

**THE DUAL PAIR  $G_2 \times PU_3(D)$   
( $p$ -ADIC CASE)**

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INTRODUCTION

Let  $F$  be a  $p$ -adic field of residue characteristic not 2, and let  $D$  be the non-split quaternion algebra over  $F$ . Associated to  $D$ , there is a linear adjoint algebraic group  $\underline{H}_D$  over  $F$  of type  $E_7$ , and relative rank 4. We shall let  $H_D$  denote the group of  $F$ -points of  $\underline{H}_D$ . There is a reductive dual pair

$$G_2 \times PU_3(D) = G \times G' \subset H_D$$

Here,  $G_2$  is split and  $PU_3(D)$  is an inner form of  $PGSp_6$  of relative rank one. In this paper, we study the dual pair correspondence which arises from the restriction of the minimal representation  $\Pi$  of  $H_D$  to the dual pair  $G_2 \times PU_3(D)$ . Recall that the minimal representation is the analogue of the Weil representation of the metaplectic group. As usual, if  $\sigma$  is an irreducible admissible representation of  $G_2$ , we let  $\Theta(\sigma)$  denote the set of irreducible admissible representations  $\sigma'$  of  $PU_3(D)$ , counted with multiplicities, such that  $\sigma \otimes \sigma'$  is a quotient of  $\Pi$ . Similarly, we have the set  $\Theta(\sigma')$ . Then we determine the sets  $\Theta(\sigma)$  and  $\Theta(\sigma')$  when  $\sigma$  and  $\sigma'$  are non-cuspidal representations.

The techniques used in this work can already be found in [MS], where, among other things, the correspondence of tempered spherical representations was determined for the dual pair  $G_2 \times PGSp_6$  in the split adjoint group of type  $E_7$ . The point is that, to determine the correspondence of non-cuspidal representations, one reduces to the determination of the Jacquet modules of  $\Pi$  with respect to the maximal parabolic subgroups of  $G_2$  and  $PU_3(D)$ . There are essentially two steps involved in this. First, we determine the restriction of  $\Pi$  to certain maximal parabolic subgroups of  $H_D$ . Unlike the case of the Weil representation, where we have several different smooth models of the representation at hand, we compute the restriction just by using the fact that  $\Pi$  is minimal. The second step is essentially an orbit computation, involving the internal modules of the groups in question.

Let us describe the main results of the paper. Let  $\pi'$  be an irreducible (finite-dimensional) representation of  $D^\times$ . In [JL], Jacquet and Langlands associated to  $\pi'$  a square-integrable representation  $\pi = JL(\pi')$  of  $GL_2$ , an inner form of  $D^\times$ . Let  $Q$  be the minimal parabolic subgroup of  $PU_3(D)$ . The Levi factor  $L$  of  $Q$  is given by:

$$L \cong D^\times \times D^\times / F^\times$$

where  $F^\times$  is embedded via  $t \mapsto (t, t^{-1})$ . Using unnormalized induction throughout the paper, let

$$\sigma' = \text{Ind}_Q^{G'} \delta_Q^{1/2}(\pi' \times \pi')$$

where  $\delta_Q$  is the modulus character of  $Q$ . Also, we let  $Q_1 = L_1U_1$  (respectively  $Q_2 = L_2U_2$ ) be the maximal parabolic subgroup of  $G_2$  whose Levi factor is generated by the long root (respectively short root) of  $G_2$ , and let

$$\begin{cases} \sigma_1 = \text{Ind}_{Q_1}^G \delta_{Q_1}^{1/2} \pi \\ \sigma_2 = \text{Ind}_{Q_2}^G \delta_{Q_2}^{1/2} \pi \end{cases}$$

For the ease of exposition, let us suppose that  $\pi$  is supercuspidal and both  $\sigma_1$  and  $\sigma'$  are irreducible, which is true generically. Then we shall show:

$$\begin{cases} \Theta(\sigma_1) = \{\sigma'\} \\ \Theta(\sigma') = \{\sigma_1\} \end{cases}$$

and

$$\Theta(\sigma_2) = \emptyset.$$

Also, let  $St$  (respectively  $St'$ ) be the Steinberg representation of  $G$  (respectively  $G'$ ). Then

$$\Theta(St') = \{St\}.$$

Note that the above results are predicted by the natural inclusion of  $L$ -groups, that is, they respect Langlands functoriality. We refer the reader to [G], for more precise results, and a global variant of this correspondence.

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#### 1. REPRESENTATIONS OF $\ell$ -GROUPS

In this first section, we establish some notation and discuss some basic facts on induced representations that are required later.

Let  $\mathfrak{G}$  be the  $F$ -points of a reductive algebraic group over  $F$ . Then recall that  $\mathfrak{G}$  is an  $\ell$ -group, in the terminology of [BZ]. Let  $\text{Alg}(\mathfrak{G})$  be the category of smooth representations of  $\mathfrak{G}$ , and let  $\text{Irr}(\mathfrak{G})$  be the set of simple objects of  $\text{Alg}(\mathfrak{G})$ .

Recall that if  $\mathfrak{P} = \mathfrak{M}\mathfrak{N}$  is a parabolic subgroup of  $\mathfrak{G}$ , then we have an exact functor

$$\text{Ind}_{\mathfrak{P}}^{\mathfrak{G}} : \text{Alg}(\mathfrak{M}) \longrightarrow \text{Alg}(\mathfrak{G})$$

whose left and right adjoints are given by the functors of co-invariants (Jacquet functors). More precisely, let  $\bar{\mathfrak{N}}$  be the opposite unipotent radical. Then

$$(1.1) \quad \begin{cases} \text{Hom}_{\mathfrak{G}}(\pi, \text{Ind}_{\mathfrak{P}}^{\mathfrak{G}}(\delta^{1/2}\sigma)) = \text{Hom}_{\mathfrak{M}}(\pi_{\mathfrak{N}}, \delta^{1/2}\sigma) \\ \text{Hom}_{\mathfrak{G}}(\text{Ind}_{\mathfrak{P}}^{\mathfrak{G}}(\delta^{1/2}\sigma), \pi) = \text{Hom}_{\mathfrak{M}}(\bar{\delta}^{1/2}\sigma, \pi_{\bar{\mathfrak{N}}}) \end{cases}$$

where,  $\delta$  and  $\bar{\delta}$  are the moduli characters of  $\mathfrak{P}$  and  $\bar{\mathfrak{P}}$ .

