\vee$	1 representation on $U(V)$	trivial
non-dihedral discrete series (with respect to k/k_0)	irreducible conjugate-symplectic	1 representation on $U(V)$ and 1 on $U(V')$	$\mathbb{Z}/2\mathbb{Z}$
dihedral principal series (with respect to k/k_0)	$2 \cdot M'$, M' conjugate-symplectic	2 representations on $U(V)$	$\mathbb{Z}/2\mathbb{Z}$
dihedral discrete series (with respect to k/k_0)	$M_1 + M_2$, $M_1 \not\cong M_2$ conjugate-symplectic	2 representations on $U(V)$ and 2 on $U(V')$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

If the conjugate-symplectic representation M is of the last two types in the above table, we shall call M dihedral with respect to k/k_0 . If it is of the first two type, we shall call it non-dihedral with respect to k/k_0 .

From the above table, we see that $\#\Pi_M = \#A_M = \#\mathrm{Irr}(A_M)$. To index the representations in Π_M by $\mathrm{Irr}(A_M)$, we need to fix a generic character of $U(V)$. According to [GGP, Prop. 12.1(2)], a generic character of $U(V)$ is specified by giving a nontrivial additive character

$$\psi : k/k_0 \rightarrow \mathbb{S}^1.$$

Via the description of $\mathrm{GU}(V)$ in terms of $\mathrm{GL}_2(k_0)$ given above, this then corresponds to a generic character of $\mathrm{GL}_2(k_0)$, which is given by an additive character

$$\psi^0 : k_0 \rightarrow \mathbb{S}^1.$$

To describe the precise relation between ψ and ψ^0 , we need to start with an explicit embedding $k \rightarrow M_2(k_0)$. We do this in a standard way, by choosing a trace zero element e of k , which gives

$$k = k_0 \cdot 1 + k_0 \cdot e \quad \text{and} \quad \mathrm{End}_{k_0}(k) \cong M_2(k_0).$$

The multiplication action of k on itself then gives an embedding $k \hookrightarrow M_2(k_0)$. Moreover, the action of σ on k gives rise to an element b in $\text{End}_{k_0}(k)$, so that

$$M_2(k_0) = k + k \cdot b.$$

After some calculations with these explicit data, which we will omit here, one sees that ψ and ψ^0 are related by

$$\psi(x) = \psi^0(\text{Tr}(e^{-1} \cdot x)) \quad \text{for all } x \in k.$$

We stress again that the above relation depends crucially on the choice of the trace zero element e , though we will not need to make use of this relation in this paper.

In any case, with $\psi : k/k_0 \rightarrow \mathbb{S}^1$ and hence $\psi^0 : k_0 \rightarrow \mathbb{S}^1$ fixed, we note that $\tau|_{\text{GL}_2(k_0)^+}$ has a unique ψ^0 -generic constituent and hence the Vogan packet Π_M has a unique ψ -generic element. We then decree that

- (i) the trivial character of A_M corresponds to ψ -generic element in Π_M ;
- (ii) a character of A_M corresponds to a representation of $\text{U}(V)$ if and only if it is trivial on the image of the central element $-1 \in {}^L\text{U}(V)$.

From the above table, we see that these requirements completely determine the bijection

$$\Pi_M \leftrightarrow \text{Irr}(A_M),$$

except in the last case, where τ is a dihedral (with respect to k/k_0) discrete series representation of $\text{GL}_2(k_0)$. In that case, if τ' denotes the Jacquet-Langlands lift of τ to D^\times , then we do not know how to label the two summands of $\tau'|_{(D^\times)^+}$ using the two characters of A_M which are nontrivial on the central -1 . However, in §8, we shall resolve this issue when we describe an alternative construction of these Vogan packets using theta correspondence.

Finally, we consider the case when $\dim V = 3$. In this case, the only other pure inner form of $\text{U}(V)$ is the group $\text{U}(V')$ where V' is the hermitian space obtained from V via scaling by an element of $k_0^\times \setminus \mathbb{N}k^\times$. In this case, the Vogan packets have been defined by Rogawski [Ro] via base change to $\text{GL}(3)$ over k using the stable trace formula.

The L -parameters are conjugate-orthogonal representations M of $WD(k)$ of dimension 3. When M is irreducible, the associated Vogan packet is said to be stable; it consists of a representation of $\text{U}(V)$ and the same representation regarded as a representation of $\text{U}(V')$. The component group A_M is $\mathbb{Z}/2\mathbb{Z}$ and we decree that the trivial character corresponds to the representation of $\text{U}(V)$. On the other hand, when M is reducible, the associated Vogan packet is said to be endoscopic. In §8, we shall describe a construction of the endoscopic packets, and the labelling of their representations by $\text{Irr}(A_M)$, via the approach of theta correspondence.

7. THETA CORRESPONDENCE

The goal of this section is to review the necessary background and framework for the theta correspondence for unitary groups. This is necessary for the construction of endoscopic Vogan packets of $U(2)$ and $U(3)$ which will be given in the following section.

Let V be a hermitian space and W a skew-hermitian space over k . To consider the theta correspondence for the dual pair $U(V) \times U(W)$, one requires certain additional data:

- (i) an additive character $\psi_0 : k_0 \rightarrow \mathbb{S}^1$;
- (ii) a character $\mu : k^\times \rightarrow \mathbb{C}^\times$ such that $\mu|_{k_0^\times} = \omega_{k/k_0}$.

To elaborate, the tensor product $\text{Res}_{k/k_0}(V \otimes_k W)$ has a natural symplectic form defined by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \text{Tr}_{k/k_0}(\langle v_1, v_2 \rangle_V \cdot \langle w_1, w_2 \rangle_W).$$

Note that many authors (for example [HKS]) include a factor $1/2$ on the right hand side, but we shall not follow this convention here. In any case, there is a natural map

$$i : U(V) \times U(W) \longrightarrow \text{Sp}(V \otimes W/k_0).$$

One has the metaplectic \mathbb{S}^1 -cover $\text{Mp}(V \otimes W)$ of $\text{Sp}(V \otimes W)$, and the character ψ_0 (together with the form $\langle -, - \rangle$ on $V \otimes W$) determines a Weil representation ω_{ψ_0} of $\text{Mp}(V \otimes W)$. To obtain a representation of $U(V) \times U(W)$ from ω_{ψ_0} , however, one needs to specify a splitting of the map i to the metaplectic cover. This is quite subtle, but was completely understood by Gelbart-Rogawski [GRO], Kudla [K] and Harris-Kudla-Sweet [HKS]; it requires the additional data above.

More precisely, the data (V, ψ_0, μ) determines a splitting

$$i_{V, \mu, \psi_0} : U(W) \hookrightarrow \text{Mp}(V \otimes W),$$

whereas the data (W, ψ_0, μ) determines a splitting

$$i_{W, \mu, \psi_0} : U(V) \hookrightarrow \text{Mp}(V \otimes W)$$

whose image commutes with that of i_{V, μ, ψ_0} . In [HKS], the above splittings can be defined for the choice of any pair of characters (χ, χ') of k^\times satisfying

$$\chi|_{k_0^\times} = \omega_{k/k_0}^{\dim V} \quad \text{and} \quad \chi'|_{k_0^\times} = \omega_{k/k_0}^{\dim W}.$$

In their terminology, our splittings are relative to the pair of characters

$$\chi = \mu^{\dim V} \quad \text{and} \quad \chi' = \mu^{\dim W}.$$

In particular, by [HKS, Corollary A.8], a property of this splitting is that the images of the centers of $U(V)$ and $U(W)$ are identified, so that the theta correspondence we consider here preserves central characters.

Using the above splittings, one obtains a Weil representation

$$\omega_{\psi_0, \mu} = \omega_{\psi_0} \circ (i_{W, \mu, \psi_0, \delta} \times i_{V, \mu, \psi_0})$$

of $U(V) \times U(W)$. Moreover, the Weil representation $\omega_{\psi_0, \mu}$ depends only on the orbit of ψ_0 under $\mathbb{N}k^\times$. Thus, given an irreducible representation π of $U(W)$, we have its big and small theta lift $\Theta_{\psi_0, \mu}(\pi)$ and $\theta_{\psi_0, \mu}(\pi)$ on $U(V)$.

It would appear that, by restricting (χ_1, χ_2) (as in [HKS]) to have the special form taken here, we are losing one degree of freedom. However, this lost degree of freedom can be regained by allowing twisting of the theta lifts by 1-dimensional characters of $U(V)$, i.e. if we consider $\theta_{\psi_0, \mu}(\pi) \otimes (\chi \circ \det)$ as well.

It is also useful to consider the theta correspondence for similitude groups. Let

$$R \subset \mathrm{GU}(V) \times \mathrm{GU}(W)$$

be the subgroup consisting of elements (g, h) such that $\mathrm{sim}(g) \cdot \mathrm{sim}(h) = 1$. Then the Weil representation $\omega_{\psi_0, \mu}$ has a natural extension to R . Now observe that

$$R \subset \mathrm{GU}^+(V) \times \mathrm{GU}^+(W)$$

where $\mathrm{GU}^+(V)$ consists of those elements $g \in \mathrm{GU}(V)$ such that $\mathrm{sim}(g)$ lies in the image of the similitude map of $\mathrm{GU}(W)$, and analogously for $\mathrm{GU}^+(W)$. Then one may consider the induced representation

$$\Omega_{\psi_0, \mu} = \mathrm{ind}_R^{\mathrm{GU}^+(V) \times \mathrm{GU}^+(W)} \omega_{\psi_0, \mu}$$

of $\mathrm{GU}^+(V) \times \mathrm{GU}^+(W)$, which depends only on the orbit of ψ_0 under $\mathbb{N}k^\times$ (and may even be independent of ψ_0 in some cases). We can now consider the theta correspondence for $\mathrm{GU}^+(V) \times \mathrm{GU}^+(W)$ associated to $\Omega_{\psi_0, \mu}$. In particular, for a representation π of $\mathrm{GU}^+(W)$, we have its big and small theta lifts $\Theta_{\psi, \mu}(\pi)$ and $\theta_{\psi, \mu}(\pi)$ on $\mathrm{GU}^+(V)$.

In this paper, we will be considering the theta correspondence for $U(V) \times U(W)$ with $|\dim V - \dim W| \leq 1$. In this case, there are some rather precise conjectures about the behavior of the theta correspondence in the literature (see for example [HKS, §7] and [P5]). We formulate these as the following working hypothesis.

Working hypothesis: Let V be a hermitian space and let W be a skew-hermitian space, and consider the theta correspondence for $U(V) \times U(W)$ relative to the data (ψ_0, μ) . For an irreducible representation π of $U(V)$, let $\theta_{\psi_0, \mu}(\pi)$ denote the (small) theta lift of π to $U(W)$.

- (a) If $\dim V = \dim W$, then the Langlands parameters of π and $\theta_{\psi_0, \mu}(\pi)$ are the same (if the latter is nonzero). For a given L -parameter M , the theta correspondence induces a permutation of the Vogan packet Π_M to itself. This bijection is given by translation by a character of the component group A_M , as given in [P5].

- (b) If $\dim V = \dim W - 1$, then the Langlands parameters M of π and N of $\theta_{\psi_0, \mu}(\pi)$ are related to each other by:

$$N = \mu^{-1}M + \mu^{\dim V}.$$

The theta correspondence relative to (ψ_0, μ) gives an injection

$$\theta_{\psi_0, \mu, V, W} : \Pi_{V, M} \hookrightarrow \Pi_{W, N}.$$

This injection can be naturally described in terms of the characters of the component groups of M and N as follows. Assume for simplicity that $\mu^{\dim V}$ does not occur in $\mu^{-1}M$, so that $A_N = \mathbb{Z}/2\mathbb{Z} \times A_M$. For an appropriately normalized Langlands-Vogan parametrization, the above injection is described by the natural map

$$\text{Irr}(A_M) \longrightarrow \text{Irr}(A_N) = \{\pm 1\} \times \text{Irr}(A_M)$$

given by

$$\rho \mapsto (\epsilon, \rho)$$

where the sign ϵ is completely determined by ρ and the space W .

Moreover, as V and W vary over all hermitian and skew-hermitian spaces of the specified dimensions, one has

$$\Pi_N = \bigcup_{V, W} \theta_{\psi_0, \mu, V, W}(\Pi_{V, M}),$$

where the union is disjoint and we ignore the theta lifts which are zero. The disjointness of the union is in fact a consequence of the main result of [HKS] on theta dichotomy.

The above working hypothesis can be made more precise, especially its relation with the Langlands-Vogan parameterization. In the following, we shall consider the low rank cases, with $\dim V \leq 2$ and $\dim W \leq 3$. In these cases, we shall verify the above working hypothesis in its precise form. We note that these low rank cases are the only ones in which the Langlands-Vogan parameterization is fully understood for $U(V)$ and $U(W)$.

For example, statement (a) for $\dim V = 1$ is a result of Moen [Mo], Rogawski [Ro2] and Harris-Kudla-Sweet [HKS] (see Theorem 9.1 below), whereas the case when $\dim V = 2$ is verified in Theorem 11.2 below. On the other hand, statement (b) for $\dim V = 1$ is easy to check, and the case of $\dim V = 2$ is due to Gelbart-Rogawski-Soudry [GRS].

