# A SIEGEL-WEIL FORMULA FOR AUTOMORPHIC CHARACTERS: CUBIC VARIATION OF A THEME OF SNITZ

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## 1. Introduction

This paper is largely inspired by the 2003 Ph.D. thesis [S] of Kobi Snitz, written under the direction of S. Kudla at the University of Maryland. Let us recall his results briefly.

Let $F$ be a number field with adele ring $\mathbb{A}$. Let $D$ be a quaternion division algebra defined over $F$ and ramified precisely at a finite set $S_D$ of places of $F$. Let $V_D$ be the 3-dimensional quadratic space $(D_0, -N_D)$ where $D_0$ is the space of trace zero elements in $D$ and $N_D$ is the reduced norm of $D$. Then the special orthogonal group $SO(V_D)$ is naturally isomorphic to $PD^\times = D^\times/F^\times$ and the reduced norm induces a natural map

$$N_D : PD^\times(\mathbb{A}) \longrightarrow \mathbb{A}^\times/\mathbb{A}^{\times 2}.$$

Each étale quadratic algebra $K$ of $F$ gives rise to a quadratic idele class character $\chi_K$ which we may regard by composition with $N_D$ as a character of $PD^\times(\mathbb{A})$. Recall that $PD^\times \times \tilde{SL}_2$ forms a dual pair, with its associated Weil representation realized on the space $S(V_D(\mathbb{A}))$ of Schwarz functions on $V_D(\mathbb{A})$. Hence one may consider the theta lift of $\chi_K$ to $\tilde{SL}_2$. Namely, if $f \in S(V_D(\mathbb{A}))$, set

$$\theta_D(\chi_K, f)(g) = \int_{PD^\times(F)\backslash PD^\times(\mathbb{A})} \theta(f)(gh) \cdot \chi_K(h)\, dh$$

and let $V_D(\chi_K)$ be the space of automorphic forms on $\tilde{SL}_2$ obtained for all $f \in S(V_D(\mathbb{A}))$.

If $\chi_K = \mathbf{1}$ is the trivial character (so that $K = F \times F$), the Siegel-Weil formula associates to each $f$ an explicit Eisenstein series $E_D(f)$ such that

$$\theta_D(\mathbf{1}, f) = E_D(f).$$

This should be thought of as providing an alternative construction of $\theta_D(\mathbf{1}, f)$. In [S], Snitz considered the analogous problem when $\chi_K$ is a non-trivial character. In this case, the space $V_D(\chi_K)$ is non-zero if and only if $K$ is non-split at all places in $S_D$. The function $\theta_D(\chi_K, f)$ is contained in the space of cusp forms and one would like an alternative construction of this cusp form.

To do so, one considers the rank 1 quadratic space $V_K = (K_0, -N_K)$, where $K_0$ is the space of trace zero elements in $K$ and $N_K$ is the norm map of $K/F$. The orthogonal group $O(V_K)$ is naturally isomorphic to the finite group scheme $S_K$ with

$$S_K(F_v) = \mathrm{Gal}(K_v/F_v) \cong \mathbb{Z}/2\mathbb{Z}.$$

Now let $\chi_D = \otimes_v \chi_{D,v}$ be the character of $S_K(\mathbb{A})$ such that $\chi_{D,v}$ is non-trivial precisely when $v \in S_D$. Observe that since $\#S_D$ is even, $\chi_D$ is a character of $S_K(F)\backslash S_K(\mathbb{A})$. Now $S_K \times \tilde{SL}_2$ forms a dual pair with its associated Weil representation realized on $S(V_K(\mathbb{A}))$. Thus for each Schwarz function

$f' \in S(V_K(\mathbb{A}))$, one has the analogous theta lift

$$\theta_K(\chi_D, f')(g) = \int_{S_K(F)\backslash S_K(\mathbb{A})} \theta(f')(gh) \cdot \chi_D(h)\, dh$$

and the space $V_K(\chi_D)$ that these lifts generate.