Finally, we record below an easy lemma:

**Lemma 1.2.** *Let  $0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$  be an exact sequence of  $\mathfrak{G}$ -modules. If the center of  $\mathfrak{G}$  acts on  $V_1$  and  $V_3$  by different eigenvalues, then the sequence is split.*

## 2. JACQUET-LANGLANDS CORRESPONDENCE AND WEIL REPRESENTATION

In this section, we recall the local Jacquet-Langlands correspondence from [JL]. First of all, there is a natural bijection between regular elliptic conjugacy classes of  $GL_2(F)$  and  $D^\times$ . For each  $\pi$  in  $\text{Irr}(GL_2(F))$ , we write  $\text{ch}_\pi$  for its character, which is, by a well-known result of Harish-Chandra, a locally integrable function, locally constant on the set of all regular conjugacy classes. Then there exists a bijection  $\pi \leftrightarrow \pi'$  between the set of all classes of irreducible essentially square integrable representations of  $GL_2(F)$  and the set of all classes of irreducible representations of  $D^\times$  characterized by

$$-\text{ch}_\pi = \text{ch}_{\pi'}$$

on the set of regular elliptic conjugacy classes. Note that  $\pi$  is supercuspidal if the dimension of  $\pi$  is greater than one. Otherwise,  $\pi$  is a special representation in the terminology of [JL].

Let  $\psi$  be an additive (unitary) character  $\psi$  of  $F$  of conductor  $\mathcal{O}_F$ , the ring of integers of  $F$ . As in the introduction,  $D$  is the unique non-split quaternion algebra over the  $p$ -adic field  $F$ . Let  $\text{Tr}$  and  $\mathbb{N}$  be the reduced trace and reduced norm on  $D$ . For  $x \in D$ ,  $\bar{x}$  denotes its conjugate with respect to the canonical anti-involution on  $D$ . The Jacquet-Langlands correspondence can be constructed using (a particular case of) the Weil representation  $W$  of  $SL_2(F)$ . The space of  $W$  is the space of Schwarz function  $S(D)$  on  $D$ , and the action of  $SL_2(F)$  is completely specified by:

$$\begin{cases} W \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \Phi(x) = |a|^2 \Phi(ax) \\ W \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi(x) = \psi(b\mathbb{N}(x)) \Phi(x) \\ W \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi(x) = -\hat{\Phi}(x) \end{cases}$$

Here,  $\hat{\Phi}$  denotes the Fourier transform of  $\Phi$  with respect to the Haar measure on  $D$  determined by the character  $\psi \circ \text{Tr}$  of  $D$ .

Let  $\widetilde{R} = GL_2(F) \times (D^\times \times D^\times / F^\times)$ . The action of  $SL_2$  on  $S(D)$  defined above, extends to the action (also denoted by  $W$ ) of

$$R = \{(g, \alpha, \beta) \in \widetilde{R} : \det(g) = \mathbb{N}(\alpha\beta)\}.$$

To describe this action, it suffices to do so for elements of the form  $(g, \alpha, \beta)$ , with

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the action is given by

$$W(g, \alpha, \beta)\Phi(x) = |a|\Phi(\bar{\alpha}x\beta).$$

Now let

$$\widetilde{W} = \text{ind}_R^{\widetilde{R}} W$$

For a general discussion of the representation  $\widetilde{W}$ , we refer the reader to [Ro], where theta correspondences for similitude groups is treated at length. We now state the result of Jacquet and Langlands in terms of the action of  $\widetilde{R}$  on  $\widetilde{W}$ :

**Theorem 2.1.** *Let  $\pi'$  be an irreducible representation of  $D^\times$ , and let  $\tilde{\pi}$  be the contragredient of  $JL(\pi')$ . Let  $\Theta(\pi' \otimes \pi')$  be the set of irreducible admissible representations  $\sigma$  of  $GL_2(F)$  such that  $\pi' \otimes \pi' \otimes \sigma$  is a quotient of  $\widetilde{W}$ . Then*

$$\begin{cases} \Theta(\pi' \otimes \pi') = \{\tilde{\pi}\} \\ \Theta(\tilde{\pi}) = \{\pi' \otimes \pi'\} \end{cases}$$

### 3. GROUPS AND DUAL PAIRS

In this section, we describe the various groups that will be studied in this paper. In particular, we shall describe the group  $H_D$ , and the dual pair  $G_2 \times PU_3(D)$ .

Let  $J = J(D)$  be the Jordan algebra of 3-by-3 hermitian matrices with coefficients in  $D$ :

$$X = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix}.$$

Note that the algebra  $J$  has a natural decomposition

$$J = \oplus J_{ij} \quad (1 \leq i \leq j \leq 3)$$

where  $J_{ij}$  consists of matrices  $X$  in  $J$  whose entries are 0 at all positions different from  $(i, j)$  and  $(j, i)$ . In particular,

$$\begin{cases} J_{ii} \cong F \\ J_{ij} \cong D \text{ if } i < j. \end{cases}$$

Recall that  $J$  has a natural cubic form  $\det$ , the determinant form, which in turn gives rise to a symmetric trilinear form such that  $(X, X, X) = 6 \det(X)$ . The value  $(X, Y, Z)$  is given by

$$(3.1) \quad \begin{aligned} (X, Y, Z) &= 2 \operatorname{Tr}(XYZ) + \operatorname{Tr}(X) \operatorname{Tr}(Y) \operatorname{Tr}(Z) \\ &\quad - \operatorname{Tr}(X) \operatorname{Tr}(YZ) - \operatorname{Tr}(Y) \operatorname{Tr}(XZ) - \operatorname{Tr}(Z) \operatorname{Tr}(XY) \end{aligned}$$

If  $X$  and  $Y$  are in  $J$ , then let  $X \times Y \in J^*$  be the element such that

$$(X \times Y)(Z) = (X, Y, Z)$$

for all  $Z \in J$ . Finally, we note that  $(X, Y) = \operatorname{Tr}(XY)$  defines a symmetric bilinear form, which can be used to identify  $J$  and  $J^*$ .

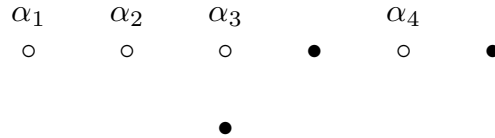
Let  $L_D$  be the algebraic group of linear transformation on  $J$  which preserves the determinant form. Then it is well-known that

$$L_D \cong SL_3(D)/\mu_2$$

(as algebraic groups). The action of  $L_D$  on  $J$  is given, for  $X \in J$ , by:

$$X \mapsto gX\bar{g}^t$$

where  $g \in SL_3(D)$ . Moreover  $GL_3(D)$  also acts on  $J$  by the same formula, and preserves the form  $\det$  up to scaling. Note that  $L_D$  has center  $\mu_3$ , and relative root system of type  $A_2$ . We let  $\mathfrak{l}_D \cong \mathfrak{sl}_3(D)$  be its Lie algebra. Associated to  $D$ , there is a linear algebraic group  $H_D$  over  $F$  which is adjoint, of type  $E_7$  and relative rank 4. The Satake diagram of  $H_D$  is:



The group  $H_D$  has relative root system  $F_4$ , and each short root space has dimension 4, and can be given the structure of  $D$ . Moreover [T; pg. 66],  $H_D$  has unique (up to conjugacy) special maximal compact subgroup  $K$ , whose reductive quotient is a finite group of type  ${}^2G_m \times {}^2E_6$ , where  ${}^2G_m$  is the group of norm one elements in the quadratic extension  $\mathbb{K}$  of the residue field  $\mathbb{F}$  of  $F$ .