8. ENDOSCOPIC PACKETS AND THETA CORRESPONDENCE

The goal of this section is to describe an alternative construction of the endoscopic packets of the unitary group $U(V)$, via theta correspondence, when $\dim V = 2$ or 3 . We shall rely heavily on the framework and notation of the previous two sections.

Our first case of interest is the theta correspondence for a skew-hermitian space W and a hermitian space V with

$$\dim W = 1 \quad \text{and} \quad \dim V = 2.$$

We shall use the associated theta correspondence to construct certain Vogan packets on $U(V)$. We shall fix once and for all a trace zero element $\delta \in k^\times$, so that

$$\delta^\sigma = -\delta.$$

Recall that in §6, we have given a construction of the rank 2 hermitian spaces V_B in terms of quaternion algebras B over k_0 . Suppose that

$$M = M_1 + M_2$$

is a 2-dimensional conjugate-symplectic representation of $WD(k)$, with M_i conjugate-symplectic (but not necessarily distinct). As we explained in the previous section, such an M gives rise to a Vogan packet Π_M of $U(V_B)$. If we fix an additive character

$$\psi : k/k_0 \longrightarrow \mathbb{S}^1$$

then there should be an associated bijection

$$J(\psi) : \Pi_M \longleftrightarrow \text{Irr}(A_M).$$

It is the Vogan packet Π_M , together with the bijection $J(\psi)$, that we would like to construct using theta correspondence. In fact, since the Vogan packets on $U(V_B)$ are defined by restriction from $\text{GU}(V_B)$, it will be better to consider the theta correspondence for the similitude groups $\text{GU}(W) \times \text{GU}^+(V_B)$, with

$$\text{GU}(W) \cong k^\times \quad \text{and} \quad \text{GU}^+(V_B) = ((B^\times)^+ \times k^\times)/k_0^\times.$$

To set up the theta correspondence, we need to fix the data ψ_0 and μ . Since we have fixed the trace zero element $\delta \in k$, there is a unique additive character ψ_0 of k_0 such that

$$\psi(x) = \psi_0\left(\frac{1}{2} \cdot \text{Tr}(\delta x)\right).$$

It follows that for any trace zero element $x \in k$,

$$\psi(x) = \psi_0(\delta x).$$

In the rest of the paper, we shall assume that ψ and ψ_0 are related as above, via the element δ .

We set

$W =$ the rank 1 skew-hermitian space with discriminant δ ,

and let W' be the other rank 1 skew-hermitian space. For any $a \in k_0^\times$, we let W_a denote the rank 1 skew-hermitian space obtained from W by scaling by a . Finally, with $M = M_1 + M_2$ as above, we set

$$\mu = M_1,$$

and let χ be any character of k^\times such that

$$\chi/\chi^\sigma = M_1 \cdot M_2.$$

This is possible since $M_1 \cdot M_2$ is a character of k^\times/k_0^\times . The choice of χ is not unique but any two choices differ by a character of k^\times which is σ -invariant, or equivalently that factors through the norm map to k_0^\times . In any case, we have

$$M = \mu + \chi/\chi^\sigma \cdot \mu^{-1},$$

and the packet Π_M is obtained by the restriction of $\tau \boxtimes \chi$, where τ is the representation of B^\times with L -parameter

$$N = \text{Ind}_{WD(k)}^{WD(k_0)} \mu \chi^{-1}.$$

Now we may consider the theta correspondence associated to the Weil representation $\Omega_{\psi_0, \mu}$ of $\text{GU}(W_a) \times \text{GU}^+(V_B)$. Regarding χ as a character of $\text{GU}(W_a)$, we have the theta lift

$$\Theta_{W_a, V_B, \psi_0, \mu}(\chi) = \theta_{W_a, V_B, \psi_0, \mu}(\chi)$$

on $\text{GU}^+(V_B)$. With $B^\times = \text{GL}_2(k_0)$, the character ψ determines a generic character of $\text{GU}^+(V_B)$. We let τ^+ be the constituent of $\tau|_{\text{GL}_2(k_0)^+}$ such that the representation $\tau^+ \boxtimes \chi$ of $\text{GU}^+(V_B)$ is ψ -generic, and let τ^- denote the other constituent. We also let τ' be the Jacquet-Langlands lift of τ to D^\times , if it exists.

With these notations, we have:

Proposition 8.1. *If B is split, so that $V_B = V$, then*

$$\begin{cases} \theta_{\psi_0, \mu, W, V}(\chi) = \tau^+ \boxtimes \chi, \\ \theta_{\psi_0, \mu, W', V}(\chi) = \tau^- \boxtimes \chi. \end{cases}$$

If B is non-split, so that $V_B = V'$, then

$$\theta_{\psi_0, \mu, W, V'}(\chi) + \theta_{\psi_0, \mu, W', V'}(\chi) = \tau' \boxtimes \chi,$$

where the RHS is interpreted as 0 if τ' does not exist. In particular, upon restriction to $\text{U}(V)$ or $\text{U}(V')$, the set

$$\{\theta_{\psi_0, \mu, V, W}(\chi), \theta_{\psi_0, \mu, V, W'}(\chi), \theta_{\psi_0, \mu, V, W'}(\chi), \theta_{\psi_0, \mu, V, W'}(\chi)\}$$

is the Vogan packet Π_M associated to the L -parameter

$$M = M_1 + M_2 = \mu + \mu^{-1} \chi / \chi^\sigma.$$

Using the above construction of endoscopic packets of $U(V)$, we can define the bijection

$$J(\psi) : \Pi_M \longleftrightarrow \text{Irr}(A_M),$$

as follows. Consider the case when $M_1 \neq M_2$, so that $A_M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; this is the only case where the bijection $\Pi_M \leftrightarrow \text{Irr}(A_M)$ has some ambiguity. We set

$$\begin{cases} \pi^{++} = \theta_{\psi_0, \mu, V, W}(\chi) \\ \pi^{--} = \theta_{\psi_0, \mu, V, W'}(\chi) \\ \pi^{+-} = \theta_{\psi_0, \mu, V', W'}(\chi) \\ \pi^{-+} = \theta_{\psi_0, \mu, V', W}(\chi). \end{cases}$$

In other words, the recipe for labelling is that

$$\pi^{\epsilon_1, \epsilon_2} = \theta_{\psi, \mu, W_a, V_B}(\chi)$$

where

$$\epsilon_1 \cdot \epsilon_2 = \epsilon(B) = \begin{cases} 1 & \text{if } B \text{ is split;} \\ -1, & \text{if } B \text{ is not split,} \end{cases}$$

and

$$\epsilon_2 = \omega_{k/k_0}(a).$$

Equivalently, if η is a character of A_M , then

$$\pi_\eta = \theta_{\psi, \mu, W_a, V_B}(\chi)$$

if and only if

$$\begin{cases} \eta(a_1) = \epsilon(B) \cdot \omega_{k/k_0}(a), \\ \eta(a_2) = \omega_{k/k_0}(a). \end{cases}$$

We leave it to the reader to verify that under this system of bijections $J(\psi)$, the various desiderata of the Vogan parameterization listed in [GGP, §9 and §10] are satisfied. In particular, the trivial character of A_M corresponds to the unique ψ -generic representation of the packet, and if ψ' belongs to the other $\mathbb{N}k^\times$ -orbit, then the unique ψ' -generic representation corresponds to the character

$$\eta_0(a_i) = (-1)^{\dim M_i}.$$

Indeed, when M is irreducible, η_0 is trivial, whereas when $M = M_1 + M_2$ is reducible, then η_0 is the character $(--)$ of $A_M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

It will be useful to convert the above classification into the setting of rank 2 skew-hermitian spaces. Using the trace zero element δ , let $W_{B, \delta}$ be the skew-hermitian space obtained from V_B by scaling by δ , and we shall frequently write W_B for $W_{B, \delta}$. Then we have

$$\text{GU}(W_B) = \text{GU}(V_B)$$

as subsets of $\text{End}_k(B)$. Moreover, the notions of L -parameters and L -packets are the same for $\text{U}(V_B)$ and $\text{U}(W_{B,\delta})$. The only difference lies in the data needed to specify a bijection of a Vogan packet with the set of characters of the component group. In the case of V_B , we used an additive character

$$\psi : k/k_0 \longrightarrow \mathbb{S}^1,$$

whereas for the case of $W_{B,\delta}$, one needs an additive character of k_0 . However, it is easy to check that if a representation π of $\text{U}(V_B)$ is generic with respect to ψ , then regarded as a representation of $\text{U}(W_{B,\delta})$, π is generic with respect to the character given by

$$\psi_{00}(y) = \psi(\delta^{-1}y), \quad \text{for } y \in k_0,$$

or equivalently

$$\psi(x) = \psi_{00}(\delta x) = \psi_{00}\left(\frac{1}{2}\text{Tr}(\delta x)\right) \quad \text{for trace zero } x \in k.$$

In other words, the character ψ_{00} is precisely the character ψ_0 which we have fixed, and the bijection

$$J(\psi) : \Pi_M \longleftrightarrow \text{Irr}(A_M)$$

for $\text{U}(V_B)$ is the bijection $J(\psi_0)$ for $\text{U}(W_{B,\delta})$. For a character η of A_M , we then have

$$\pi_\eta = \theta_{\psi_0, \mu, W_{B,\delta}, V_a}(\chi)$$

where μ and χ are obtained from M as before, V_a is the rank 1 hermitian space with discriminant a , and

$$\begin{cases} \eta(a_1) = \epsilon(B) \cdot \omega_{k/k_0}(a) \\ \eta(a_2) = \omega_{k/k_0}(a). \end{cases}$$

Finally, we consider the endoscopic Vogan packets of $\text{U}(V)$ when $\dim V = 3$. Hence, we fix a rank 3 hermitian space V and let V' denote the other rank 3 hermitian space. More generally, for any $a \in k_0^\times$, we let V_a denote the hermitian space obtained from V by scaling the hermitian form by a .

Consider L -parameters of $\text{U}(V)$ of the form

$$M = M_1 + M_2$$

where M_i are conjugate-orthogonal representations of $WD(k)$ of dimension i . Unless $M \cong 3 \cdot M_1$, we may further assume that M_1 is distinct from any irreducible constituent of M_2 . It was shown in [GRS] that the Vogan packet Π_M can be constructed using theta correspondence from $\text{U}(W_B)$, where W_B is the rank 2 skew-hermitian space introduced above, together with twisting by 1-dimensional characters of $\text{U}(V)$.

To specify the data needed for theta correspondence, note that M_1 is a character of k^\times/k_0^\times and thus we can hope to find a conjugate-symplectic character μ of k^\times such

that

$$M_1 = \mu^2.$$

This cannot always be achieved, but if we allow ourselves to replace M by a twist, then it can certainly be done. Indeed, one would simply pick a conjugate-symplectic μ and twist M by $\mu^2 M_1^{-1}$. Since the Langlands-Vogan parametrization is compatible with twisting, there is no loss of generality in assuming that M_1 has a square root μ which is conjugate-symplectic.

Now set

$$N = M_2 \cdot \mu$$

so that N is conjugate-symplectic and is an L -parameter for $U(W_B)$, and

$$M = \mu^2 + N \cdot \mu^{-1}.$$

For the additive character ψ_0 of k_0 , we can now consider the theta correspondence associated to the Weil representation $\Omega_{\psi_0, \mu, W_B, V_a}$.

More precisely, let Π_N be the Vogan packet associated to N , together with the bijection

$$J(\psi_0) : \Pi_N \longleftrightarrow \text{Irr}(A_N)$$

associated to the additive character ψ_0 . Then for $\eta \in \text{Irr}(A_N)$, we may consider the theta lift

$$\theta_{\psi_0, \mu, W_B, V_a}(\pi_\eta),$$

where $\pi_\eta \in \Pi_N$ is the representation of $U(W_B)$ (this uniquely specifies B) indexed by η under $J(\psi_0)$. As the element a varies over the two representatives of $k_0^\times / \mathbb{N}k^\times$, and the character η varies over $\text{Irr}(A_N)$, we obtain a collection of $2 \cdot \#\Pi_N$ representations (possibly zero). It was shown by Gelbart-Rogawski-Soudry [GRS] that this set of representations so obtained is the Vogan packet associated to the endoscopic parameter Π_M .

The following lemma, which was shown in [GRS], addresses more precisely the issue of non-vanishing of these theta lifts.

Lemma 8.2. *Let $M = M_1 + M_2 = \mu^2 + N \cdot \mu^{-1}$ as above. If $M \not\cong 3M_1$, assume without loss of generality that M_1 is distinct from any irreducible constituent of M_2 .*

(i) *If $M \not\cong 3M_1$, then the representation $\theta_{\psi_0, \mu, W_B, V_a}(\pi_\eta)$ is always nonzero.*

(ii) *If $M = 3M_1$, then $N = 2 \cdot \mu^3$ and $A_N \cong \mathbb{Z}/2\mathbb{Z}$, so that we may regard $\eta = \pm 1$, depending on whether η is trivial or not. The representation $\theta_{\psi_0, \mu, W_B, V_a}(\pi_\eta)$ is nonzero if and only if*

$$\omega_{k/k_0}(\text{disc}V_a) = \eta.$$

In each case above, the non-zero representations are mutually distinct. Moreover, the representation $\theta_{\psi_0, \mu, W_B, V_a}(\pi_\eta)$ is generic if and only if π_η is generic with respect to $\psi_{0, \text{disc}(V_a)}$.