Now the main result of [S] is:

**Theorem 1.1.** *Given $f \in S(V_D(\mathbb{A}))$, there is a unique $f' \in S(V_K(\mathbb{A}))[\chi_D]$ (the $\chi_D$-isotypic component of $S(V_K(\mathbb{A}))$ such that*

$$\theta_D(\chi_K, f) = \theta_K(\chi_D, f').$$

In fact, results of this type were first obtained some twenty years ago by Schulze-Pillot [SP1,2,3] in a more classical language and applied to the arithmetic of quadratic forms. Snitz's theorem provides a refinement of the results of Schulze-Pillot and exposes the representation theoretic underpinnings.

The map $f \mapsto f'$ can be explicitly described at the local level and can be expressed as a weighted orbital integral. Namely, we fix an $F$-algebra embedding $\iota : K \hookrightarrow D$, which gives rise to a map of quadratic spaces $V_K \hookrightarrow V_D$; this exists because of our assumption that $K_v$ is a field for all $v \in S_D$. Fix also a non-zero element $x_0$ in the 1-dimensional $F$-vector space $V_K$. Then, for each place $v$, a Schwarz function $f_v \in S(V_{D_v})$ is mapped to the function $f'_v \in S(V_{K_v})$ given by:

$$f'_v(r) = |\frac{r}{x_0}|_v \cdot \chi_{K_v}(\frac{r}{x_0}) \int_{\iota(K_v^\times)\backslash D_v^\times} f_v(h_v^{-1} \cdot \iota(r) \cdot h_v) \cdot \chi_{K_v}(N_{D_v}(h_v))\, dh_v, \qquad \text{if } r \neq 0.$$

Since any two such embeddings as $\iota$ are conjugate under $PD_v^\times$ by the Skolem-Noether theorem, the above definition is independent of the choice of $\iota$. In any case, the above formula determines $f'_v$ uniquely. Note that though this local map depends on the choice of $x_0 \in V_K$, the associated global map (which is the tensor product of the local ones) does not.

The purpose of this paper is to prove a cubic analog of the above theorem. As we explain below, there are a number of subtleties which are absent in the quadratic case and which makes the formulation of the result in the cubic case somewhat interesting.

More precisely, assume henceforth that $D$ is a division algebra of degree 3 over $F$, ramified precisely at the finite set $S_D$ of places of $F$. Unlike the case of quaternions, $D$ is not uniquely specified by the set $S_D$. For each place $v \in S_D$, one needs to specify a local invariant

$$\mathrm{inv}(D_v) = \pm 1 \in \mathbb{Z}/3\mathbb{Z} \quad \text{with} \quad \sum_{v \in S_D} \mathrm{inv}(D_v) = 0 (mod\, 3).$$

The definition of the local invariant $\mathrm{inv}(D_v)$ and the condition of global coherence they satisfy requires the specification of the global Artin isomorphism

$$art_F : \pi_0(F^\times \backslash \mathbb{A}^\times) \longrightarrow W_F^{ab}.$$

There are after all two natural choices of this (which are inverses of each other), depending on whether local uniformizers at each finite place are sent to the arithmetic or geometric Frobenius elements. In any case, we fix $art_F$ once and for all. If $D$ is associated to the data $(S_D, \{\epsilon_v\})$, then the opposite algebra $D^{op}$ of $D$ has data $(S_D, \{-\epsilon_v\})$.

The automorphism group of $D$ is naturally isomorphic to $PD^\times$. One subtlety in the cubic case is that since

$$D^\times \cong (D^{op})^\times \quad \text{via} \quad x \mapsto x^{-1},$$

there is a natural isomorphism $i_D : PD^\times \cong (PD^{op})^\times$. Thus the correspondence $D \mapsto PD^\times$ is two-to-one. To resolve this, we should think of $D$ as giving the group $PD^\times$ as well as a distinguished homomorphism

$$N_D : PD^\times(\mathbb{A}) \longrightarrow \mathbb{A}^\times/\mathbb{A}^{\times 3}$$

induced by the reduced norm map. We shall write $G_D$ for the group $PD^\times$ equipped with the map $N_D$.