Now we describe the Lie algebra  $\mathfrak{h}$  of  $H_D$ , following the construction in [Ru]. Notice that  $\mathfrak{h}$  has a subalgebra  $\mathfrak{h}_0 = \mathfrak{sl}_3 \oplus \mathfrak{sl}_3(D)$ . This is obtained by covering the vertex  $\alpha_2$  in the Satake diagram. Via the adjoint action,  $\mathfrak{h}$  decomposes as a  $\mathfrak{h}_0$ -module:

$$(3.2) \quad \mathfrak{h} = \mathfrak{sl}_3 \oplus \mathfrak{sl}_3(D) \oplus (V \otimes J) \oplus (V^* \otimes J^*)$$

where

$$\begin{cases} V = \langle e_1, e_2, e_3 \rangle \\ V^* = \langle e_1^*, e_2^*, e_3^* \rangle \end{cases}$$

are the standard representation of  $\mathfrak{sl}_3(F)$  and its dual.

As for the bracket relations, most of them are obvious, except for the bracket between two elements of  $V \otimes J$ , and the bracket between an element of  $V \otimes J$  and an element of  $V^* \otimes J^*$ . For the former, we have

$$[e_i \otimes X, e_j \otimes Y] = \pm e_k^* \otimes X \times Y$$

where  $\pm$  is the sign of permutation  $(i, j, k)$  of  $(1, 2, 3)$ . For the latter, we refer the reader to [Ru] for details.

Using the realization (3.2), we can describe the dual pair  $G_2 \times PU_3(D)$  very easily. Indeed, let  $e$  be a generic element of  $J$ , i.e.  $\det(e) \neq 0$ , and let  $G' \subset L_D$  be the subgroup which fixes  $e$ . Then  $G'$  is isomorphic to  $PU_3(D)$ . Let  $G$  be the closed subgroup of  $H_D$  with Lie algebra:

$$\mathfrak{g} = \mathfrak{sl}_3 \oplus V \otimes e \oplus V^* \otimes e^*$$

where  $2e^* = e \times e$ . Then  $G$  is isomorphic to a split group of type  $G_2$ . It is easy to see that  $G$  and  $G'$  are mutual commutants in  $H_D$ , and so  $G \times G'$  is a reductive dual pair in  $H_D$ . Moreover, the choice of  $e$  is not important, as all such  $e$ 's are in the same orbit of  $GL_3(D)/\mu_2$ . Hence, for definiteness, we fix the following choice of  $e$ :

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

This gives a fixed embedding:

$$G \times G' \hookrightarrow H_D.$$

Finally, we note that under the identification of  $J$  and  $J^*$  defined by  $\text{Tr}(XY)$ , the element  $e$  corresponds to  $e^*$ .

#### 4. PARABOLIC SUBGROUPS

In this section, we describe various maximal parabolic subgroups of  $H_D$ . First we have the Heisenberg maximal parabolic subgroup  $P_2 = M_2 \cdot N_2$ , which corresponds to the vertex  $\alpha_1$  in the Satake diagram. Then  $N_2$  is a Heisenberg group with center  $Z_2$ . Note that  $N_2/Z_2$  is a representation of  $M_2$ , and we let  $\Omega$  be the minimal non-trivial  $M_2$ -orbit in  $\bar{N}_2/\bar{Z}_2$ ; it is the orbit generated by the highest weight vector. For a discussion of  $P_2$ , we refer the reader to [MS].

Now let  $P_1 = M_1 \cdot N_1$  be the maximal parabolic subgroup of  $H_D$  corresponding to the vertex  $\alpha_2$ . Then  $N_1$  is a 3-step nilpotent group. On the level of Lie algebras, let

$$s_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix},$$

and  $\mathfrak{h}_1(i) = \{x \in \mathfrak{h} : [s_1, x] = ix\}$ . Then the Lie algebra  $\mathfrak{p}_1 = \mathfrak{m}_1 \oplus \mathfrak{n}_1$  of  $P_1$  is given by

$$\begin{cases} \mathfrak{m}_1 = \mathfrak{h}_1(0) = \mathfrak{gl}_2 \oplus \mathfrak{l}_D \\ \mathfrak{n}_1 = \bigoplus_{i \geq 0} \mathfrak{h}_1(i), \end{cases}$$

where

$$\begin{cases} \mathfrak{h}_1(1) = \langle e_1, e_2 \rangle \otimes J \\ \mathfrak{h}_1(2) = \langle e_3^* \rangle \otimes J^* \cong \det \otimes J^* \\ \mathfrak{h}_1(3) \cong \det \otimes \langle e_1, e_2 \rangle \subset \mathfrak{sl}_3. \end{cases}$$

Here  $\langle e_1, e_2 \rangle$  is the standard representation of  $L_1 \cong GL_2$ , and  $\det$  is the usual determinant of 2-by-2 matrices. We let  $\Omega_1$  (respectively  $\Omega_2$ ) be the minimal non-trivial orbit of  $M_1$  on  $\mathfrak{h}_1(-1)$  (respectively  $\mathfrak{h}_1(-2)$ ).

Consider now the intersection of  $G \times G'$  with  $P_1$ . We have:

$$(G \times G') \cap P_1 = Q_1 \times G'$$

where  $Q_1 = L_1 \cdot U_1$  is the maximal parabolic subgroup of  $G = G_2$  whose Levi factor is generated by the long root. Then the Lie algebra  $\mathfrak{u}_1$  of  $U_1$  can be identified as the subspace of  $\mathfrak{n}_1$ , the Lie algebra of  $N_1$ , given by:

$$\begin{cases} \mathfrak{u}_1(1) = \langle e_1, e_2 \rangle \otimes \langle e \rangle \subset \mathfrak{h}_1(1) \\ \mathfrak{u}_1(2) = \langle e_3^* \otimes e^* \rangle \subset \mathfrak{h}_1(2) \\ \mathfrak{u}_1(3) = \mathfrak{h}_1(3) \end{cases}$$

Let  $V_i$  be the orthogonal complement of  $\mathfrak{u}_1(i)$  in  $\mathfrak{h}_1(-i)$ , for  $i = 1, 2$ . Then

$$\begin{cases} V_1 = \langle e_1^*, e_2^* \rangle \otimes J_0^* \\ V_2 = \det^* \otimes J_0^*. \end{cases}$$

where  $J_0$  is the subspace of  $J$  orthogonal to  $e^*$ , and  $J_0^*$  is the subspace of  $J^*$  orthogonal to  $e$ . We have