We may now define a labeling of the elements in Π_M by the irreducible characters of A_M .

(i) If $M \cong 3M_1$, and M_1 does not occur in M_2 , then

$$A_M = A_{M_1} \times A_{M_2} = A_{M_1} \times A_N.$$

For a character $\chi = (\epsilon, \eta) \in \text{Irr}(A_{M_1}) \times \text{Irr}(A_N)$, we set

$$\pi^\chi = \pi^{\epsilon, \eta} = \theta_{\psi_0, \mu, W_B, V_a}(\pi_{\eta_V})$$

with

$$\epsilon \cdot \eta(-1) = \omega_{k/k_0}(a),$$

and

$$\eta_V = \begin{cases} \eta, & \text{if } \omega_{k/k_0}(\text{disc}V) = 1; \\ \eta \cdot \eta_{N,0}, & \text{if } \omega_{k/k_0}(\text{disc}V) = -1, \end{cases}$$

where $\eta_{N,0}$ is the character of A_N which indexes the ψ' -generic element of Π_N . More simply, when $\text{disc}(V) = 1$, we have

$$\chi(a_1) = \omega_{k/k_0}(a) \cdot \eta(-1) = \omega_{k/k_0}(a) \cdot \epsilon(B)$$

and

$$\chi|_{A_{M_2}} = \eta.$$

In particular, for a character χ of $A_M = A_{M_1} \times A_N$, π_χ is a representation of $U(V)$ if and only if $\chi(-1, -1) = 1$.

(ii) If $M = 3M_1 = 3\mu^2$, then

$$A_M \cong A_N = \mathbb{Z}/2\mathbb{Z}.$$

For a character $\eta = \pm$ of A_M , we set

$$\pi^\eta = \theta_{\psi_0, \mu, W_B, V_a}(\pi_{\eta \cdot \omega_{k/k_0}(\text{disc}V)})$$

with

$$\omega_{k/k_0}(a) = \eta.$$

By part (ii) of the above lemma, this condition ensures that the theta lift above is nonzero. In particular, the trivial character of A_M corresponds to a representation of $U(V)$ whereas the nontrivial character corresponds to the same representation regarded on $U(V')$.

Note that since $\dim V = 3$, there is only one orbit of generic characters for $U(V)$, and hence the Vogan parameterization in this case is canonical. So it is instructive to observe that the above parameterization is independent of the choice of ψ_0 . We leave this to the reader, as well as the verification that the above definition satisfies the desiderata of the Vogan parameterization listed in [GGP, §9 and §10].

9. SKEW-HERMITIAN CASE: $U(1) \times U(1)$

Having explicated the Langlands-Vogan parameterization of the unitary groups $U(V)$ with $\dim V \leq 3$, we are now in a position to verify instances of [GGP, Conjecture 16.3].

In this section, we consider the case of a pair of skew-hermitian spaces $W \subseteq V$, with $\dim W = \dim V = 1$. Without loss of generality, we assume that

$$\delta = \text{the discriminant of } V.$$

As in the previous section, we let $W' = V'$ be the other rank 1 skew-hermitian space, and more generally, for any $a \in k_0^\times$, we let W_a be the skew-hermitian space obtained from W by scaling by a .

To specify the restriction problem in this setting, fix an additive character

$$\psi_0 : k_0 \rightarrow \mathbb{S}^1,$$

and a character μ of k^\times with

$$\mu|_{k_0^\times} = \omega_{k/k_0}.$$

These determine a Weil representation $\omega_{W_a, \psi_0, \mu}$ of $U(W_a)$. Also, fix two conjugate-orthogonal representations M and N of dimension 1, which gives rise to characters $\alpha \times \beta$ of $U(W_a) \times U(W_a)$. We are interested in determining

$$\text{Hom}_{U(W_a)}(\alpha \cdot \beta, \omega_{W_a, \psi_0, \mu}).$$

This question has been resolved by Moen [Mo], Rogawski [Ro2] and Harris-Kudla-Sweet [HKS], and we state the result from [HKS, Corollary 8.5] as:

Theorem 9.1. *For each $a \in k_0^\times$, let W_a be the rank 1 hermitian space with discriminant $a \cdot \delta$, and for each $b \in k_0^\times$, let V_b be the rank 1 hermitian space with discriminant b . Given a character η of k^\times/k_0^\times , which can be regarded as a character of $U(W_a)$, we have*

$$\text{Hom}_{U(W_a)}(\eta, \omega_{W_a, V_b, \psi_0, \mu}) \neq 0 \iff \epsilon(\eta \cdot \mu^{-1}, \psi_0(\text{Tr}(\delta-))) = \omega_{k/k_0}(a \cdot b).$$

Remark: We note that our convention here differs from [HKS] in two aspects. Namely, we have adopted the convention that on $W_a \otimes V_b$, the symplectic form is $\text{Tr}(\langle -, - \rangle_{W_a} \otimes \langle -, - \rangle_{V_b})$. In [HKS], the symplectic form is

$$\frac{1}{2} \cdot \text{Tr}(\langle -, - \rangle_{W_a}^\sigma \otimes \langle -, - \rangle_{V_b}).$$

Besides the factor of $1/2$, the skew-hermitian form on W_a is conjugated by σ , which is necessitated by the convention adopted by [HKS] that skew-hermitian forms are linear

in the second variable and hermitian forms are linear in the first variable. Conjugating the form on W_a by σ has the effect of replacing δ by $-\delta$ in [HKS, Corollary 8.5].

To apply the above theorem to [GGP, Conjecture 16.3], set $\eta = \alpha \cdot \beta$ in the theorem, and note that the distinguished character χ_0 of $A_M \times A_N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given in [GGP, Conjecture 16.3] satisfies

$$\chi_0(-1, 1) = \chi_0(1, -1) = \epsilon(M \otimes N(\mu^{-1}), \psi_0(\text{Tr}(\delta-))).$$

Thus, Theorem 9.1 implies that

$$\chi_0 \text{ is trivial} \iff \text{Hom}_{\text{U}(W)}(\alpha \cdot \beta, \omega_{W, \psi_0, \mu}) \neq 0$$

and

$$\chi_0 \text{ is nontrivial} \iff \text{Hom}_{\text{U}(W)}(\alpha' \cdot \beta', \omega_{W', \psi_0, \mu}) \neq 0.$$

This verifies [GGP, Conjecture 16.3] for this case.

10. RESTRICTION FROM $\text{U}(2)$ TO $\text{U}(1)$

In this section, we consider the restriction problem from $\text{U}(2)$ to $\text{U}(1)$. This problem has been studied by H. Saito [Sa2] and T. Konno [Ko], but we shall give an independent treatment here and relate the result to [GGP, Conjecture 16.3].

Recall that in §6, we have given a construction of rank 2 hermitian spaces V_B using quaternion algebras B over k_0 , together with a non-degenerate rank 1 subspace:

$$L_B \hookrightarrow V_B,$$

such that

$$L_B^\perp = \langle 1 \rangle.$$

When B is split, this gives a pair of split hermitian spaces $L \subset V$, with

$$\text{disc}(L) = -1.$$

On the other hand, if B is the quaternion division algebra D , one obtains a relevant pair $L' \subset V'$ with V' anisotropic. The groups

$$G = G(V) \times G(L) \quad \text{and} \quad G' = G(V') \times G(L')$$

are relevant pure inner forms of each other.

Suppose that M is a conjugate-symplectic 2-dimensional representation of $WD(k)$, with component group A_M , so that M determines a Vogan packet Π_M of $\text{U}(V)$. We fix an additive character ψ of k/k_0 , and translate it by $-2 \cdot \text{disc}(L) = 2$, using the resulting character ψ_2 to fix the bijection

$$J(\psi_2) : \Pi_M \leftrightarrow \text{Irr}(A_M).$$

Recall that the parameter M gives rise to a representation

$$\tau \boxtimes \chi \text{ of } \text{GU}(V) = (\text{GL}_2(k_0) \times k^\times) / \Delta k_0^\times.$$

If τ' denotes the Jacquet-Langlands lift of τ to D^\times , then Π_M is simply the set of irreducible constituents in the restriction of $\tau \boxtimes \chi$ and $\tau' \boxtimes \chi$ to $U(V)$ and $U(V')$ respectively.

Similarly, suppose that N is a 1-dimensional conjugate-orthogonal representation of $WD(k)$. Then N determines a character η of k^\times/k_0^\times , which may be regarded as a character of $U(L_B)$. We are interested in determining

$$\mathrm{Hom}_{U(L_B)}(\pi_B \otimes \eta, \mathbb{C})$$

for $\pi_B \in \Pi_{M,B}$ and η the character of $U(L_B)$ corresponding to N .

Since the embedding

$$U(L_B) \hookrightarrow U(V_B) \subset GU^+(V_B)$$

is given by the diagonal map

$$k^\times/k_0^\times \hookrightarrow (B^\times \times k^\times)/\Delta k_0^\times,$$

we see that

$$\begin{aligned} & \bigoplus_B \bigoplus_{\pi_B \in \Pi_{M,B}} \mathrm{Hom}_{U(L_B)}(\pi_B \otimes \eta, \mathbb{C}) \\ &= \mathrm{Hom}_{k^\times}(\tau, \chi^{-1}\eta^{-1}) + \mathrm{Hom}_{k^\times}(\tau', \chi^{-1}\eta^{-1}). \end{aligned}$$

Now we note the following theorem of Waldspurger [Wa2], Tunnell [Tu] and Saito [Sa]:

Theorem 10.1. *Let τ be a representation of $GL_2(k_0)$ with L -parameter $N(\tau)$ and Jacquet-Langlands lift τ' on D^\times . For any character ν of k^\times , with $\nu|_{k_0^\times} = \omega_\tau$, we have*

$$\dim \mathrm{Hom}_{k^\times}(\tau, \nu) + \dim \mathrm{Hom}_{k^\times}(\tau', \nu) = 1.$$

Moreover,

$$\mathrm{Hom}_{k^\times}(\tau, \nu) \neq 0 \iff \epsilon(N(\tau)|_{WD(k)} \otimes \nu^{-1}, \psi) = 1.$$

Applying this theorem to the case at hand, with $\nu = \chi^{-1} \cdot \eta^{-1}$, we immediately deduce [GGP, Conjecture 16.1] (multiplicity one in L -packets). In fact, when τ is not dihedral with respect to k/k_0 , this theorem also implies [GGP, Conjecture 16.3]. Indeed, in this case, $\tau \boxtimes \chi$ remains irreducible when restricted to $U(V)$, so that

$$\Pi_M = \{\pi_M, \pi'_M\}.$$

Moreover, $A_M \cong A_N \cong \mathbb{Z}/2\mathbb{Z}$ and the distinguished character χ_0 of $A_M \times A_N$ satisfies

$$\chi_0(-1, 1) = \chi_0(1, -1) = \epsilon(N(\tau)|_{WD(k)} \otimes \chi \cdot \eta, \psi).$$

Hence we deduce that

$$\chi_0 \text{ is trivial} \iff \mathrm{Hom}_{U(L)}(\pi_M \otimes \eta, \mathbb{C}) \neq 0$$

and

$$\chi_0 \text{ is nontrivial} \iff \mathrm{Hom}_{U(L')}(\pi'_M \otimes \eta, \mathbb{C}) \neq 0.$$

Suppose then that τ is dihedral with respect to k/k_0 , so that

$$N(\tau)|_{WD(k)} = \alpha + \alpha^\sigma$$

for a character α of k^\times . In this case, τ is the sum of two distinct irreducible summands when restricted to $\mathrm{GL}_2(k_0)^+$ and the same holds for its Jacquet-Langlands lift τ' (if it exists). Now we have the following refinement of Theorem 10.1, due to the third author [P3].

Theorem 10.2. *Fix an additive character ψ of k/k_0 . Suppose that τ is dihedral with respect to k/k_0 .*

(i) *The two irreducible summands of $\tau_{\mathrm{GL}_2(k_0)^+}$ can be indexed as τ^{++} and τ^{--} such that the following holds. For any character β of k^\times with*

$$\beta|_{k_0^\times} = \omega_\tau = \alpha|_{k_0^\times} \cdot \omega_{k/k_0},$$

we have

$$\mathrm{Hom}_{k^\times}(\tau^{++}, \beta) \neq 0 \iff \epsilon(\alpha \cdot \beta^{-1}, \psi) = \epsilon(\alpha^\sigma \cdot \beta^{-1}, \psi) = +1,$$

and

$$\mathrm{Hom}_{k^\times}(\tau^{--}, \beta) \neq 0 \iff \epsilon(\alpha \cdot \beta^{-1}, \psi) = \epsilon(\alpha^\sigma \cdot \beta^{-1}, \psi) = -1.$$

(ii) *Similarly, the two irreducible summands of $\tau'|_{(D^\times)^+}$ can be labelled as $(\tau')^{+-}$ and $(\tau')^{-+}$ such that the following holds. For any character β of k^\times as above,*

$$\mathrm{Hom}_{k^\times}((\tau')^{+-}, \beta) \neq 0 \iff \epsilon(\alpha \cdot \beta^{-1}, \psi) = +1 \quad \text{and} \quad \epsilon(\alpha^\sigma \cdot \beta^{-1}, \psi) = -1,$$

and

$$\mathrm{Hom}_{k^\times}((\tau')^{-+}, \beta) \neq 0 \iff \epsilon(\alpha \cdot \beta^{-1}, \psi) = -1 \quad \text{and} \quad \epsilon(\alpha^\sigma \cdot \beta^{-1}, \psi) = +1.$$

This theorem immediately implies [GGP, Conjecture 20.1], which is a variant of [GGP, Conjecture 16.3]. To obtain the full [GGP, Conjecture 16.3], one would need to relate the labelling supplied by the above theorem with the bijection $\Pi_M \leftrightarrow \mathrm{Irr}(A_M)$ determined by the additive character ψ_2 . Recall that this bijection was constructed in §8 using theta correspondence.