Now let $E$ be a cyclic cubic field extension of $F$. Unlike the case of quadratic extensions, where $K$ and $\chi_K$ determine each other, $E/F$ determines a pair of cubic characters $\{\chi_E, \chi_E^{-1}\}$ of $Gal(E/F)$. This is another subtlety in the cubic case. Using the isomorphism $art_F$, we obtain by composition an idele class character $\chi_E \circ art_F$. Since $art_F$ is fixed, we shall suppress it from the notations and denote this idele class character by $\chi_E$ as well. Then we may regard $\chi_E = \otimes_v \chi_{E,v}$ as a character of $G_D(\mathbb{A})$ by composition with the distinguished map $N_D$. Observe that

$$\chi_E \circ N_{D^{op}} \circ i_D = \chi_E^{-1} \circ N_D$$

as characters on $PD^\times$. Thus $i_D$ identifies the pair $(D, \chi_E^{-1})$ with $(D^{op}, \chi_E)$.

Now let $G_2$ denote the split exceptional group of type $G_2$. Then $G_D \times G_2$ is a dual pair in an inner form $H_D$ of type $E_6$. One may consider the theta lift of the character $\chi_E$ of $G_D$ to $G_2$, using the minimal representation $\Pi_D$ of $H_D$. This theta lift is non-zero iff $E_v$ is a field for all $v \in S_D$, in which case the lift is cuspidal. Assuming this, for each $f \in \Pi_D$, one thus has a cusp form $\theta_D(\chi_E, f)$ on $G_2$. Our main result gives an alternative construction of $\theta_D(\chi_E, f)$.

We now describe the alternative construction. As in the quadratic case, the field $E$ gives rise to a finite group scheme $S_E$ which is a twisted form of $S_3$; $S_E$ is simply the automorphism group scheme of the $F$-algebra $E$. We have:

$$S_E(F_v) = \mathrm{Gal}(E_v/F_v) \cong \begin{cases} A_3, & \text{if } E_v \text{ is a field;} \\ S_3, & \text{if } E_v \text{ is split.} \end{cases}.$$

Now we want to define an automorphic character

$$\chi_E^D = \prod_v \chi_{E,v}^{D_v}$$

of $S_E(\mathbb{A})$. Recall that if $E_v$ is a field, then $\chi_{E,v}$ is a character of $S_E(F_v)$ by definition. We define the character $\chi_E^D$ by

$$\chi_{E,v}^{D_v} = \chi_{E,v}^{inv(D_v)}$$

where we recall that $inv(D_v)$ is the local invariant of $D_v$ and for $v \notin S_D$, $inv(D_v) = 0$. Observe that the character $\chi_E^D$ is trivial when restricted to $S_E(F)$ by the condition of global coherence and that

$$\chi_E^D = (\chi_E^{D^{op}})^{-1}.$$

Now we have the dual pair

$$S_E \times G_2 \subset H_E \rtimes S_E := Spin_8^E \rtimes S_E.$$

Given an element $f' \in \Pi_E$ (the minimal representation of $H_E$), one thus has an automorphic form $\theta_E(\chi_E^D, f')$. Our first result is:

**Theorem 1.2.** *Assume that $E_v$ is unramified for all $v \in S_D$. Given $f \in \Pi_D$, there is a unique element $f' \in \Pi_E[\chi_E^D]$ (the $\chi_E^D$-isotypic part of $\Pi_E$) such that*

$$\theta_D(\chi_E, f) = \theta_E(\chi_E^D, f').$$

Observe that if we replace the Artin isomorphism $art_F$ by $art_F \circ inv$ (where $inv$ is the inverse map on $W_F^{ab}$), then the characters $\chi_E$ and $\chi_E^D$ will be replaced by their inverses. Thus the theorem does not depend on the choice of $art_F$.