**Lemma 4.1.**

(1)  $\Omega_1 \cap V_1$  is the minimal non-trivial  $L_1 \times G'$  orbit in  $V_1$ :

$$\Omega_1 \cap V_1 = \{0 \neq v \otimes X \in \langle e_1, e_2 \rangle \otimes J_0 : \text{rank}(X) = 1\}.$$

(2)  $\Omega_2 \cap V_2$  is the minimal non-trivial  $L_1 \times G'$  orbit in  $V_2$ :

$$\Omega_2 \cap V_2 = \{X \in J_0^* : \text{rank}(X) = 1\}.$$

Now we do the same for another maximal parabolic subgroup  $P = M \cdot N$  of  $H_D$ . Here,  $P$  corresponds to the vertex  $\alpha_4$ , and  $N$  is a 2-step nilpotent group. The Lie algebra  $\mathfrak{p}$  of  $P$  can be described as follows. Let

$$s = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \in \mathfrak{sl}_3(D),$$

and define the spaces  $\mathfrak{h}(i)$  analogously as before. Then, for example,

$$\begin{cases} \mathfrak{h}(1) = \mathfrak{l}_D(1) \oplus V \otimes J_{12} \oplus V^* \otimes J_{23}^* \\ \mathfrak{h}(2) = \mathfrak{l}_D(2) \oplus V \otimes J_{11} \oplus V^* \otimes J_{33}^* \end{cases}$$

where

$$\begin{cases} \mathfrak{l}_D(1) \cong D \oplus D \\ \mathfrak{l}_D(2) \cong D \end{cases}$$

are two summands of the nilpotent radical of the minimal parabolic subalgebra of  $\mathfrak{sl}_3(D)$  consisting of the upper-triangular matrices.

Now we have:

$$(G \times G') \cap P = G \times Q$$

where  $Q = L \cdot U$  is the minimal parabolic subgroup of  $G'$ . The Lie algebra of  $U$  can be identified as:

$$\begin{cases} \mathfrak{u}(1) = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x + \bar{z} = 0 \right\} \subset \mathfrak{l}_D(1) \\ \mathfrak{u}(2) = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \text{Tr}(y) = 0 \right\} \subset \mathfrak{l}_D(2) \end{cases}$$

The Levi factor  $L$  can be identified with  $D^\times \times D^\times / F^\times$  where  $F^\times$  is embedded into  $D^\times \times D^\times$  via  $t \mapsto (t, t^{-1})$ . The adjoint action of  $(\alpha, \beta)$  in  $L$  on  $\mathfrak{u}(1) \cong D$  and  $\mathfrak{u}(2) \cong D^0$  (where  $D^0$  is the 3-dimensional space of traceless quaternions) is given by

$$\begin{cases} (\alpha, \beta) : x \mapsto \beta x \bar{\alpha} \\ (\alpha, \beta) : y \mapsto \mathbb{N}(\alpha\beta)y. \end{cases}$$

Note that with these identifications, the modulus character of  $Q$  is given by:

$$\delta_Q(\alpha, \beta) = |\mathbb{N}(\alpha \cdot \beta)|^5.$$

Let  $V'_i$  be the orthogonal complement of  $\mathfrak{u}(i)$  in  $\mathfrak{h}(-i)$ , for  $i = 1, 2$ . Then, as a representation of  $G \times L$ , we have:

$$\begin{cases} V'_1 = \mathbb{O}^0 \otimes D \\ V'_2 = \mathbb{O}^0 \otimes \mathbb{N}. \end{cases}$$

Here,  $\mathbb{O}^0$  is the 7-dimensional space of traceless octonions, on which  $G$  acts, and the actions of  $D^\times \times D^\times / F^\times$  on  $D$  and  $\mathbb{N}$  are given respectively by:

$$\begin{cases} (\alpha, \beta) : x \mapsto \bar{\alpha}^{-1} x \beta^{-1} \\ (\alpha, \beta) \mapsto \mathbb{N}(\alpha\beta)^{-1} \end{cases}$$

**Lemma 4.2.** *Let  $\Omega'$  be the minimal  $M$ -orbit on  $\mathfrak{h}(2)$ . Then  $\Omega' \cap V_2'$  is the minimal  $G \times L$ -orbit in  $V_2'$ .*

## 5. THE MINIMAL REPRESENTATION $\Pi$

In this section, we describe a particular representation  $\Pi$  of  $H_D$ , known as the minimal representation. Recall that for any irreducible admissible representation  $\pi$  of a reductive  $p$ -adic group  $\mathfrak{H}$ , the character  $\text{Tr}(\pi)$  of  $\pi$  is an invariant distribution, i.e. a linear functional on the space of locally constant, compactly supported functions on  $\mathfrak{H}$ , which is invariant under conjugation. In a small neighborhood of 1, we can regard  $\text{Tr}(\pi)$  as a distribution in a neighborhood of 0 in the Lie algebra  $\mathfrak{h}$  of  $\mathfrak{H}$ . Then a well-known result of Harish-Chandra says that:

$$\text{Tr}(\pi) = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}$$

where the sum above is over the set of nilpotent orbits  $\mathcal{O}$  in  $\mathfrak{h}$  and  $\hat{\mu}_{\mathcal{O}}$  is the Fourier transform of the (suitably normalized) invariant measure on  $\mathcal{O}$ . The wave-front set  $WF(\pi)$  of  $\pi$  is then the closure of the union of all those orbits  $\mathcal{O}$  such that  $c_{\mathcal{O}}$  is non-zero.

In the case of  $H_D$ , there is a unique minimal nilpotent orbit  $\mathcal{O}_{\min}$ . A representation  $\pi$  of  $H_D$  is said to be minimal if the only non-zero  $c_{\mathcal{O}}$ 's in the above character expansion are the ones corresponding to the minimal orbit and the trivial orbit. In other words,  $\pi$  is minimal if its wave-front set is the union of the trivial orbit and the minimal orbit. See [MS] for a justification of the term minimal.