To recall the construction, note that when τ is dihedral with respect to k/k_0 , we have

$$M = M_1 + M_2$$

with M_i conjugate-symplectic (not necessarily distinct). We assume that $M_1 \neq M_2$, since the case $M_1 \cong M_2$ is similar. Then we have

$$A_M = \mathbb{Z}/2\mathbb{Z}a_1 \times \mathbb{Z}/2\mathbb{Z}a_2.$$

Setting

$$\mu = M_1 \quad \text{and} \quad \chi/\chi^\sigma = M_1 \cdot M_2,$$

the packet Π_M consists of the representations

$$\pi_\rho = \theta_{\psi_{0,2},\mu,V_B,W_a}(\chi)$$

where W is the rank 1 skew-hermitian space of discriminant δ , and ψ and ψ_0 are related by

$$\psi(x) = \psi_0\left(\frac{1}{2} \cdot \text{Tr}(\delta \cdot x)\right).$$

Moreover, we have

$$\rho(a_1) = \epsilon(B) \cdot \omega_{k/k_0}(a)$$

and

$$\rho(a_2) = \omega_{k/k_0}(a).$$

Now we consider the see-saw diagram

$$\begin{array}{ccc} \text{U}(L_{-b} + L_1) & & \text{U}(W_a) \times \text{U}(W_a) \\ & \searrow & \nearrow \\ & & \Delta\text{U}(W_a) \\ & \nearrow & \searrow \\ \text{U}(L_{-b}) \times \text{U}(L_1) & & \end{array}$$

where L_{-b} denotes the rank 1 hermitian space with discriminant $-b$. Note that the rank 2 hermitian space $L_{-b} + L_1$ is isomorphic to V_B with

$$\epsilon(B) = \omega_{k/k_0}(b),$$

and the pair $L_{-b} \subset L_{-b} + L_1$ is isomorphic to $L_B \subset V_B$. We start with the representation $\chi|_{\text{U}(W_a)}$ on $\Delta\text{U}(W_a)$ and the character η^{-1} on $\text{U}(L_{-b})$, and consider the theta correspondence with respect to the additive character $\psi_{0,2}$. Then the seesaw identity gives

$$\text{Hom}_{\text{U}(L_{-b})}(\pi_\rho, \eta^{-1}) = \text{Hom}_{\text{U}(W_a)}(\theta_{\psi_0^{-1},\mu,W_a,L_{-b}}(\eta^{-1}) \otimes \omega_{\psi_{0,2},\mu,W_a}, \chi).$$

Hence,

$$\text{Hom}_{\text{U}(L_{-b})}(\pi_\rho, \eta^{-1}) \neq 0$$

if and only if the following two conditions hold:

(a)

$$\theta_{\psi_{0,2},\mu,W_a,L_{-b}}(\eta^{-1}) \neq 0,$$

in which case, $\theta_{\psi_{0,2},\mu,W_a,L_{-b}}(\eta^{-1}) = \eta^{-1}$;

(b)

$$\text{Hom}_{\text{U}(W_a)}(\eta^{-1} \otimes \omega_{\psi_{0,2},\mu,W_a}, \chi) \neq 0.$$

But both (a) and (b) are special cases of Theorem 9.1 [HKS, Corollary 8.5]. We deduce that (a) holds if and only if

$$\epsilon(\mu^{-1}\eta^{-1}, \psi_{0,2}(\mathrm{Tr}(\delta-))) = \omega_{k/k_0}(-b) \cdot \omega_{k/k_0}(a)$$

or equivalently

$$\epsilon(M_1 \otimes N, \psi) = \omega_{k/k_0}(b) \cdot \omega_{k/k_0}(a) = \rho(a_1).$$

Similarly, (b) holds if and only if

$$\epsilon(\mu^{-1} \cdot \eta \cdot \chi/\chi^\sigma, \psi_{0,2}(\mathrm{Tr}(\delta-))) = \omega_{k/k_0}(a),$$

or equivalently

$$\epsilon(M_2 \otimes N, \psi) = \omega_{k/k_0}(a) = \rho(a_2).$$

Thus, we conclude that

$$\mathrm{Hom}_{\mathrm{U}(L_b)}(\pi_\rho, \eta^{-1}) \neq 0$$

if and only if ρ is the distinguished character χ_0 of [GGP, Conjecture 16.3], where the local root numbers are computed using the additive character ψ of k/k_0 .

11. THETA CORRESPONDENCE FOR $\mathrm{U}(2) \times \mathrm{U}(2)$

Before moving on to the next case of [GGP, Conjecture 16.3], we need to establish some results about the theta correspondence for $\mathrm{U}(2) \times \mathrm{U}(2)$. More precisely, let V_B be the rank 2 hermitian space introduced in §6. and let $W_{B'}$ be the rank 2 skew-hermitian space obtained from $V_{B'}$ by scaling by the trace zero element $\delta \in k^\times$. Then we are interested in establishing the theta correspondence for the dual pair

$$\mathrm{U}(V_B) \times \mathrm{U}(W_{B'})$$

relative to the data (ψ_0, μ) .

The first result is the following proposition due to Harris [Ha, Lemma 4.3.3] and Konno-Konno [KK, Prop. 5.3 and Thm. 5.4].

Proposition 11.1. *Let M be a 2-dimensional conjugate-symplectic representation of $WD(k)$ which gives rise to a L -packet $\Pi_{M,B}$ for $\mathrm{U}(V_B)$ and $\Pi_{M,B'}$ for $\mathrm{U}(W_{B'})$.*

(i) *For any $\pi \in \Pi_{M,B}$,*

$$\theta_{\psi_0, V_B, W_{B'}, \mu}(\pi) \neq 0 \iff \epsilon(M \otimes \mu^{-2}, \psi) = \epsilon(B) \cdot \epsilon(B').$$

Note that the root number above is independent of the choice of the additive character ψ of k/k_0 .

(ii) *If the condition of (i) holds, then $\theta_{\psi_0, V_B, W_{B'}, \mu}(\pi)$ belongs to $\Pi_{M,B'}$. In other words, the theta correspondence is the identity map on L -parameters.*

Thus, under the theta correspondence for (ψ_0, μ) , there is a unique B' such that the theta lift gives a bijection

$$\Pi_{M,B} \longleftrightarrow \Pi_{M,B'}.$$

If the parameter M is non-dihedral (with respect to k/k_0), then $\#\Pi_{M,B} = 0$ or 1. Hence the above proposition completely determines the theta lift of the representations in Π_M . When M is dihedral with respect to k/k_0 , then $\#\Pi_{M,B} = 0$ or 2, and in the latter case, there are two possible bijections

$$\Pi_{M,B} \longleftrightarrow \Pi_{M,B'},$$

which the above proposition does not resolve. In [P5], the third author has given a precise conjecture addressing this issue. The following theorem confirms the conjecture in [P5] for this case:

Theorem 11.2. *Suppose that $M = M_1 + M_2$ is dihedral with respect to k/k_0 . Fix the additive character ψ of k/k_0 which gives bijections*

$$\Pi_M \longleftrightarrow \text{Irr}(A_M),$$

and let ψ_0 be the additive character of k_0 such that

$$\psi(x) = \psi_0\left(\frac{1}{2} \cdot \text{Tr}(\delta x)\right).$$

Then the permutation of Π_M induced by the theta correspondence associated to (ψ_0, μ, δ) is given by multiplication by the character ρ_0 of A_M defined by

$$\rho_0(a_i) = \epsilon(M_i \otimes \mu^{-2}, \psi_2)$$

with

$$\psi_2(x) = \psi_0(\text{Tr}(\delta x)).$$

Proof. Consider first the case where B' is split whereas B is arbitrary. In this case, then two elements in $\Pi_{M,B'}$ can be distinguished by the Whittaker models they support. Computing Whittaker models of the Weil representation $\omega_{\psi_0, V_B, W_{B'}, \mu}$, one sees that for $\pi_\rho \in \Pi_{M,B}$, the representation $\theta_{\psi_0, V_B, W_{B'}, \mu}(\pi_\rho)$ of $U(W_{B'})$ is ψ_0 -generic if and only if

$$\text{Hom}_{U(L_B)}(\pi_\rho^\vee, \mu^{-2}) \neq 0.$$

By the result of the previous section, this holds if and only if

$$\rho(a_1) = \epsilon(M_1 \otimes \mu^{-2}, \psi_2) \quad \text{and} \quad \rho(a_2) = \epsilon(M_2 \otimes \mu^{-2}, \psi_2),$$

as desired. This establishes the result when one of B or B' is split.

The only remaining case is where B and B' are both non-split, so that

$$\epsilon(M_1 \otimes \mu^{-2}, \psi) \cdot \epsilon(M_2 \otimes \mu^{-2}, \psi) = 1.$$

In this case, the desired result can be proved by a global method. We give a brief sketch of this.

Let π be a representation in $\Pi_{M,B}$, so that $\theta_{\psi_0, \mu}(\pi)$ belongs to $\Pi_{M,B}$ also. We may find:

- (1) a number field F with at least one real place v_∞ and such that $F_{v_0} = k_0$ for some finite place v_0 of F ;
- (2) an additive character Ψ of \mathbb{A}_F/F such that $\Psi_{v_0} = \psi_0$;
- (3) a quadratic extension E of F such that $E_{v_\infty} \cong \mathbb{C}$ and $E_{v_0} \cong k$;
- (4) a trace zero element $\Delta \in E$ such that $\Delta_v = \delta$;
- (5) an idele class character Σ of E^\times such that $\Sigma_{v_0} = \mu$ and $\Sigma|_{\mathbb{A}_F^\times} = \omega_{E/F}$;
- (6) a global quaternion algebra \mathbb{B} over F ramified precisely at $\{v_\infty, v_0\}$, so that $\mathbb{B}_{v_0} = B$; this gives a hermitian space $V_{\mathbb{B}}$ over F which is isomorphic to V_B over F_{v_0} ;
- (7) a cuspidal representation Π of $U(V_{\mathbb{B}})$ such that
 - (a) $\Pi_{v_0} = \pi$;
 - (b) Π belongs to a global endoscopic packet;
 - (c) $L(BC_{E/F}(\Pi) \otimes \Sigma^{-2}, 1/2) \neq 0$

In particular, it follows that the set

$$S = \{v : \epsilon(BC_{E_v/F_v}(\Pi_v) \otimes \Sigma_v, \psi) = -1\}$$

has even cardinality and does not contain the place v_0 . Let \mathbb{B}' be the quaternion algebra over F such that

$$\epsilon(\mathbb{B}'_v) \neq \epsilon(\mathbb{B}_v) \iff v \in S.$$

In other words, \mathbb{B}' is obtained from \mathbb{B} by switching the local invariants of \mathbb{B} at the set S . Since $v_0 \notin S$, we have

$$\mathbb{B}'_{v_0} \cong B.$$

Moreover, for each place v of F ,

$$\Theta_{\Psi_v, \Sigma_v, V_{\mathbb{B}_v}, W_{\mathbb{B}'_v}}(\Pi_v) \neq 0.$$

By [Ha], The non-vanishing of the central L -value above implies that the global theta lift is non-vanishing as well:

$$\Theta_{\Psi, \Sigma, V_{\mathbb{B}}, W_{\mathbb{B}'}}(\Pi) \neq 0.$$

Now the assertion of the theorem has been checked for all finite places of F outside v_0 , since at least one of \mathbb{B}_v or \mathbb{B}'_v is split at any $v \neq v_0$. At the archimedean places, the result of the theorem is also known (c.f. [Pa] for example). If the result of the theorem is not true at the place v_0 , we would have a cuspidal representation $\Theta_{\Psi, \Sigma, V_{\mathbb{B}}, W_{\mathbb{B}'}}(\Pi)$ of $U(W_{\mathbb{B}'})$ which violates the Langlands-Arthur multiplicity formula for global endoscopic packets of $U(2)$. This gives the desired contradiction.

For example, suppose that S is empty so that $\mathbb{B} = \mathbb{B}'$. Then if the result of the theorem holds at all $v \neq v_0$ but fails at v_0 , the cuspidal representation $\Theta_{\Psi, \Sigma, V_{\mathbb{B}}, W_{\mathbb{B}'}}(\Pi)$

of $U(W_{\mathbb{B}'})$ would differ from the cuspidal representation Π at an odd number of places v . This is a contradiction. \square

12. TRILINEAR FORMS FOR $U(2)$

In this section, we return to the skew-hermitian case of [GGP, Conjecture 16.3]. In particular, we consider the case when

$$W \subseteq V \quad \text{with } \dim W = \dim V = 2.$$

Thus, let $W_{B,\delta}$ be the rank 2 skew-hermitian case obtained from V_B by scaling by δ . Fix an additive character ψ_0 of k_0 , and a character μ of k^\times so that

$$\mu|_{k_0^\times} = \omega_{k/k_0}.$$

This determines Weil representations $\omega_{\psi_0,\mu}$ for $U(W_B)$. Given two conjugate-symplectic representations M and N of dimension 2, with corresponding Vogan packet Π_M and Π_N , we are interested in computing

$$\text{Hom}_{U(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}}, \mathbb{C})$$

as π_M and π_N vary over all representations in Π_M and Π_N .