We hope that our exposition above makes the parallel between the quadratic and cubic cases transparent.

Our second result is the honest Siegel-Weil formula which identifies the theta lift of the trivial character as an explicit Eisenstein series. In fact, the Siegel-Weil formula was needed in our proof of Theorem 1.2. In [G1], we have obtained an instance of such a formula for the dual pair $F_4^{cpt} \times G_2$ in the quaternionic form of $E_8$, with $F_4^{cpt}$ an anisotropic group of type $F_4$. This is analogous to the situation in Weil's paper [W], in the sense that the theta integral and the Eisenstein series are both absolutely convergent. The case we treat here is analogous to that in the paper [KR] of Kudla-Rallis, where the Siegel-Weil formula was extended beyond Weil's range of convergence for anisotropic orthogonal groups.

As in [KR], the relevant Eisenstein series may a priori have a pole (of order 2 here) at the point of interest. We show however that it is holomorphic there for the standard sections we are interested in. The main difficulty in the proof of the formula here is to show that the theta lift is orthogonal to the space of cusp forms. In [G1], this was easy because the theta lift of the trivial representation turns out to be non-unitary. This is no longer the case here. Indeed, there are cuspidal representations of $G_2$ which are nearly equivalent to the theta lift of the trivial representation. These cuspidal representations belong to the so-called cubic unipotent Arthur packets introduced in [GGJ]. We overcome this difficulty by using the results of [G2] which gives a complete classification of the near equivalence class of such representations.

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## 2. **Groups and Dual Pairs**

We begin by describing the relevant groups and representations which play a role in this paper.

2.1. **The group $H_J$.** Let $J = D$ or $E$, and let $N_J$ (resp. $T_J$) denote the norm (resp. trace) map on $J$. The trace map $T_J$ induces a symmetric bilinear form on $J$ given by $T_J(x,y) = T_J(xy)$.

As mentioned in the introduction, we have a linear algebraic group $H_J$ which is adjoint of type $E_6$ (respectively simply-connected of type $D_4$) when $J = D$ (respectively $J = E$). The group $H_J$ has $F$-rank 2 and its relative root system $\Phi$ is of type $G_2$. Let $T_J$ be a maximal split torus of $H_J$ and for each root $\gamma \in \Phi$, let $U_{J,\gamma}$ be the associated root subgroup. Then

$$U_{J,\gamma} \cong \begin{cases} F, & \text{if } \gamma \text{ is long;} \\ J, & \text{if } \gamma \text{ is short.} \end{cases}$$

Indeed, when $J = E$ so that $H_E$ is quasi-split, one has a compatible system of such isomorphisms by means of a Chevalley-Steinberg system of épinglage. We choose a system of simple roots $\{\alpha, \beta\}$ for $\Phi$, with $\alpha$ short and $\beta$ long.

One can describe the Lie algebra $Lie(H_J)$ uniformly. Namely, there is a $\mathbb{Z}/3\mathbb{Z}$-grading

$$Lie(H_J) = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1$$

such that

$$\begin{cases} \mathfrak{h}_0 = \mathfrak{sl}_3 \oplus \mathfrak{l}_J \\ \mathfrak{h}_1 = F^3 \otimes J \\ \mathfrak{h}_{-1} = (F^3 \otimes J)^*. \end{cases}$$

Here,

$$\mathfrak{l}_J = \begin{cases} D_0 \oplus D_0, \text{ if } J = D; \\ E_0, \text{ if } J = E. \end{cases}$$

Using the symmetric bilinear form on $J$ and the standard pairing on $F^3$, we shall identity $\mathfrak{h}_{-1}$ with $F^3 \otimes J$. For the definition of the Lie brackets, we refer the reader to [Ru1].