Let  $K$  be the special maximal compact subgroup of  $H_D$ , and let  $K_1$  be its pro-unipotent radical. Then, as we have mentioned,  $K/K_1$  is a finite group of type  ${}^2\mathbb{G}_m \times {}^2E_6$ . Now the minimal representation  $\Pi$  can also be characterized by the fact that it is the unique irreducible admissible representation of  $H_D$  such that  $\Pi^{K_1}$  is isomorphic to the minimal representation of the finite group  $K/K_1$ . This representation of  $K/K_1$  is denoted by  $\phi'_{2,4}$  in the notation of [Ca]. The representation  $\phi'_{2,4}$  has dimension  $q^{11} - q^8 + q^7 + q^5 - q^4 + q$ , and is a unipotent principal series representation whose space of Borel-fixed vectors is 2-dimensional. Note in particular that  $\Pi$  is not  $K$ -spherical. Moreover, it is known that [Ru]

$$c_{\mathcal{O}_{\min}} = 1.$$

For the rest of this section, we review the results of Mœglin and Waldspurger [MW] on the values of coefficients of leading terms in the Harish-Chandra character formula. Let  $\pi$  be an irreducible representation of  $\mathfrak{H}$ . Let  $\mathcal{O} \subset \mathfrak{h}$  be a nilpotent orbit such that if  $\mathcal{O}'$

is an orbit with strictly bigger dimension than that of  $\mathcal{O}$ , then  $c_{\mathcal{O}'} = 0$  in the character expansion of  $\pi$ . Pick an element  $f$  in  $\mathcal{O}$ , and let  $s \in \mathfrak{h}$  be a semisimple element such that

$$[s, f] = -2f,$$

and such that  $ad(s)$  has integral eigenvalues. Existence of one such  $s$  is guaranteed by the Jacobson-Morozov theorem, but there are more choices as we shall see in our applications.

Abusing the notation, let

$$\mathfrak{h}(i) = \{x \in \mathfrak{h} \mid [s, x] = ix\}$$

and define

$$\begin{cases} \mathfrak{n} = \bigoplus_{i>0} \mathfrak{h}(i) \\ \bar{\mathfrak{n}} = \bigoplus_{i<0} \mathfrak{h}(i). \end{cases}$$

Let  $N$  and  $\bar{N}$  be the corresponding unipotent subgroups of  $\mathfrak{H}$ . Note that  $f$  is contained in  $\mathfrak{h}(-2)$ . We have two cases:

(I)  $\mathfrak{h}(1) \equiv 0$ . In this case the formula

$$(5.1) \quad \psi_f(\exp(x)) = \psi(\langle f, x \rangle)$$

defines a character of  $N$ . Here  $\langle \cdot, \cdot \rangle$  is the Killing form on  $\mathfrak{h}$ , and  $\psi$  a non-trivial additive character of  $F$ . The main result of [MW] is

$$(5.2) \quad \dim \pi_{N, \psi_f} = \dim \text{Hom}_N(\pi, \psi_f) = c_{\mathcal{O}}.$$

Here  $\pi_{N, \psi_f}$  is the maximal quotient of  $\pi$  on which  $N$  acts as a multiple of  $\psi_f$  [BZ; 2.30].

(II)  $\mathfrak{h}(1) \neq 0$ . Then Let  $\mathfrak{n}' \subset \mathfrak{n}$  be the radical of the skew symmetric bilinear form

$$(5.3) \quad (x, y) := \langle [x, y], f \rangle,$$

where  $x$  and  $y$  are in  $\mathfrak{n}$ . Note that  $\mathfrak{n}' \supseteq \bigoplus_{i>1} \mathfrak{h}(i)$  and the formula (5.1) defines a character  $\Psi_f$  of  $N' = \exp \mathfrak{n}'$ . Let  $N''$  be the kernel of this character. Then  $N/N''$  is a Heisenberg group. Let  $W_f$  be the smooth irreducible representation of  $N/N''$  with central character  $\psi_f$ . In this case, the result of [MW] is that

$$(5.4) \quad \dim \text{Hom}_N(\pi, W_f) = c_{\mathcal{O}}.$$

## 6. JACQUET MODULES I

The bulk of the work of this paper is the computation of various Jacquet modules, which will be carried out in this and the next two sections. As in [MS], this computation will be based solely on the assumption that  $\Pi$  is minimal. While calculations in [MS] were based

on (5.2) only, here we have to use both (5.2) and (5.4), due to more complicated structure of the maximal parabolic subgroups.

In this section, we compute  $\Pi_{U_1}$ , where  $U_1$  is the unipotent radical of the maximal parabolic subgroup  $Q_1 = L_1 \cdot U_1$  of  $G$ . But first, we need to determine the restriction of  $\Pi$  to the maximal parabolic subgroup  $P_1$ . Recall that  $N_1$  is a 3-step nilpotent group:

$$\{1\} = N_1(4) \subset N_1(3) \subset N_1(2) \subset N_1(1) = N_1$$

with:

$$N_1(i)/N_1(i+1) \cong \mathfrak{h}_1(i)$$

as groups. Hence we have a filtration:

$$0 = \Pi_4 \subset \Pi_3 \subset \Pi_2 \subset \Pi_1 \subset \Pi_0 = \Pi$$

of  $P$ -modules such that:

$$E_i := \Pi_i/\Pi_{i+1} \cong (\Pi_i)_{N_1(i)}$$

Our task is to describe the  $P$ -modules  $E_i$ .

First, we know that  $E_0 = \Pi_{N_1}$ , and the structure of this  $M_1$ -module is easily computed by restricting the two-dimensional representation of the Iwahori Hecke algebra corresponding to  $\Pi$  to the Iwahori Hecke algebra of  $M_1$ . Furthermore, the center of  $M_1$  coincides with the center of  $L_1 \cong GL_2$ . It can be checked that the central character of  $\pi_{N_1}$  is  $|z|^9$ .

Next, we describe  $E_1$ . Note that  $(E_1)_{N_1} = 0$ , and  $(E_1)_{N_1(2)} = E_1$  by definition. So we can regard  $E_1$  as a representation of  $N_1/N_1(2) \cong \mathfrak{h}_1(1)$ . Let  $f_1 \in \mathfrak{h}_1(-1)$ . Then, as described by (5.1),  $f_1$  defines a character  $\psi_{f_1}$  of  $N_1/N_1(2)$ , and by (5.2), we deduce that

$$(6.1) \quad \dim(E_1)_{N_1, \psi_{f_1}} = \begin{cases} 1 & \text{if } f \in \Omega_1 \\ 0 & \text{if not.} \end{cases}$$

Let  $f_1$  in  $\Omega_1$ . Let  $M_{f_1} \subset M_1$  be the stabilizer of  $f_1$  in  $M_1$ . Then  $M_{f_1}$  acts on the 1-dimensional space  $\Pi_{N_1, \psi_{f_1}}$ , via a character which we denote by  $\delta_1$ . From (6.1) and arguing as in [MS, Thm. 6.1], we have

$$E_1 \cong \text{ind}_{M_{f_1} N_1}^{P_1} (\delta_1 \otimes \psi_{f_1}) \cong \mathcal{C}_0^\infty(\Omega_1)$$