Note that the representation $\omega_{\psi_0,\mu}$ is not an irreducible representation of $U(W_B)$. However, we may decompose $\omega_{\psi_0,\mu}$ according to central characters

$$\omega_{\psi_0,\mu} = \bigoplus_{\chi} \omega_{\psi_0,\mu}[\chi]$$

as χ runs over characters of $Z_{U(W_B)} \cong k^\times/k_0^\times$. In fact, this decomposition is simply the decomposition of the Weil representation for the dual pair $U(V_1) \times U(W_B)$. Thus, each summand $\omega_{\psi_0,\mu}[\chi]$ is an irreducible representation of $U(W_B)$. Moreover, it belongs to an endoscopic packet of $U(W_B)$ constructed in Proposition 8.1.

Now, because of central character reasons, it is clear that

$$\text{Hom}_{U(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}[\chi]}, \mathbb{C}) = 0$$

unless

$$\det M \cdot \det N = \chi.$$

For this χ , we have

$$\text{Hom}_{U(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}}, \mathbb{C}) = \text{Hom}_{U(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}[\chi]}, \mathbb{C}).$$

In particular, [GGP, Conjecture 16.3] amounts to a question about invariant trilinear forms on $U(W_B)$.

Given that the group $U(W_B)$ can be described in terms of $GL_2(k_0)$ and its inner form, we shall see this question can be related to a question about invariant trilinear forms for GL_2 . This has been addressed in a series of paper by the third author [P1,2,6]; we recall his result here:

Theorem 12.1. *Let N_1, N_2 and N_3 be 2-dimensional representations of $WD(k_0)$, with associated representations $\pi_{i,B}$ of B^\times . Assume that $\det N_1 \cdot \det N_2 \cdot \det N_3 = 1$. Then*

$$\sum_B \dim \text{Hom}_{B^\times}(\pi_{1,B} \otimes \pi_{2,B} \otimes \pi_{3,B}, \mathbb{C}) = 1.$$

Moreover,

$$\text{Hom}_{B^\times}(\pi_{1,B} \otimes \pi_{2,B} \otimes \pi_{3,B}, \mathbb{C}) \neq 0 \iff \epsilon(N_1 \otimes N_2 \otimes N_3, \psi) = \epsilon(B).$$

To apply this theorem to the case of $U(W_B)$, we need to consider the group $(B^\times)^+$ and calculate

$$\dim \text{Hom}_{(B^\times)^+}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}).$$

More generally, let G be a subgroup of $GL_2(k_0)$ containing $SL_2(k_0)$. The group G is uniquely determined by the subgroup

$$k_G^\times \subset k_0^\times$$

consisting of determinants of elements of G . Thus, for any quaternion algebra B , it makes sense to define a corresponding subgroup G_B inside B^\times containing $SL_1(B)$. Restricting representations of B^\times to G_B , one gets a notion of L -packet of representations of G_B . It is known that representations of $GL_2(k_0)$ restrict to G with multiplicity 1, but this need not be the case for representations of B^\times if B is non-split. For a representation π_B of G_B , let $m(\pi_B)$ denote the multiplicity with which it appears in the restriction of an irreducible representation of B^\times .

Now we have:

Theorem 12.2. *For $i = 1, 2$ and 3 , let N_i be a 2-dimensional representation of $WD(k_0)$ with associated representation $\tilde{\pi}_{B,i}$ of B^\times . Assume that $\prod_i \det N_i = 1$. Then*

$$\sum_B \dim \text{Hom}_{G_B}(\tilde{\pi}_{B,1} \otimes \tilde{\pi}_{B,2} \otimes \tilde{\pi}_{B,3}, \mathbb{C}) = \#(k_0^\times / k_0^{\times 2} k_G^\times).$$

In particular,

$$\sum_B \sum_{\pi_{B,1}, \pi_{B,2}, \pi_{B,3}} m(\pi_{B,1}) \cdot m(\pi_{B,2}) \cdot m(\pi_{B,3}) \cdot \dim \text{Hom}_{G_B}(\pi_{B,1} \otimes \pi_{B,2} \otimes \pi_{B,3}, \mathbb{C})$$

is equal to

$$\#(k_0^\times / k_0^{\times 2} k_G^\times),$$

where the inner sum is taken over irreducible representations $\pi_{B,i}$ of G_B which are contained in the representations $\tilde{\pi}_i$ of B^\times .

Proof. Clearly,

$$\mathrm{Hom}_{G_B}(\tilde{\pi}_{B,1} \otimes \tilde{\pi}_{B,2} \otimes \tilde{\pi}_{B,3}, \mathbb{C}) \cong \sum_{\chi: k_0^\times/k_G^\times \rightarrow \mathbb{Z}/2} \mathrm{Hom}_{G_B}(\tilde{\pi}_{B,1} \otimes \tilde{\pi}_{B,2} \otimes \tilde{\pi}_{B,3}, \mathbb{C}_\chi),$$

where the χ 's range over characters of B^\times trivial on G_B , and \mathbb{C}_χ denotes the 1-dimensional representation $\chi \circ \mathbb{N}_B$ of B^\times . But by Theorem 12.1, we have

$$\sum_B \dim \mathrm{Hom}_{G_B}(\tilde{\pi}_{B,1} \otimes \tilde{\pi}_{B,2} \otimes \tilde{\pi}_{B,3}, \mathbb{C}_\chi) = 1,$$

for all characters χ of order ≤ 2 (by absorbing χ in one of the π_i 's). Adding up the contribution of the various χ 's, we get the conclusion of the theorem. \square

Specializing this theorem to the case $G_B = (B^\times)^+$ and noting that, in this case, $m(\pi_{B,i}) = 1$ for each B , we obtain:

Corollary 12.3. *In the context of Theorem 12.2, let $G = \mathrm{GL}_2(k_0)^+$. Then one has,*

$$\sum_B \sum_{\pi_{B,1}, \pi_{B,2}, \pi_{B,3}} \dim \mathrm{Hom}_{G_B}(\pi_{B,1} \otimes \pi_{B,2} \otimes \pi_{B,3}, \mathbb{C}) = 2,$$

where the inner sum is taken over irreducible representations $\pi_{B,i}$ of G_B which are contained in the representations $\tilde{\pi}_{B,i}$ of B^\times .

We can now apply the corollary to the group $\mathrm{GU}^+(W_B)$ or equivalently $\mathrm{U}(W_B)$.

Corollary 12.4. *Let M_i be conjugate-symplectic representations of $WD(k)$ with associated L -packet $\Pi_{M_i, B}$ of $\mathrm{U}(W_B)$. Assume that $\det M_1 \cdot \det M_2 \cdot \det M_3 = 1$. Then*

(i)

$$\sum_B \sum_{\pi_i \in \Pi_{M_i, B}} \dim \mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) = 2.$$

(ii) *If one of the M_i 's, say M_1 , is dihedral with respect to k/k_0 , so that $\#\Pi_{M_1, B_0} = 2$ for B_0 split, then*

$$\dim \mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) \leq 1$$

for each B . If the above Hom space is nonzero, then

$$\dim \mathrm{Hom}_{\mathrm{U}(W_{B'})}(\pi'_1 \otimes \pi'_2 \otimes \pi'_3, \mathbb{C}) = 0$$

for $B' \neq B$.

Proof. The first assertion follows immediately from the previous corollary and the definition of L -packets for $U(W)$ given in §6. To deduce the last assertion, note that if

$$\mathrm{Hom}_{U(W_B)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) \neq 0,$$

then we also have

$$\mathrm{Hom}_{U(W_B)}(\pi_1^c \otimes \pi_2^c \otimes \pi_3^c, \mathbb{C}) \neq 0,$$

where π_i^c denotes the conjugate of π_i by an element $c \in \mathrm{GU}(W_B) \setminus \mathrm{GU}^+(W_B)$. Since

$$\dim \mathrm{Hom}_{U(W_B)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) + \dim \mathrm{Hom}_{U(W_B)}(\pi_1^c \otimes \pi_2^c \otimes \pi_3^c, \mathbb{C}) \leq 2,$$

each of these dimensions must be equal to 1, and all other Hom spaces must be 0. \square

Remark: Since $k_0^\times/k_0^{\times 2}$ is a 2-group whose cardinality can be made arbitrarily large by choosing k_0 appropriately, and since the L -packet of representations of $\mathrm{SL}_2(k)$ is bounded by 4 [LL], it follows that

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(k)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C})$$

can be made arbitrarily large.

Now we can return to [GGP, Conjecture 16.3], so that M and N are two 2-dimensional conjugate-symplectic representations of $WD(k)$ which determine Vogan packets Π_M and Π_N of $U(W_B)$. For a fixed additive character ψ_0 of k_0 , we have obtained a bijection

$$J(\psi_0) : \Pi_M \longleftrightarrow \mathrm{Irr}(A_M)$$

and similarly for Π_N . We are interested in computing

$$\mathrm{Hom}_{U(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0, \mu}})$$

for $\pi_M \in \Pi_M$ and $\pi_N \in \Pi_N$.

If M and N are non-dihedral (with respect to k/k_0), so that Π_M and Π_N both contain at most one representation of each $U(W_B)$ (as B varies), then [GGP, Conjecture 16.3] is a consequence of Theorem 12.1. Indeed, we have

$$A_M \times A_N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and the distinguished character χ_0 satisfies

$$\chi_0(-1, 1) = \chi_0(1, -1) = \epsilon(M \otimes N(\mu^{-1}), \psi)$$

for any character ψ of k/k_0 . On the other hand, if Π_M is obtained by the restriction of the representation $\tau_M \boxtimes \chi_M$ of $\mathrm{GU}(W_B)$ and Π_N is obtained from $\tau_N \boxtimes \chi_N$, then the

epsilon factor occurring in Theorem 12.1 is

$$\begin{aligned}
& \epsilon(\rho_M \otimes \rho_N \otimes \text{Ind}(\mu^{-1}\chi_M\chi_N), \psi_0) \\
&= \epsilon(\rho_M|_{WD(k)} \otimes \rho_N|_{WD(k)} \otimes \mu^{-1} \cdot \chi_M \cdot \chi_N, \psi_0(\text{Tr})) \\
&= \epsilon(M \otimes N(\mu^{-1}), \psi_0(\text{Tr})) \\
&= \epsilon(M \otimes N(\mu^{-1}), \psi).
\end{aligned}$$

This verifies [GGP, Conjecture 16.3] in this case.

When at least one of M or N is dihedral with respect to k/k_0 , we may appeal to the theta correspondence. Since the case when exactly one of them is dihedral with respect to k/k_0 is similar and easier, we shall give the details only when both M and N are dihedral with respect to k/k_0 . Thus, let

$$M = M_1 + M_2 \quad \text{and} \quad N = N_1 + N_2,$$

with M_i and N_i conjugate-symplectic (not necessarily distinct), and write their component groups as

$$A_M = \mathbb{Z}/2\mathbb{Z}e_1 \times \mathbb{Z}/2\mathbb{Z}e_2 \quad \text{and} \quad A_N = \mathbb{Z}/2\mathbb{Z}f_1 \times \mathbb{Z}/2\mathbb{Z}f_2.$$

In this case, the packet Π_M can be obtained by theta correspondence from $U(1)$. Set

$$\nu = M_1$$

and

$$M_1 \cdot M_2 = \eta/\eta^\sigma,$$

for some character η of k^\times . If L_a denote the rank 1 hermitian space with discriminant a , then

$$\Pi_M = \{\theta_{\psi_0, \nu, W_{B, \delta}, L_a}(\eta|_{U(L_a)}) : a \in k_0^\times/\mathbb{N}k^\times, \epsilon(B) = \pm 1\}.$$

Relative to the additive character ψ of k/k_0 , we have the labelling

$$\pi_{\rho_M} = \theta_{\psi_0, \nu, W_{B, \delta}, L_a}(\eta|_{U(L_a)})$$

if and only if

$$\rho_M(e_1) = \epsilon(B) \cdot \omega_{k/k_0}(a) \quad \text{and} \quad \rho_M(e_2) = \omega_{k/k_0}(a).$$

Similarly, a representation in Π_N has the form π_{ρ_N} , so that

$$\rho_N(f_1) \cdot \rho_N(f_2) = \epsilon(B).$$

Now consider the see-saw diagram

$$\begin{array}{ccc}
 \mathrm{U}(L_a + L_{-1}) & & \mathrm{U}(W_B) \times \mathrm{U}(W_B) \\
 | & \searrow & | \\
 \mathrm{U}(L_a) \times \mathrm{U}(L_{-1}) & & \Delta\mathrm{U}(W_B)
 \end{array}$$

and note that the rank 2 hermitian space $L_a + L_{-1}$ is isomorphic to $V_{B'}$ with $\epsilon(B') = \omega_{k/k_0}(a)$. We start with the representation $\eta|_{\mathrm{U}(L_a)}$ of $\mathrm{U}(L_a)$, so that the representation we obtain on $\mathrm{U}(W_B)$ is precisely

$$\pi_{\rho_M} = \theta_{\psi_0, \nu, W_B, \delta, L_a}(\eta|_{\mathrm{U}(L_a)}).$$