2.2. **Parabolic subgroups.** Since $H_J$ has $F$-rank 2, it has two conjugacy classes of $F$-rational maximal parabolic subgroups. One of them is the Heisenberg parabolic $P_J = M_J N_J$. The unique simple root of its Levi subgroup $M_J$ is the short root $\alpha$. The unipotent radical $N_J$ is a Heisenberg group with 1-dimensional center $Z_J$. If $\bar{N}_J$ denotes the opposite unipotent radical, then

$$\bar{V}_J := \bar{N}_J/\bar{Z}_J = U_{J,-\beta} \times U_{J,-\alpha-\beta} \times U_{J,-2\alpha-\beta} \times U_{J,-3\alpha-\beta} \cong F \times J \times J \times F.$$

We denote an element of $\Omega_J$ by $(a, x, y, d)$ with $a, d \in F$ and $x, y \in J$.

The representation of $M_J$ on $\bar{V}_J$ is irreducible and we let $\Omega_J$ denote the minimal non-trivial $M_J$-orbit. It is the orbit of the highest weight vector and is a cone in $\bar{V}_J$. The Zariski closure of $\Omega_J$ is $\overline{\Omega}_J = \Omega_J \cup \{0\}$. The variety $\overline{\Omega}_J$ can be described as those $(a, x, y, d) \in \bar{V}_J$ satisfying:

$$\begin{cases} (xy + yx)/2 = ad \\ x^\# = a \cdot y \\ y^\# = d \cdot x \end{cases}$$

Here $x \mapsto x^\#$ is the quadratic map on $J$ determined by the condition that

$$x^\# x = N_J(x).$$

Note that the product

$$x * y = \frac{xy + yx}{2}$$

makes $J$ into a Jordan algebra.

Since $J$ is a division algebra, $N_J(x)$ is non-zero if $x$ is non-zero. Thus, if $(a, x, y, d) \in \Omega$, then $a$ and $d$ are both non-zero, and

$$(a, x, y, d) = a \cdot (1, x, x^\#, N_J(x)).$$

Thus the variety $\Omega$ is birational to $F^\times \times J$, via

$$(a, x, y, d) \mapsto (a, x/a).$$

The other maximal parabolic subgroup is denoted by $Q_J = L_J U_J$. The unique simple root of its Levi subgroup $L_J$ is the long root $\beta$. When $J = D$, we have

$$L_D = GL_2 \times_{\mu_3} (SL_1(D) \times_{\mu_3} SL_1(D)).$$

Thus there is a natural projection map of algebraic groups

$$GL_2 \times (SL_1(D) \times_{\mu_3} SL_1(D)) \longrightarrow L_D.$$

Since the unipotent radical $U_J$ is 3-step nilpotent, we shall often refer to $Q_J$ as the 3-step parabolic.

Note that if $\chi \in \Omega_J$, then the stabilizer in $M_J$ of the line spanned by $\chi$ is the parabolic subgroup $M_J \cap Q_J$. Hence the stabilizer of $\chi$ itself is a codimension one subgroup of $M_J \cap Q_J$.

2.3. **Dual pairs.** Now we can describe the dual pair $G_J \times G_2 \subset H_J$. If $e$ denotes the identity element in $J$, then

$$\mathfrak{sl}_3 \oplus (F^3 \otimes e) \oplus (F^3 \otimes e) \subset \mathfrak{sl}_3 \oplus \mathfrak{l}_J \oplus (F^3 \otimes J) \oplus (F^3 \otimes J)$$

is a Lie subalgebra isomorphic to $Lie(G_2)$. Its centralizer is trivial when $J = E$, but when $J = D$, the centralizer is $Lie(G_D) = D_0$ which is embedded into $\mathfrak{l}_J = D_0 \oplus D_0$ diagonally. This describes the dual pair

$$G_D \times G_2 \hookrightarrow H_D.$$

When $J = E$, the outer automorphism group $Out(H_E)$ of $H_E$ is isomorphic to $S_E$. We fix such an isomorphism as follows. As we have noted before, using a Chevalley-Steinberg system of épinglage for the quasi-split group $H_E$, one has a natural isomorphism