Finally, we consider  $E_2$ . By definition,  $(E_2)_{N_1(3)} = E_2$ , and  $(E_2)_{N_1(2)} = 0$ . Let  $f_2$  be in  $\mathfrak{h}_1(-2)$ . It defines a character  $\psi_{f_2}$  of  $N_1(2)/N_1(3)$ , and an irreducible representation  $W_{f_2}$  of  $N_1$ . Then, by (5.4), we deduce that

$$(6.2) \quad (E_2)_{N_1(2), \psi_{f_1}} = \begin{cases} W_{f_2} & \text{if } f_2 \in \Omega_2 \\ 0 & \text{if not.} \end{cases}$$

as  $N_1$ -modules. Now let  $f_2$  in  $\Omega_2$ . Let  $M_{f_2} \subset M_1$  be the stabilizer of  $f_2$  in  $M_1$ . Then  $M_{f_2}$  acts on  $W_{f_2}$ . Again, from (6.2) and arguing as in [MS, Thm. 6.1], we have

$$E_1 \cong \text{ind}_{M_{f_2} N_1}^{P_1} (W_{f_2}) \cong \mathcal{C}_0^\infty(\Omega_2; W_{f_2})$$

where  $\mathcal{C}_0^\infty(\Omega_2; W_{f_2})$  is the space of all locally constant, compactly supported sections of the  $P_1$ -equivariant vector bundle on  $\Omega_2$  whose fiber at  $f_2$  is  $W_{f_2}$ .

Thus we have shown:

**Proposition 6.3.** *The module  $\Pi_{N_1(3)}$  has a  $P_1$ -equivariant filtration, with successive quotients*

$$\begin{aligned} & \Pi_{N_1} \\ & \text{ind}_{M_{f_1} N_1}^{P_1}(\gamma_1 \otimes \psi_{f_1}) \\ & \text{ind}_{M_{f_2} N_1}^{P_1}(W_{f_2}). \end{aligned}$$

Next we use this proposition to compute the Jacquet module  $\Pi_{U_1}$ . Using the fact that the Jacquet functor is exact, and that  $U_1(3) = N_1(3)$  we see that we need to compute  $U_1$ -coinvariants for the subquotients in the proposition. Obviously,  $U_1$  acts trivially on  $\Pi_{N_1}$ . Next,

$$(6.4) \quad \mathcal{C}_0^\infty(\Omega_1)_{U_1} = \mathcal{C}_0^\infty(V_1 \cap \Omega_1).$$

Now recall that  $V_1 \cap \Omega_1$  is the orbit of the highest weight vector in the irreducible  $L_1 \times G'$ -module  $\langle e_1^*, e_2^* \rangle \otimes J_0^*$ . It follows from (6.4) that

$$\text{ind}_{M_{f_1} N_1}^{P_1}(\gamma_1 \otimes \psi_{f_1})_{U_1} = \text{Ind}_{B_1 \times Q}^{L_1 \times G'}(\delta_1 \otimes \mathcal{C}_0^\infty(F^\times))$$

where  $B_1$  is a Borel subgroup of  $L_1$  and  $F^\times$  is the line in  $V_1 \cap \Omega_1$  stabilized by  $B_1 \times Q$ .

Now we proceed to compute  $(\text{ind}_{M_{f_2} N_1}^{P_1} W_{f_2})_{U_1}$ . First, we have

$$(6.5) \quad \mathcal{C}_0^\infty(\Omega_2; W_{f_2})_{U_1(2)} = \mathcal{C}_0^\infty(V_2 \cap \Omega_2; W_{f_2})$$

where  $V_2 \cap \Omega_2$  is the orbit of the highest weight vector in the irreducible  $L_1 \times G'$ -module  $\langle e_3 \rangle \otimes J_0$ . Identifying  $\langle e_3 \rangle \otimes J_0$  with  $J_0$ , we now pick

$$f_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To finish the calculation we have to calculate  $(W_{f_2})_{U_1}$ . In the terminology of Section 4,  $N_{f_2} = N_1$ ,  $N'_{f_2} = N_1(2)$ , and the skew-linear form on  $N_1/N_1(2) \cong \langle e_1, e_2 \rangle \otimes J$  defined by the formula (5.3) is

$$(u \otimes X, v \otimes Y) = (u, v) \cdot (f_2, X, Y)$$

where  $(u, v)$  is the standard symplectic form on  $\langle e_1, e_2 \rangle$ . A direct computation using (3.1) shows that the kernel  $\Delta$  of the bilinear form  $(f_2, X, Y)$  consists of the elements in the form the form

$$X = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & 0 & 0 \\ y & 0 & 0 \end{pmatrix}.$$

It follows that  $N''_{f_2}$  is the inverse image of the space  $\langle e_1, e_2 \rangle \otimes \Delta \subset \mathfrak{h}_1(1)$  under the projection map from  $N_1$  to  $N_1/N_1(2) \cong \mathfrak{h}_1(1)$ . Let  $\Delta^\perp$  be the complement of  $\Delta$  in  $J$  consisting of all elements of the form

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & x \\ 0 & \bar{x} & c \end{pmatrix}.$$

On these elements, the quadratic form  $(f_2, X, X)$  is given by

$$(f_2, X, X) = 2bc - 2\mathbb{N}(x).$$

It follows that the representation  $W_{f_2}$  is associated to the 12-dimensional symplectic space  $\langle e_1, e_2 \rangle \otimes \Delta^\perp$ . We fix a polarization of this space consisting of all elements of the form

$$e_1 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix} + e_1 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_1 \end{pmatrix} + e_2 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_2 \end{pmatrix}.$$

We realize  $W_{f_2}$  on the space of locally constant, compactly supported functions  $\Phi(x, c_1, c_2)$ . The action of  $U_1/U_1(2)$  is given by

$$\Pi(u)\Phi(x, c_1, c_2) = \psi(u_1c_1 + u_2c_2)\Phi(x, c_1, c_2),$$

where  $u = (u_1e_1 + u_2e_2) \otimes e$  under the identification  $U_1/U_1(2) \cong \langle e_1, e_2 \rangle \otimes \langle e \rangle$ . It follows that

$$(6.6) \quad (W_{f_2})_{U_1/U_1(2)} \cong \mathcal{C}_0^\infty(D) \cong W$$

where  $W$  is the Heisenberg representation associated to the symplectic space  $\langle e_1, e_2 \rangle \otimes D$ . Now note that  $L_1 \times L = \tilde{R}$  and  $M_{f_2} \cap (L_1 \times L) = R$ , which were introduced in Section 2. Furthermore, the action of  $M_{f_2}$  on  $W_{f_2}$  induces an action of  $R$  on  $W$  which, by the Schur Lemma, must be a twist by a character of  $R$  of the action described in the Section 2.