On the other side of the see-saw, we start with the representation $\mu \cdot \nu \cdot \pi_{\rho_N}^\vee$ of $\mathrm{U}(W_B)$. Note that taking contragredient has the following effect on the Vogan parameterization: for any character ρ_N of A_N , the representation π_{ρ_N} has Vogan parameter

$$(N^\vee, \rho_N \cdot \beta_0)$$

where β_0 is the character of $A_{N^\vee} = A_N$ given by

$$\beta_0(b_i) = \omega_{k/k_0}(-1).$$

Now the see-saw identity gives:

$$\begin{aligned}
 & \mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_{\rho_M} \otimes \omega_{\psi_0^{-1}, \nu, W_B}, \mu \cdot \nu \cdot \pi_{\rho_N}^\vee) \\
 &= \mathrm{Hom}_{\mathrm{U}(L_a)}(\Theta_{\psi_0, \nu^2, W_B, L_a + L_{-1}}(\mu\nu\pi_{\rho_N}^\vee), \eta|_{\mathrm{U}(L_a)}).
 \end{aligned}$$

Since

$$\omega_{\psi_0^{-1}, \nu, W_B} = \omega_{\psi_0^{-1}, \mu^{-1}, W_B} \otimes \mu\nu = \overline{\omega_{\psi_0, \mu}} \otimes \mu\nu,$$

we see that the LHS of this identity is equal to the desired space

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_{\rho_M} \otimes \pi_{\rho_N} \otimes \overline{\omega_{\psi_0, \mu}}, \mathbb{C}).$$

On the other hand, the RHS is nonzero if and only if conditions (a) and (b) below are satisfied:

- (a) $\Theta_{\psi_0, \nu^2, W_B, L_a + L_{-1}}(\mu\nu(\pi_{\rho_N})^\vee) \neq 0$. According to Theorem 11.2, this holds if and only if

$$\epsilon(N^\vee \otimes \mu\nu\nu^{-2}, \psi) = \epsilon(B) \cdot \omega_{k/k_0}(a),$$

or equivalently

$$\epsilon(N \otimes M_1(\mu^{-1}), \psi) = \epsilon(B) \cdot \omega_{k/k_0}(a) = \rho_M(e_1).$$

If this is satisfied, then by Theorem 11.2, the theta lift is equal to the representation

$$\pi_{\rho_{N^\vee}} \cdot \mu\nu$$

of $U(L_a + L_{-1})$, with

$$\rho_{N^\vee}(f_i) = \rho_N(f_i) \cdot \omega_{k/k_0}(-1) \cdot \epsilon(N_i \otimes M_1(\mu^{-1}), \psi_{-2}).$$

- (b) $\text{Hom}_{U(L_a)}(\pi_{\rho_{N^\vee}} \mu \nu, \eta/\eta^\sigma) \neq 0$. This is a branching problem for $U(2) \times U(1)$ which we have resolved in §10. Using the results there, we see that the desired non-vanishing holds if and only if

$$\rho_{N^\vee}(f_i) = \rho_N(f_i) \cdot \omega_{k/k_0}(-1) \cdot \epsilon(N_i \otimes M_1(\mu^{-1}), \psi_{-2}) = \epsilon(N_i \otimes M_2(\mu^{-1}), \psi_2)$$

or equivalently

$$\rho_N(f_i) = \epsilon(N_1 \otimes M(\mu^{-1}), \psi_2) = \epsilon(N_1 \otimes M(\mu^{-1}), \psi).$$

Finally, since

$$\rho_M(-1) = \rho_N(-1) = \epsilon(B),$$

we conclude that

$$\rho_M(e_2) = \epsilon(N \otimes M_2(\mu^{-1}), \psi).$$

Thus we conclude that

$$\text{Hom}_{U(W_B)}(\pi_{\rho_M} \otimes \pi_{\rho_N} \otimes \overline{\omega_{\psi_0, \mu}}, \mathbb{C}) \neq 0$$

if and only if $\rho_M \times \rho_N$ is the distinguished character χ_0 of [GGP, Conjecture 16.3].

13. RESTRICTION FROM $U(3)$ TO $U(2)$: ENDOSCOPIC CASE

In this section, we consider the restriction problem for $U(3) \times U(2)$. Using theta correspondence, we establish [GGP, Conjecture 16.3] for endoscopic packets of $U(3)$. In the following section, we shall consider the stable packets of $U(3)$.

We fix a pair

$$W \subset V$$

of split hermitian spaces of dimension 2 and 3 respectively. Without loss of generality, we assume that $\text{disc} V = 1$, so that V/W is a rank 1 hermitian space L_1 of discriminant 1. Let $W' \subset V'$ be hermitian spaces of dimension 2 and 3, such that $V/W \cong V'/W'$.

More concretely, for each quaternion algebra B over k_0 , we have a rank 2 hermitian space V_B . Then the rank 3 hermitian space

$$V_{B,b} = V_B + L_b$$

has discriminant satisfying

$$\omega_{k/k_0}(\text{disc}(V_{B,b})) = \epsilon(B) \cdot \omega_{k/k_0}(b).$$

If we take $b = 1$, then as B varies, the pair

$$V_B \subset V_{B,1}$$

gives the pairs $W \subset V$ and $W' \subset V'$.

Suppose first that N is a 2-dimensional conjugate-symplectic representation of $WD(k)$ with associated Vogan packet Π_N of $U(V_B)$. If $N = \oplus_i N_i$, then we write

$$A_N = \prod_i A_{N_i} = \prod_i \mathbb{Z}/2\mathbb{Z}f_i.$$

For the fixed given additive character ψ of k/k_0 , we translate ψ by $-2 \cdot \text{disc}(V) = -2$ and use the resulting character ψ_{-2} to fix the Vogan parameterization

$$J(\psi_{-2}) : \Pi_N \longleftrightarrow \text{Irr}(A_N).$$

Now consider a 3-dimensional conjugate-orthogonal representation

$$M = M_1 + M_2$$

with $\dim M_i = i$ and such that each M_i is conjugate-orthogonal. Unless, $M \cong 3M_1$, we may further assume that M_1 does not occur in M_2 . We shall assume that this is the case, since the other case is similarly handled. Then

$$A_M = A_{M_1} \times A_{M_2}$$

and we write:

$$A_{M_1} = \mathbb{Z}/2\mathbb{Z}e \quad \text{and} \quad A_{M_2} = \prod_i \mathbb{Z}/2\mathbb{Z}e_i$$

if $M_2 = \oplus_i M_{2,i}$.

Moreover, we shall assume that the conjugate-orthogonal character M_1 has a conjugate-symplectic square root. This can be achieved by twisting M , and since this twist can be absorbed into N for the purpose of the restriction problem, there is no loss of generality in making this assumption on M_1 . Under this assumption on M_1 , we have described in §8 a construction of the Vogan packet Π_M as well as a bijection

$$\Pi_M \longleftrightarrow \text{Irr}(A_M)$$

which is canonical in this case (i.e. independent of the choice of any additive character). To recall the construction briefly, we set

$$M_1 = \mu^2$$

for some conjugate-symplectic character μ and set

$$N' = M_2 \cdot \mu,$$

so that N' is conjugate-symplectic and $A_{N'} = A_{M_2}$. Then, for quaternion algebras B and B' over k_0 , one considers the theta correspondence for

$$U(W_{B'}) \times U(V_{B,1})$$

relative to the data $(\psi_{0,-2}, \mu)$, where ψ_0 is our fixed additive character of k_0 . The packet Π_M is then the theta lift of the packet $\Pi_{N'}$ of $U(W_{B'})$. For the labelling of the representations in Π_M by $\text{Irr}(A_M)$, we refer the reader to the end of §8.

Now we would like to determine

$$\mathrm{Hom}_{\mathrm{U}(V_B)}(\pi_M \otimes \pi_N, \mathbb{C}),$$

for $\pi_M \in \Pi_M$ and $\pi_N \in \Pi_N$. We examine this restriction problem using the see-saw diagram

$$\begin{array}{ccc} \mathrm{U}(V_B + L_1) & & \mathrm{U}(W_{B'}) \times \mathrm{U}(W_{B'}) \\ | & \searrow & | \\ \mathrm{U}(V_B) \times \mathrm{U}(L_1) & & \mathrm{U}(W_{B'}). \end{array}$$

On $\mathrm{U}(W_{B'})$, we start with a representation $\pi_{N'}^\eta \in \Pi_{N'}$ indexed by a character η of $A_{N'}$, so that

$$\eta(-1) = \epsilon(B').$$

On $\mathrm{U}(V_B)$, we start with a representation $\pi_{\rho_N}^\vee$ associated to a character ρ_N of A_N , so that

$$\rho_N(-1) = \epsilon(B).$$

Then we have the see-saw identity:

$$\mathrm{Hom}_{\mathrm{U}(V_B)}(\Theta_{\psi_0, -2, \mu}(\pi_{N'}^\eta) \otimes \pi_{\rho_N}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(W_{B'})}(\Theta_{\psi_0, -2, \mu^2, V_B, W_{B'}}(\pi_{\rho_N}^\vee) \otimes \omega_{\psi_0, -2, \mu, L_1, W_{B'}}(\pi_{N'}^\eta)).$$

Note that

$$\pi_{\rho_M} = \theta_{\psi_0, -2, \mu}(\pi_{N'}^\eta)$$

with

$$\rho_M|_{A_{N'}} = \eta \quad \text{and} \quad \rho_M(e) = \epsilon(B') \cdot \eta(-1) = \epsilon(B) \cdot \epsilon(B').$$

Moreover, $\pi_{\rho_N}^\vee$ has Vogan parameter (relative to $J(\psi_0, -2)$)

$$(N^\vee, \rho_{N^\vee}) = (N^\vee, \rho_N \cdot \beta_0)$$

with

$$\beta_0(f_i) = \omega_{k/k_0}(-1).$$

Then the see-saw identity reads:

$$\mathrm{Hom}_{\mathrm{U}(V_B)}(\pi_{\rho_M} \otimes \pi_{\rho_N}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(W_{B'})}(\theta_{\psi_0, -2, \mu^2, V_B, W_{B'}}(\pi_{\rho_{N^\vee}}) \otimes \omega_{\psi_0, -2, \mu, W_{B'}}(\pi_{N'}^\eta)).$$

The RHS is nonzero if and only if (i) and (ii) below hold.

(i) $\theta_{\psi_0, -2, \mu^2, V_B, W_{B'}}(\pi_{\rho_{N^\vee}}) \neq 0$. By proposition 11.1, this holds if and only if

$$\epsilon(N^\vee \mu^{-2}, \psi_{-2}) = \epsilon(B) \cdot \epsilon(B') = \rho_M(e),$$

or equivalently

$$\epsilon(N \otimes M_1, \psi) = \rho_M(e).$$

Moreover, by Theorem 11.2, when this holds, we have

$$\theta_{\psi_0, \mu^2, V_B, W_{B'}}((\pi_{\rho_{N^\vee}})) = \pi_{\rho_{N^\vee} \cdot \rho_0}$$

where ρ_0 is the character of $A_{N^\vee} = A_N$ given by

$$\rho_0(f_i) = \epsilon(N_i^\vee \mu^{-2}, \psi_{-1}) = \epsilon(N_i \otimes M_1, \psi).$$

(ii) $\text{Hom}_{\text{U}(W_{B'})}(\pi_{\rho_{N^\vee} \cdot \rho_0} \otimes \omega_{\psi_0, -2, \mu, W_{B'}}, \pi_{N'}^\eta) \neq 0$. This question was addressed in the previous section, and we deduce that the desired non-vanishing holds if and only if the character

$$(\rho_N \cdot \rho_0, \eta) \in \text{Irr}(A_N) \times \text{Irr}(A_{N'})$$

is the distinguished character χ_0 in [GGP, Conjecture 16.3] for the skew-hermitian case for $(W_{B'}, \mu)$. More precisely, the desired non-vanishing holds if and only if

$$\rho_N(f_i) \cdot \epsilon(N_i \otimes M_1, \psi) = \epsilon(N_i^\vee \otimes (N')^\vee(\mu), \psi_{-1}) = \epsilon(N_i \otimes M_2, \psi),$$

so that

$$\rho_N(f_i) = \epsilon(N_i \otimes M, \psi),$$

and

$$\eta(e_i) = \epsilon((N'_i)^\vee \otimes N^\vee(\mu), \psi_{-1}) = \epsilon(M_{2,i} \otimes N, \psi).$$

This shows that

$$\text{Hom}_{\text{U}(V_B)}(\pi_{\rho_M} \otimes \pi_{\rho_N}, \mathbb{C}) \neq 0$$

if and only if the character $\rho_M \times \rho_N$ is the distinguished character χ_0 of [GGP, Conjecture 16.3], computed using the additive character ψ of k/k_0 .

14. RESTRICTION FROM U(3) TO U(2): STABLE CASE

We now consider the restriction problem for stable Vogan packets of U(3). We preserve the notation of the previous sections. In particular, we have the pairs of spaces $W \subset V$ and $W' \subset V'$, with $\text{disc}(V) = 1 = \text{disc}(V/W)$. Moreover, we use the additive character ψ_{-2} to normalize the Vogan parameterization for U(W).