$$U_\alpha \cong Res_{E/F}\mathbb{G}_a.$$

The isomorphism

$$Out(H_E) \cong S_E$$

is such that $S_E(F) = \mathrm{Gal}(E/F)$ acts in the natural way on $U_\alpha(F) = E$. Moreover, with the Chevalley-Steinberg system of épinglage, one obtains a splitting

$$S_E \cong Out(H_E) \hookrightarrow Aut(H_E).$$

As a result, one has a dual pair

$$S_E \times G_2 \subset H_E \rtimes S_E.$$

We may consider the intersection of $G_2$ with the two parabolic subgroups introduced above. By working on the level of Lie algebras, one sees that

$$G_2 \cap P_J = P$$

where $P = M \cdot N$ is a Heisenberg parabolic of $G_2$. Here, $M \cong GL_2$ whereas $N$ is a 5-dimensional Heisenberg group whose center is that of $N_J$. If we set $\bar{V} = \bar{N}/\bar{Z}$, then the $M$-orbits on $\bar{V}$ are parametrized naturally by cubic $F$-algebras, with the generic orbits corresponding to étale cubic algebras.

The embedding $\bar{V} \hookrightarrow \bar{V}_J$ is given by

$$F \times F \times F \times F \hookrightarrow F \times J \times J \times F.$$

There is a natural $M$-equivariant projection $\bar{V}_J \longrightarrow \bar{V}$ by

$$(a, x, y, d) \mapsto (a, T_J(x), T_J(y), d).$$

Given an element $\chi = (a, b, c, d) \in \bar{V}$, let $\Omega_J(\chi)$ be the fiber of this projection map over $\chi$.

**Lemma 2.1.** *(i) One has*

$$\Omega_J(\chi) = \{(a(1, x, x^{\#}, N_J(x)) : x \text{ has characteristic polynomial } x^3 - (b/a)x^2 + (c/a)x - (d/a)\}.$$

*Thus, if the $M$-orbit of $\chi$ corresponds to an étale cubic algebra $A$, then $\Omega_J(\chi)$ is in bijection with the set of algebra embeddings of $A$ into $D$. In particular, when $J = E$, $\Omega_E(\chi)$ is non-empty iff $A = E$, whereas when $J = D$, $\Omega_D(\chi)$ is non-empty iff $A_v$ is a field for all $v \in S_D$.*

*(ii) When $\Omega_J(\chi)$ is non-empty, $G_J$ acts transitively on $\Omega_J(\chi)$. If $J = E$, this action is simply transitive, whereas if $J = D$, the stabilizer of of a point in $\Omega_D(\chi)$ is the maximal torus $PA^\times \hookrightarrow PD^\times$ associated to the corresponding embedding $A \hookrightarrow D$.*

Likewise, the intersection of $G_2$ with the 3-step parabolic $Q_J$ is equal to $Q = L \cdot U$, which is the other maximal parabolic subgroup of $G_2$. The Levi subgroup $L$ is isomorphic to $GL_2$ and its unipotent radical $U$ is a 3-step nilpotent group.

## 3. **Representations**

Now let $v$ be a place of $F$ and fix a non-trivial additive character $\psi_v$ of $F$. The group $H_J(F_v)$ has a distinguished representation $\Pi_{J,v}$ known as the minimal representation. For a systematic introduction to minimal representations, we refer the reader to [GS2]. We shall content ourselves with stating the properties of $\Pi_{J,v}$ that we need.

3.1. **Minimal representations.** Assume first that $v$ is a finite place of $F$; this assumption will hold up to and including §3.4. Unlike the Weil representation, we do not have a completely explicit smooth model for $\Pi_{J,v}$, but we do have a rather neat description of it which suffices for some purposes.