**Lemma 6.7.** *Any character of  $R$  is a restriction of a character of  $\tilde{R}$  of the form  $|\det|^t \otimes \chi$  where  $t$  is a real number and  $\chi$  a unitary character of  $D \times D/F^\times$ .*

It follows from (6.5) and (6.6) that

$$(\text{ind}_{M_{f_2}N_1}^{P_1} W_{f_2})_{U_1} \cong \text{Ind}_{L_1 \times Q}^{L_1 \times G'} \delta_2 \otimes \tilde{W}$$

where  $\delta_2$  is a character of  $L_1 \times L$  as in the Lemma.

We summarize what we have shown in the following proposition:

**Proposition 6.8.** *As a representation of  $L_1 \times G'$ , the module  $\Pi_{U_1}$  has a filtration with successive quotients*

$$\begin{aligned} & \Pi_{N_1} \\ & \text{Ind}_{B_1 \times Q}^{L_1 \times G'} \delta_1 \otimes \mathcal{C}_0^\infty(F^\times) \\ & \text{Ind}_{L_1 \times Q}^{L_1 \times G'} \delta_2 \otimes \widetilde{W}, \end{aligned}$$

where  $\delta_2 = |\det|^{t_2} \otimes \chi_2$  is a character of  $L_1 \times L$  as in Lemma 6.7. The central character of  $L_1 \cong GL_2$  on  $\Pi_{N_1}$  is  $|z|^9$ .

We shall show that  $t_2 = 5$  and that the character  $\chi_2$  is trivial in Section 9, where we investigate the local correspondence.

## 7. JACQUET MODULES II

In this section, we compute  $\Pi_{U_2}$ , where  $U_2$  is the unipotent radical of the maximal parabolic subgroup  $Q_2 = L_2 \cdot U_2$  of  $G$ . Since the computation is entirely similar to the case of split groups in [MS], we shall content ourselves with just stating the results. There is an exact sequence of  $P_2$ -modules

$$0 \longrightarrow \text{ind}_{M_f N_2}^{P_2}(\gamma \otimes \psi_f) \longrightarrow \Pi_{Z_2} \longrightarrow \Pi_{N_2} \longrightarrow 0$$

where  $f$  is in the orbit  $\Omega$ . This then implies the following proposition.

**Proposition 7.1.** *As a representation of  $L_2 \times G'$ , the module  $\Pi_{U_2}$  has a filtration with successive quotients*

$$\begin{aligned} & \Pi_{N_2} \\ & \text{Ind}_{B_2 \times Q}^{L_2 \times G'} \delta \otimes \mathcal{C}_0^\infty(F^\times). \end{aligned}$$

The central characters of  $L_2 \cong GL_2$  on  $\Pi_{N_2}$  are  $|z|^4$  and  $|z|^6$ , the latter corresponding to a one-dimensional summand of  $\Pi_{N_2}$ .

## 8. JACQUET MODULES III

In this section, we compute the Jacquet module of  $\Pi$  with respect to the subgroup  $U$  of  $G'$ . Since the computation is similar to that in Section 6, we shall be brief.

As before, we first determine the restriction of  $\Pi$  to the maximal parabolic  $P = M \cdot N$ . Since now  $N$  is a 2-step nilpotent group

$$\{1\} = N(3) \subset N(2) \subset N(1) = N,$$

there is a filtration of  $P$ -modules:

$$0 = \Pi_3 \subset \Pi_2 \subset \Pi_1 \subset \Pi_0 = \Pi$$

with  $E_i$  defined as before. Hence,  $E_0$  is again  $\Pi_N$ . As for  $E_1$ , we find that it is now equal to 0, since the minimal nilpotent orbit has empty intersection with  $\mathfrak{h}(-1)$ . Thus we only need to describe  $E_2$ , and as in Section 6 we find that

$$E_2 \cong \text{ind}_{M_f N}^P W_f$$

where  $f$  is in the minimal orbit  $\Omega'$ . We summarize the results without a detailed proof:

**Proposition 8.1.** *As a representation of  $G \times L$ , the module  $\Pi_U$  has a filtration with successive quotients:*

$$\Pi_N$$

$$\text{Ind}_{P_1 \times L}^{G \times L} \delta' \otimes \widetilde{W}$$

where  $\delta' = |\det|^{t'} \otimes \chi'$  is a character of  $L_1 \times L$  as in Lemma 6.7. The central character of  $L \cong D \times D/F^\times$ , on  $\Pi_N$  is

$$(z_1, z_2) \mapsto |z_1 z_2|^6.$$

We shall show that  $t' = 5$  and that the character  $\chi'$  is trivial in Section 9, where we investigate the local correspondence.

## 9. LOCAL CORRESPONDENCE

In this section we prove the correspondence of representations discussed in the introduction, and also determine the characters  $\delta_2$  and  $\delta'$  from the previous sections. Basically, the hard work has been done in the last three sections.

Let  $\pi = JL(\pi')$  be an irreducible square-integrable representation of  $GL_2$  with unitary central character, and let

$$\pi(s) = \pi \otimes |\det|^s$$

where  $s \in \mathbb{R}$ . Consider the representations

$$\begin{cases} I_1(\pi, s) = \text{Ind}_{P_1}^G \delta_{P_1}^{1/2} \pi(s) \\ I_2(\pi, s) = \text{Ind}_{P_2}^G \delta_{P_2}^{1/2} \pi(s). \end{cases}$$

where

$$\begin{cases} \delta_{P_1} = |\det|^5 \\ \delta_{P_2} = |\det|^3. \end{cases}$$

If  $s > 0$ , then  $I_i(\pi, s)$  has unique (Langlands) quotient, which we denote by  $J_i(\pi, s)$ . Similarly, let

$$\pi'(s) = \pi' \otimes |\mathbb{N}(\alpha\beta)|^s$$

and let

$$I(\pi', s) = \text{Ind}_Q^{G'} \delta_Q^{1/2} \cdot \pi'(s) \otimes \pi'(s).$$

Again, if  $s > 0$ , then  $I(\pi', s)$  has unique (Langlands) quotient, which we denote by  $J(\pi', s)$ .

The calculation of the characters  $\delta'$  and  $\delta_2$  goes along with the proof of the following theorem.