Let M be an irreducible 3-dimensional conjugate-orthogonal representation of $WD(k)$, so that its associated Vogan packet has the form

$$\Pi_M = \{\pi_M, \pi'_M\},$$

where π_M is a representation of $U(V)$ and π'_M is the same representation considered on $U(V')$. If M is an irreducible representation of the Weil group $W(k)$, then the representation π_M is supercuspidal. Otherwise,

$$M = \mu \boxtimes S_3$$

where μ is a conjugate-orthogonal character of $W(k)$ and S_3 denotes the irreducible 3-dimensional representation of $SL_2(\mathbb{C})$. In this case, the representation π_M is a twisted Steinberg representation

$$\pi_M = St \otimes (\mu \circ \det).$$

On the other hand, let N be an arbitrary 2-dimensional conjugate-symplectic representation of $WD(k)$ with associated Vogan packet Π_N of $U(W)$. We would like to determine

$$\mathrm{Hom}_{U(W)}(\pi_M \otimes \pi_N, \mathbb{C})$$

for $\pi_M \in \Pi_M$ and $\pi_N \in \Pi_N$. We shall reduce this question to the case when Π_M and Π_N are both supercuspidal packets, by first treating the other cases directly. The supercuspidal case will then be handled by a global method.

We first consider the case when $M = \mu \boxtimes S_3$. Since we can absorb the twist by μ into the parameter N , we may assume without loss of generality that $\mu = 1$. In this case, $\pi_M = St$ is a quotient of a (un-normalized) principal series representation:

$$0 \longrightarrow 1 \longrightarrow \mathrm{Ind}_B^{U(V)}(1) \longrightarrow St \longrightarrow 0.$$

We have:

Proposition 14.1. *(i) If N is not the parameter of the Steinberg representation of $U(W)$, we have*

$$\mathrm{Hom}_{U(W)}(St \otimes \pi_{\rho_N}, \mathbb{C}) = \mathrm{Hom}_{U(W)}(I_B(1) \otimes \pi_{\rho_N}, \mathbb{C}) = \mathrm{Hom}_{U(L)}(\pi_{\rho_N}, \mathbb{C}).$$

In particular, $\mathrm{Hom}_{U(W)}(St \otimes \pi_{\rho_N}, \mathbb{C}) \neq 0$ if and only if

$$\rho_N(f_i) = \epsilon(N_i, \psi) = \epsilon(N_i \otimes M, \psi).$$

(ii) If N is the parameter of the Steinberg representation of $U(W)$, so that $\Pi_N = \{St_{U(W)}, 1_{U(W')}\}$, we have

$$\mathrm{Hom}_{U(W)}(St \otimes St_{U(W)}, \mathbb{C}) = \mathrm{Hom}_{U(W)}(I_B(1) \otimes St_{U(W)}, \mathbb{C}) \neq 0.$$

On the other hand,

$$\mathrm{Hom}_{U(W')}(1_{U(V')}, \mathbb{C}) + \mathrm{Hom}_{U(W')}(St_{U(V')}, \mathbb{C})$$

is equal to

$$\mathrm{Hom}_{U(W')}(1_{U(W')}, \mathbb{C}) = \mathrm{Hom}_{U(L')}(1_{U(W')}, \mathbb{C}) = \mathbb{C},$$

so that

$$\mathrm{Hom}_{\mathrm{U}(W')}(\mathrm{St}_{\mathrm{U}(V')}, \mathbb{C}) = 0.$$

This proposition verifies [GGP, Conjecture 16.3] when $M = \mu \otimes S_3$ and N is arbitrary. We may thus restrict attention to the case when M is an irreducible representation of $W(k)$, so that Π_M is a stable supercuspidal packet.

Next, we consider the case when

$$N = P + (P^\sigma)^\vee \quad \text{or} \quad \mu \otimes S_2,$$

where P and $(P^\sigma)^\vee$ are not necessarily distinct. In such cases, the associated representations of $\mathrm{U}(W)$ are contained in principal series representations of $\mathrm{U}(W)$. Thus, we need to compute:

$$\mathrm{Hom}_{\mathrm{U}(W)}(\pi_M, \mathrm{Ind}_B^{\mathrm{U}(W)}(\chi))$$

for a supercuspidal representation π_M of $\mathrm{U}(V)$. By Frobenius reciprocity, we see that this is equal to

$$\mathrm{Hom}_T((\pi_M)_Z, \chi)$$

where the unipotent radical Z of $B \subset \mathrm{U}(W)$ coincides with the center of the unipotent radical of a Borel subgroup of $\mathrm{U}(V)$. But $(\pi_M)_Z$ is isomorphic to the regular representation $\mathcal{S}(k^\times)$ of $T \cong k^\times$. Thus we have

$$\mathrm{Hom}_{\mathrm{U}(W)}(\pi_M, \mathrm{Ind}_B^{\mathrm{U}(W)}(\chi)) = \mathrm{Hom}_{k^\times}(\mathcal{S}(k^\times), \chi) = \mathbb{C}.$$

This proposition is proved by a standard application of Mackey theory. Indeed, it is a special case of [GGP, Theorem 15.1], and so we omit its proof here. The main point to note is that the proposition verifies [GGP, Conjecture 16.3] when

$$N = P + (P^\sigma)^\vee \quad \text{with } P \neq (P^\sigma)^\vee,$$

as the principal series representation on $\mathrm{U}(W)$ is irreducible. If $P = (P^\sigma)^\vee$, the parameter N is dihedral with respect to k/k_0 and the corresponding principal series representation of $\mathrm{U}(W)$ is the sum of two irreducible summands, and we have not determined which of these summands contribute to the 1-dimensional Hom space above.

Finally, when $N = \mu \otimes S_2$, we may assume without loss of generality that $\mu = 1$ (by absorbing μ into M). Then

$$\Pi_N = \{\mathrm{St}_{\mathrm{U}(W)}, 1_{\mathrm{U}(W')}\},$$

and

$$0 \longrightarrow 1_{\mathrm{U}(W)} \longrightarrow \mathrm{Ind}_B^{\mathrm{U}(W)} 1 \longrightarrow \mathrm{St}_{\mathrm{U}(W)} \longrightarrow 0.$$

The above computation shows that

$$\mathrm{Hom}_{\mathrm{U}(W)}(\pi_M, \mathrm{Ind}_B^{\mathrm{U}(W)}) = \mathbb{C}.$$

On the other hand, by [GRS], we have

$$\mathrm{Hom}_{\mathrm{U}(W)}(\pi_M, \mathbb{C}) = 0$$

and

$$\mathrm{Hom}_{\mathrm{U}(W)}(\pi'_M, \mathbb{C}) = 0.$$

Indeed, if these Hom spaces were not zero, π_M and π'_M would be obtainable as a theta lifting from some $\mathrm{U}(2)$, contradicting the fact that M is a stable parameter of $\mathrm{U}(3)$. Thus, we conclude that

$$\mathrm{Hom}_{\mathrm{U}(W)}(\pi_M, St_{\mathrm{U}(W)}) \neq 0,$$

which is what [GGP, Conjecture 16.3] predicts.

We are thus left with the case when M is an irreducible representation of $W(k)$ and N is a supercuspidal parameter. For this remaining case, we shall prove the following local theorem by global methods.

Theorem 14.2. *Let W be a 2 dimensional hermitian subspace of a hermitian space V of dimension 3 over k . Suppose that π_M (resp. π_N) is an irreducible supercuspidal representation of $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$) with irreducible Langlands parameter M (resp. N) of $W(k)$. Then $\epsilon(M \otimes N, \psi)$ for a character ψ of k/k_0 is independent of ψ , so may be denoted as $\epsilon(M \otimes N)$. Suppose that $\mathrm{Hom}_{\mathrm{U}(W)}(\pi_M \otimes \pi_N, \mathbb{C}) \neq 0$. Then*

$$\epsilon(M \otimes N) = \begin{cases} 1 & \text{if } \mathrm{U}(V) \times \mathrm{U}(W) \text{ is quasi-split} \\ -1 & \text{otherwise} \end{cases}$$

Remark : The method that we follow to prove this theorem is pretty general, but it is based on a global theorem of Ginzburg, Jiang, and Rallis [GJR3, theorem 4.6] which assumes that automorphic forms on unitary groups $\mathrm{U}(n)$ have base change to $\mathrm{GL}(n)$ something which is known at the moment only for generic automorphic representations on quasi-split unitary groups. However, by Rogawski [Ro], base change is known for any unitary group in 3 variables, which is why we have restricted ourselves to $\mathrm{U}(3)$ in the above theorem. Nonetheless, we have formulated some of the preliminary results below in greater generality.

We begin with the following globalization result about local fields, which will be applied to globalize hermitian spaces over local fields so that there is no ramification outside the place being considered, and the unitary groups at infinity are either compact, or of rank 1.

Lemma 14.3. *Let k be a quadratic extension of a non-archimedean local field k_0 . Then there exists a totally real number field F with k_0 as its completion, and a quadratic totally imaginary extension E of F with corresponding completion k such that E is unramified over F at all finite places different from k .*

Proof: Except for the requirement about E being unramified at any place of F outside the place k , this is well-known. Suppose that a quadratic extension E_1 over F_1 with

possible ramifications is constructed. Then a well-known technique, *crossing with a field*, says that after a suitable base change, one can get rid of the ramifications; we leave the details to the reader.

Lemma 14.4. *Let W be a hermitian space over k . Let F be a totally real number field with completion k_0 at a place of F , and let E be a quadratic totally imaginary extension of F with corresponding completion k . Then there is a hermitian space \mathbb{W} over E giving rise to W over k in such a way that the corresponding unitary group is quasi-split at all finite places of F except the one corresponding to the completion k_0 ; and at all but one infinite place the group is the compact group $U(n)$, and at the remaining infinite place, the group is either $U(n)$, or $U(n-1, 1)$; if n is odd, we can assume that the group is compact at all the infinite places.*

Proof : The proof of the lemma will depend on the well-known classification of a hermitian form over a number field, according to which a hermitian form over a number field is determined by

- (1) the normalized discriminant, and
- (2) the signatures at the infinite places.

Moreover, given any normalized discriminant, and signatures at infinite places (except for obvious compatibility between normalized discriminant and signatures), there is a hermitian form.

We also note the following exact sequence from classfield theory,

$$0 \rightarrow F^\times / NE^\times \rightarrow \mathbb{A}_F^\times / \mathbb{N}\mathbb{A}_E^\times \rightarrow \text{Gal}(E/F) \rightarrow 0,$$

from which it follows that one can construct an element in F^\times which is trivial in F_v^\times / NE_v^\times at all the finite places except k_0 , and which at the infinite places has the desired signs, except that the product of the signs is 1 or -1, depending on whether the element in $k_0^\times / \mathbb{N}k^\times$ is trivial or nontrivial.

The proof of the lemma is now completed by observing that a hermitian form of normalized discriminant 1 over a non-archimedean local field defines a quasi-split group, and that the normalized discriminants of the hermitian spaces $Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + \dots + Z_n\bar{Z}_n$, and $Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + \dots + Z_{n-1}\bar{Z}_{n-1} - Z_n\bar{Z}_n$ over \mathbb{C} are negative of each other, and if n is odd, the normalized discriminants of the hermitian spaces $Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + \dots + Z_n\bar{Z}_n$, and $-(Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + \dots + Z_n\bar{Z}_n)$ are negative of each other.

Corollary 14.5. *Let W be a hermitian subspace of codimension 1 of a hermitian space V over k . Let F be a totally real number field with completion k_0 at a place of F , and let E be a quadratic totally imaginary extension of F with corresponding completion k . Then there is a hermitian subspace \mathbb{W} of codimension 1 of a hermitian space \mathbb{V} over E giving rise to W and V over k in such a way that the corresponding unitary groups are quasi-split at all the finite places of F except the one corresponding to the completion k_0 ; assuming $F \neq \mathbb{Q}$, the group $U(\mathbb{V})$ is the compact group $U(n+1)$ at all but two infinite places, and at the remaining infinite places, the group is either $U(n+1)$, or $U(n, 1)$; the subgroup $U(\mathbb{W})$ is compact at all but possibly one infinite place.*

Proof : Let $V = W \oplus L_c$ where L_c is k with the hermitian structure $cZ\bar{Z}$, $c \in k_0^\times$. Globalize W by the previous lemma, and globalize c so that the normalized discriminant of V is 1 at all the finite places of F other than k_0 (so that $U(\mathbb{V})$ is quasi-split at all the finite places of F other than k_0), and so that c has arbitrary signs at infinity, with only the product of the signs pre-determined which allows for the desired conclusion.

We omit a proof of the following corollary of the lemma which follows exactly as in the previous corollary.

Corollary 14.6. *If $n \equiv 1, 2 \pmod{4}$, then hermitian spaces $W \subset V$ of dimensions $n, n+1$ can be globalized keeping them positive definite at infinity, and maximally split at all finite places other than k . For $n \equiv 2 \pmod{4}$, if W has an isotropic subspace of dimension $n/2$, and for $n \equiv 1 \pmod{4}$, if V has an isotropic subspace of dimension $(n+1)/2$, then there are an even number of real places in F , else an odd number of real places.*

Proof of Theorem 14.2: By the corollary above, we can assume that $U(\mathbb{V})$ is compact at infinity. It is then easy to see that we can globalize the representation π_M of $U(V)$ to an automorphic representation Π_1 of $U(\mathbb{V})(\mathbb{A})$ in such a way that it is unramified at all the finite places of F except k_0 .