One may identify $\bar{V}_J$ with the set of unitary characters of $N_J(F_v)$, using the Killing form and the additive character $\psi_v$. Recall that $\Omega_J$ is the minimal non-trivial $M_J$-orbit on $\bar{V}_J$. It was shown in [MS] that there is a $P_J(F_v)$-equivariant embedding

$$i_v : (\Pi_{J,v})_{Z_J} \hookrightarrow C^\infty(\Omega_J),$$

where $P_J(F_v)$ acts on $C^\infty(\Omega_J)$ by:

$$\begin{cases} (m \cdot f)(\chi) = \delta_{P_J}(m)^{s_J} \cdot f(m^{-1} \cdot \chi) \\ (n \cdot f)(\chi) = \chi(n) \cdot f(\chi) \end{cases}$$

where

$$s_J = \begin{cases} 2/11, \text{ if } J = D; \\ 1/5, \text{ if } J = E. \end{cases}$$

The image of $i$ contains $C_c^\infty(\Omega_J)$, and we have an exact sequence of $P_J(F_v)$-modules:

$$0 \longrightarrow C_c^\infty(\Omega_J) \longrightarrow (\Pi_{J,v})_{Z_J} \longrightarrow (\Pi_{J,v})_{N_J} \longrightarrow 0.$$

Moreover, one can describe $(\Pi_{J,v})_{N_J}$ as a representation of $M_J(F_v)$ as follows.

- When $J = D$, we have:

$$(\Pi_{D,v})_{N_D} \cong \begin{cases} \delta_{P_D}^{2/11} \oplus \delta_{P_D}\Pi(M_D), \text{ if } D_v \text{ is unramified;} \\ \delta_{P_D}^{2/11}, \text{ if } D_v \text{ is ramified,} \end{cases}$$

  where $\Pi(M_D)$ is the minimal representation of $M_D(F_v)$ with center acting trivially.
- When $J = E$ and $E_v$ is a field, then

$$(\Pi_{E,v})_{N_E} \cong \chi_{E,v}\delta_{P_E}^{1/5} \oplus \chi_{E,v}^{-1}\delta_{P_E}^{1/5}.$$

  When $E_v$ is split, the space of $N_{E_v}$-coinvariants is somewhat complicated; we refer the reader to [GGJ, Prop. 4.4(ii)].

From this, one sees that there is an injection

$$j_v : \Pi_{J,v} \hookrightarrow Ind_{P_J}^{H_J} \chi_J \cdot \delta_{P_J}^{s_J} \quad \text{(unnormalized induction)}$$

where $\chi_J$ is trivial unless $J_v = E_v$ is a cubic field, in which case $\chi_J$ is one of the two cubic characters associated to $E_v$.

Further, for each non-trivial character $\chi$ of $N_J(F_v)$, one has:

$$\dim \mathrm{Hom}_{N_J(F_v)}(\Pi_{J,v}, \mathbb{C}_\chi) = \begin{cases} 1, & \text{if } \chi \in \Omega_J(F_v); \\ 0, & \text{otherwise.} \end{cases}$$

A non-zero element of this Hom space for $\chi \in \Omega_J(F_v)$ is given by

$$L^0_{\chi,v}(f) = i_v(f)(\chi).$$

Note that the stabilizer of $\chi$ in $M_J(F_v)$ acts on the above Hom space, and recall from the end of §2.2 that this stabilizer is a codimension one subgroup of $(M_J \cap Q_J)(F_v)$. Let $R_{J,\chi}$ denote the derived group of this stabilizer (which is also the derived group of $M_J \cap Q_J$). Then one knows that $R_{J,\chi}(F_v)$ acts trivially on $\mathrm{Hom}_{N_J(F_v)}(\Pi_{J,v}, \mathbb{C}_\chi)$ (cf. [GS2, §11]).