**Theorem 9.1.** *Let  $\pi = JL(\pi')$  be an irreducible super-cuspidal representation of  $GL_2$  with unitary central character, and let  $s > 0$ . Then*

$$\begin{cases} \Theta(J_1(\pi, s)) = \{J(\pi', s)\} \\ \Theta(J(\pi', s)) = \{J_1(\pi, s)\} \end{cases}$$

and

$$\Theta(J_2(\pi, s)) = \emptyset.$$

Let  $1'$  be the trivial representation of  $G'$ . The one-dimensional summand of  $\Pi_{N_2}$  corresponding to the central character  $|z|^6$  is isomorphic to  $|\det|^3 \otimes 1'$ , as  $L_2 \times G'$ -modules. It follows from the Frobenius reciprocity that  $\Theta(1')$  contains a subquotient of  $\text{Ind}_{P_2}^G(|\det|^3)$ . Since the subquotients of  $\text{Ind}_{P_2}^G(|\det|^3)$  are 1, the trivial representation of  $G$ , and  $J_1(st, 5/2)$  (where  $st$  is the Steinberg representation of  $GL_2$ ) we have

$$\Theta(1') \cap \{1, J_1(st, 5/2)\} \neq \emptyset.$$

Let  $\sigma$  be an irreducible representation in the intersection above. Since  $1'$  is a submodule of  $\text{Ind}_Q^{G'} 1$ ,

$$\emptyset \neq \text{Hom}_{G \times G'}(\Pi, \sigma \otimes \text{Ind}_Q^{G'} 1) = \text{Hom}_{G \times L}(\Pi_U, \sigma \otimes 1).$$

Since the central character of  $\Pi_N$  is  $|z_1 z_2|^6$  by Proposition 8.1, and the central character of 1 is trivial,  $\sigma \otimes 1$  must be a quotient of  $\text{Ind}_{P_1 \times L}^{G \times L} \delta' \otimes \widetilde{W}$  by Lemma 1.2. A calculation, using (1.1) and Theorem 2.1 shows that  $\sigma$  is a quotient of  $\text{Ind}_{P_1}^G(st \otimes \delta')$ . This is possible only if  $\sigma = J_1(st, 5/2)$ ,  $\chi' = 1$  and  $t' = 5$ . Thus, we have determined the character  $\delta'$  and have shown that

$$\Theta(1') = \{J_1(st, 5/2)\}.$$

We are now ready to prove the following lemma.

**Lemma 9.2.** *Let  $\pi = JL(\pi')$  be an irreducible square-integrable representation of  $GL_2$ , and let  $s > 0$ . Then*

$$\begin{cases} \Theta(J_1(\pi, s)) \neq \emptyset \\ \Theta(J(\pi', s)) \subseteq \{J_1(\pi, s)\}. \end{cases}$$

*Proof.* Let  $\sigma$  be in  $\Theta(J(\pi', s))$ . Note that  $J(\pi', s)$  is unique submodule of  $I(\tilde{\pi}', -s)$ . By the Frobenius reciprocity,

$$\text{Hom}_{G \times G'}(\Pi, \sigma \otimes I(\tilde{\pi}', -s)) = \text{Hom}_{G \times L}(\Pi_U, \sigma \otimes \tilde{\pi}'(5/2 - s) \otimes \tilde{\pi}'(5/2 - s)).$$

Note that the central character of  $\tilde{\pi}'(5/2 - s) \otimes \tilde{\pi}'(5/2 - s)$  is, up to a unitary character, equal to  $|z_1 z_2|^{5-2s}$ . Since  $s$  is positive, it is different then the central character  $|z_1 z_2|^6$  of  $\Pi_N$ , it follows that

$$\text{Hom}_{G \times L}(\text{Ind}_{P_1 \times L}^{G \times L} \delta' \otimes \widetilde{W}, \sigma \otimes \tilde{\pi}'((5/2 - s) \otimes \tilde{\pi}'(5/2 - s))) \neq \emptyset.$$

Again, a calculation, using (1.1) and Theorem 2.1 shows that  $\sigma$  must be  $J_1(\pi, s)$ . The lemma is proved.

For those  $s$  for which  $I(\pi', s)$  is irreducible, we have in fact shown that  $J_1(\pi, s) \otimes I(\pi', s)$  is a quotient of  $\Pi$ . Since  $J_1(\pi, s)$  is unique submodule of  $I_1(\tilde{\pi}, -s)$ , it follows that

$$\emptyset \neq \text{Hom}_{G \times G'}(\Pi, I_1(\tilde{\pi}, -s) \otimes I(\pi', s)) = \text{Hom}_{L_1 \times G'}(\Pi_{U_1}, \tilde{\pi}(5/2 - s) \otimes I(\pi', s)).$$

Again, the central characters show that  $\tilde{\pi}(5/2 - s)$  cannot be a quotient of  $\Pi_{N_1}$ . Furthermore, if we take  $\pi$  to be supercuspidal, then  $\tilde{\pi}(5/2 - s)$  cannot be a quotient of the middle term of the filtration of  $\Pi_{U_1}$  given by Proposition 6.8. Thus,

$$\text{Hom}_{L_1 \times G'}(\text{Ind}_{L_1 \times Q}^{L_1 \times G'} \delta_2 \otimes \widetilde{W}, \tilde{\pi}(5/2 - s) \otimes I(\pi', s)) \neq \emptyset$$

and this is possible only if  $\chi_2 = 1$  and  $t_2 = 5$ . This determines  $\delta_2$ , and the following analogue of Lemma 9.2 follows.

**Lemma 9.3.** *Let  $\pi = JL(\pi')$  be an irreducible super-cuspidal representation of  $GL_2$ , and let  $s > 0$ . Then*

$$\begin{cases} \Theta(J(\pi', s)) \neq \emptyset \\ \Theta(J_1(\pi, s)) \subseteq \{J(\pi', s)\}. \end{cases}$$

The first part of the Theorem is now a simple combination of Lemma 9.2 and Lemma 9.3. For the second part note that  $J_2(\pi, s)$  is unique submodule of  $I_2(\tilde{\pi}, -s)$ . Again, by the Frobenius reciprocity,

$$\text{Hom}_G(\Pi, I_2(\tilde{\pi}, -s)) = \text{Hom}_{L_2}(\Pi_{U_2}, \tilde{\pi}(3/2 - s)).$$

Since  $\tilde{\pi}(3/2 - s)$  is supercuspidal, it cannot be a quotient of the second term of the filtration of  $\Pi_{U_2}$  given by Proposition 7.1. Also, since the central character of  $\tilde{\pi}(3/2 - s)$  is, up to a unitary character, equal to  $|z|^{3-2s}$ , it cannot be a quotient of  $\Pi_{N_2}$  if  $s > 0$ . This shows that  $J_2(\pi, s)$  is not a quotient of  $\Pi$ . The theorem is proved.

We finish this paper by calculating  $\Theta(St')$  where  $St'$  is the Steinberg representation of  $G'$ . Since  $St'$  is a submodule of  $I(1, 5/2)$ , it follows as in the proof of Lemma 9.2 that

$$\begin{cases} \Theta(St') \subseteq \{St\} \\ \Theta(St) \cap \{St', J(1, 5/2)\} \neq \emptyset. \end{cases}$$

where  $St$  is the Steinberg representation of  $G$ . Since, by Lemma 9.2,  $J(1, 5/2)$  cannot be paired with  $St$ , it follows that

$$(9.4) \quad \Theta(St') = \{St\}.$$

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