By Lemma 1 of [P6], we can globalize π_N to an automorphic representation Π_0 such that the period integral

$$\int_{U(\mathbb{W}) \backslash U(\mathbb{W})(\mathbb{A})} f_0 f_1 \neq 0,$$

for some f_0 in Π_0 , and f_1 in Π_1 .

By the theorems due to Ginzburg, Jiang, and Rallis, cf. [GJR3, theorem 4.6], since the period integral is nonzero, the central critical L -value,

$$L\left(\frac{1}{2}, \Pi_0^E \otimes \Pi_1^E\right) \neq 0,$$

where Π_0^E and Π_1^E denote base change of Π_0 and Π_1 to E .

This implies that the global root number,

$$\epsilon\left(\frac{1}{2}, \Pi_0^E \otimes \Pi_1^E\right) = 1.$$

Let

$$\Pi_0 = \otimes_w \Pi_{0,w}, \text{ and } \Pi_1 = \otimes_w \Pi_{1,w},$$

with $\Pi_{0,v} = \pi_N$, and $\Pi_{1,v} = \pi_M$. From the nonvanishing of the period integral, it follows that

$$\text{Hom}_{U(\mathbb{W}_w)}(\Pi_{0,w} \otimes \Pi_{1,w}, \mathbb{C}) \neq 0$$

for all places w of F .

Since the representations $\Pi_{1,w}$ for w , a finite place of F , not v , are unramified by construction, they are in particular quotients of principal series representations, and hence we know the validity of theorem 14.2 for such representations:

$$\epsilon_w\left(\frac{1}{2}, \Pi_{0,w}^E \otimes \Pi_{1,w}^E\right) = 1$$

for all finite places w of F except v .

Since the global epsilon factor is a product of local epsilon factors, we have

$$\epsilon\left(\frac{1}{2}, M \otimes N\right) \epsilon_\infty\left(\frac{1}{2}, \Pi_{0,\infty}^E \otimes \Pi_{1,\infty}^E\right) = 1.$$

The following lemmas then complete the proof of the theorem on noting that there are an even number of places at infinity if $U(\mathbb{W})$ is quasi-split, and odd number of places at infinity when $U(\mathbb{W})$ is not quasi-split.

Lemma 14.7. *Let W be a codimension 1 hermitian subspace of a positive definite hermitian space V of dimension $n + 1$ over \mathbb{C} . Suppose that π_0 (resp. π_1) is a finite dimensional irreducible representation of $U(V)$ (resp. $U(W)$). Let the Langlands parameter of π_0 (resp. π_1) be σ_0 (resp. σ_1). Suppose that $\text{Hom}_{U(W)}(\pi_0 \otimes \pi_1, \mathbb{C}) \neq 0$. Then*

$$\epsilon(\sigma_0 \otimes \sigma_1) = \begin{cases} 1 & \text{if } n \equiv 0, 3 \pmod{4} \\ -1 & \text{if } n \equiv 1, 2 \pmod{4}. \end{cases}$$

Proof: The proof of this lemma is a simple consequence of the well-known branching law from the compact group $U(n + 1)$ to $U(n)$, combined with the value of the epsilon factor given by the following lemma, which has been demonstrated in Proposition 2.1.

Lemma 14.8. *Let ψ be the additive character on \mathbb{C} given by $\psi(z) = e^{-2\pi iy}$ where $z = x + iy$. For n an integer, let χ_n denote the character $\chi_n(z) = (\bar{z}/z)^{n/2} = e^{-ni\theta}$ for $z = re^{i\theta} \in \mathbb{C}^\times$. Then for n odd,*

$$\epsilon(\chi_n, \psi) = \begin{cases} 1 & \text{if } n > 0 \\ -1 & \text{if } n < 0. \end{cases}$$

Lemma 14.9. *Let π_0 (resp. π_1) be a finite dimensional irreducible representation of the compact group $U(n)$ (resp. $U(n + 1)$) with L -parameter restricted to \mathbb{C}^\times given by an n -tuple of half-integers $\sigma_0 = \{-\lambda_n < -\lambda_{n-1} < \dots < -\lambda_1\}$ (resp. $\sigma_1 = \{\mu_1 < \mu_2 < \dots < \mu_{n+1}\}$ an $(n + 1)$ -tuple of half-integers), where all the λ_i 's are half-integers but not integers if n is even, and are integers if n is odd, and μ_i 's are all integers if n is even, and half-integers but not integers if n is odd. Then $\text{Hom}_{U(n)}(\pi_1 \otimes \pi_0, \mathbb{C}) \neq 0$ if and only if*

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \dots < \lambda_n < \mu_{n+1}.$$

Corollary 14.10. *With notation as in the lemma, and assuming that π_0^\vee appears in π_1*

$$\epsilon(\chi_{\mu_k} \otimes \sigma_0) = (-1)^{n-k+1}, \text{ for all } k,$$

and therefore,

$$\begin{aligned} \epsilon(\sigma_1 \otimes \sigma_0) &= \prod_{k=1}^{n+1} (-1)^{n-k+1} \\ &= (-1)^{\frac{n(n+1)}{2}}. \end{aligned}$$

Remark : It should be mentioned that the global method followed in the proof of theorem 14.2 proves that if there is an invariant linear form, then the epsilon factor has some fixed value. The natural variant of the theorem of Waldspurger in [Wa4] to unitary groups proves that such an invariant form exists on a relevant pair of unitary groups, which will then strengthen theorem 14.2 to an if and only if statement.

REFERENCES

- [AGRS] A. Aizenbud, D. Gourevitch, S. Rallis and G. Schiffmann, *Multiplicity One Theorems*, ArXiv: 0709.4215; to appear in the *Annals of Mathematics*.
- [BZ] J. Bernstein and A. Zelevinsky, *Induced Representations of Reductive p -adic groups*, Ann. Scient. Ecole Norm. Sup. 10 (1977) 441-472.
- [DL] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) 103 (1976), no. 1, 103–161.
- [DR] S. Debacker and M. Reeder, *Depth-zero supercuspidal L -packets and their stability*, Annals of Math., 169, No.3, (2009), pages 795–901.
- [DR2] S. Debacker and M. Reeder, *Some very cuspidal generic representations*, preprint, available at <http://www2.bc.edu/reederma/vcgeneric.pdf>.
- [FQ] A. Fröhlich and J. Queyrut, *On the functional equation of an Artin L -function for characters of real representations*, Inv. Math. 20 (1973) 125-138.
- [GGP] W. T. Gan, Benedict H. Gross and D. Prasad, *Symplectic local root numbers, central critical L -values and restriction problems in the representation theory of classical groups*, preprint.
- [GJR1] D. Ginzburg, D. Jiang, and S. Rallis, *On the nonvanishing of the central value of the Rankin-Selberg L -functions*, J. Amer. Math. Soc. 17 (2004), no. 3, 679–722.
- [GJR2] D. Ginzburg, D. Jiang, and S. Rallis, *On the nonvanishing of the central value of the Rankin-Selberg L -functions, II*, Automorphic Representations, L -functions and Applications: Progress and Prospects, 157-191, Ohio State Univ. Math. Res. Inst. Publ. 11, de Gruyter, Berlin 2005.
- [GJR3] D. Ginzburg, D. Jiang, and S. Rallis, *Models for certain residual representations of unitary groups*, AMS Contemporary series (Gelbart volume); to appear (2008).
- [GRO] S. Gelbart and J. Rogawski, *L -functions and Fourier-Jacobi coefficients for the unitary group $U(3)$* , Invent. Math. 105 (1991), no. 3, 445–472.
- [GRS] S. Gelbart, J. Rogawski, and D. Soudry, *Endoscopy, theta-liftings, and period integrals for the unitary group in three variables*. Annals of Math, 145 (1997), 419-476.

- [GP1] B. Gross and D. Prasad, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}* . Canad. J. Math. 44 (1992), no. 5, 974–1002.
- [GP2] B. Gross and D. Prasad, *On irreducible representations of $SO_{2n+1} \times SO_{2m}$* . Canad. J. Math. 46 (1994), no. 5, 930–950.
- [GR] B. Gross and M. Reeder, *From Laplace to Langlands via representations of orthogonal groups*, Bulletin of the AMS 43 (2006) 163–205.
- [Ha] M. Harris, *L-functions of 2×2 unitary groups and factorization of periods of Hilbert modular forms*, Journal of AMS 6 (1993), 637–719.
- [HKS] M. Harris, S. Kudla and W. J. Sweet, *Theta dichotomy for unitary groups*, J. of the AMS, vol. 9, No. 4 (1996), 941–1004.
- [II] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, to appear in Geometric and Functional Analysis.
- [K] S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math. 87 (1994), no. 1-3, 361–401.
- [KK] K. Konno and T. Konno, *CAP automorphic representations of $U_{E/F}(4)$ I. Local a -packets*, Kyushu Univ. Preprint Series in Mathematics 2003–2004.
- [Ko] T. Konno, *Local Gross-Prasad conjecture for $U(2)$* , slides of a talk given in *Workshop on Representation theory and Automorphic Forms*, National Univ. of Singapore, June (2008).
- [LL] J.-P. Labesse and R. Langlands, *L-indistinguishability for $SL(2)$* , Canad. J. Math. 31 (1979), no. 4, 726–785.
- [Mo] C. Moen, *The dual pair $(U(3), U(1))$ over a p -adic field*, Pacific J. Math. 127:1 (1987), 141–154.
- [MVW] C. Mœglin, M.-F. Vignéras and J.-L. Waldspurger, *Correspondances de Howe sur un corps p -adique*, Lecture Notes in Mathematics, 1291. Springer-Verlag, Berlin, 1987. viii+163 pp.
- [P1] D. Prasad, *Trilinear forms for representations of $GL(2)$ and local epsilon factors*, Compositio Math, vol. 75 (1990), 1–46.
- [P2] D. Prasad, *Invariant forms for representations of GL_2 over a local field*, Amer. J. Math. 114 (1992), no. 6, 1317–1363.
- [P3] D. Prasad, *On an extension of a theorem of Tunnell*, Compositio 94 (1994), 19–28.
- [P4] D. Prasad, *Some applications of seesaw duality to branching laws*, Math. Annalen, vol. 304 (1996), 1–20.
- [P5] D. Prasad, *Theta correspondence for Unitary groups*, Pacific J. of Math. 194 (2000), 427–438.
- [P6] D. Prasad, *Relating invariant linear form and local epsilon factors via global methods, with an appendix by H. Saito*, Duke J. Math 138 (2007) no. 2, 233–261.
- [PT] D. Prasad and R. Takloo-Bighash, *Bessel models for $GSp(4)$* , preprint.
- [Pa] A. Paul, *Howe correspondence for real unitary groups*, J. Funct. Anal. 159 (1998), no. 2, 384–431.
- [R] S. Rallis, *On the Howe duality conjecture*, Compositio Math. 51 (1984), no. 3, 333–399.
- [Ro] J. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol 123, Princeton University Press, 1990.
- [Ro2] J. Rogawski, *The multiplicity formula for A -packets*, in *The zeta functions of Picard modular surfaces*, edited by R. Langlands, and D. Ramakrishnan, 395–419, . Univ. Montral, Montreal, QC, 1992.
- [Sa] H. Saito, *On Tunnell’s formula for characters of $GL(2)$* . Compositio Math. 85 (1993), no. 1, 99–108.
- [Sa2] H. Saito, *Two remarks on a theorem of Dipendra Prasad*, Pacific J. Math. 234 (2008), no. 1, 185–199.
- [T] J. Tate, “Number theoretic background,” in Automorphic forms, Representations, and L -functions, Proc. Symposia Pure Math., AMS, 33, 3–26 (1979).

- [Th] E. Thoma, *Die Einschränkung der Charakter von $GL(n, \mathbb{F}_q)$ auf $GL(n-1, \mathbb{F}_q)$* . Math Z. 119 (1971) 321-338.
- [Tu] J. Tunnell, *Local ϵ -factors and characters of $GL(2)$* . Amer. J. Math. 105 (1983), no. 6, 1277–1307.
- [Wa1] J.-L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures Appl. (9) 59 (1980), no. 1, 1–132.
- [Wa2] J.-L. Waldspurger, *Correspondances de Shimura et quaternions*, Forum Math. 3 (1991), no. 3, 219–307.
- [Wa3] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symtrie*. Compositio Math. 54 (1985), no. 2, 173–242.
- [Wa4] J.-L. Waldspurger, *Une Formule intégrale reliée à la conjecture locale de Gross-Prasad*, arXiv: 0902.1875.
- [Wa5] J.-L. Waldspurger, *Une Formule intégrale reliée à la conjecture locale de Gross-Prasad, 2ème partie: extension aux représentations tempérées*, arXiv: 0904.0314.

W.T.G.: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT SAN DIEGO, 9500 GILMAN DRIVE, LA JOLLA, 92093
E-mail address: wgan@math.ucsd.edu

B.H.G: DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138
E-mail address: gross@math.harvard.edu

D.P.: SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, COLABA, MUMBAI-400005, INDIA
E-mail address: dprasad@math.tifr.res.in