3.2. **A $P_J$-model.** The above gives a rather complete description of $(\Pi_{J,v})_{Z_J}$ as a $P_J(F_v)$-module. On the other hand, if $\chi$ is a non-trivial character of $Z_H(F_v)$, then as a representation of the derived group $P_{J,der}$ of $P_J$,

$$(\Pi_{J,v})_{Z_H,\chi} \cong \omega_\chi$$

where $\omega_\chi$ is the Weil representation $Mp(N_J/Z_J) \ltimes N_J$ (with central character $\chi$) pulled back to $P_{H,der}(F_v)$. Thus, we have a short exact sequence of $P_J(F_v)$-modules:

$$0 \longrightarrow ind_{P_{J,der}}^{P_J} \omega_\chi \longrightarrow \Pi_{J,v} \longrightarrow (\Pi_{J,v})_{Z_J} \longrightarrow 0.$$

In fact, one has

$$ind_{P_{J,der}}^{P_J} \omega_\chi \subset \Pi_{J,v} \subset Ind_{P_{J,der}}^{P_J} \omega_\chi.$$

The latter space can be realized as the space of $P_J(F_v)$-smooth functions on $F_v^\times$ taking values in $\omega_\chi$, while the former is the subspace of compactly supported such functions.

3.3. **A unitary model.** We do not know how to characterize the subspace $\Pi_{J,v}$ in the above space of $P_{J,v}$-smooth functions. However, we do know that the functions in $\Pi_{J,v}$ are square-integrable for the natural inner product on $Ind_{P_{J,der}}^{P_J} \omega_\chi$ and its unitary completion is

$$\widehat{\Pi_{J,v}} = L^2(F_v^\times; \widehat{\omega_\chi}).$$

On this unitary model, one can write down the action of $H_J(F_v)$ by giving the action of the Weyl group element $w_\beta$ (which together with $P_J(F_v)$ generates $H_J(F_v)$). Let us realize the unitary Weil representation $\widehat{\omega_\chi}$ on $L^2(F_v \times J_v)$; this is the Schrodinger model. Then $\widehat{\Pi_{J,v}}$ can be realized on $L^2(F_v^\times \times F_v \times J_v)$ and the action of $P_J(F_v)$ on this unitary model can be found in [Ru1, Prop. 43]. The action of $w_\beta$ is now given by [Ru1, Prop. 47]:

$$(w_\beta \cdot f)(t, a, x) = \chi\left(\frac{N_J(x)}{a}\right) \cdot f(-\frac{a}{t}, -a, x).$$

3.4. **A $Q_J$-model.** One can analogously give a model for $\Pi_{J,v}$ in which the action of $Q_J(F_v)$ can be neatly described. This is easier to describe when $J_v$ is a division algebra and so we shall restrict ourselves to this situation here.

Recall that the center $C_J$ of $U_J$ is two-dimensional and contains $Z_J$. Indeed, $C_J = Z_J \times U_{3\alpha+\beta}$. Moreover, the Levi subgroup $L_J(F_v)$ acts transitively on the non-trivial characters of $C_J(F_v)$. Let $\chi$ be the character of $C_J(F_v)$ defined by

$$\begin{cases} \chi|_{Z_J} = \text{trivial}; \\ \chi|_{U_{3\alpha+\beta}} = \psi_v. \end{cases}$$

The stabilizer in $Q_J$ of the line spanned by $\chi$ is precisely the minimal parabolic $F$-subgroup

$$B_J = P_J \cap Q_J.$$

Now by a version of Mackey theory, we have the following short exact sequence of $Q_J(F_v)$-modules:

$$0 \longrightarrow ind_{B_J}^{Q_J}\left(C_c^\infty(F_v^\times) \otimes (\Pi_{J,v})_{C_J,\chi}\right) \longrightarrow \Pi_{J,v} \longrightarrow (\Pi_{J,v})_{C_J} \longrightarrow 0.$$

To make this completely explicit, it remains to describe the $B_J$-action on $C_c^\infty(F_v^\times) \otimes (\Pi_{J,v})_{C_J,\chi}$ and the structure of $(\Pi_{J,v})_{C_J}$.

For this, note that both $(\Pi_{J,v})_{C_J,\chi}$ and $(\Pi_{J,v})_{C_J}$ are quotients of $(\Pi_{J,v})_{Z_J}$. From our description of $(\Pi_{J,v})_{Z_J}$ given in (3.1) and Lemma 2.1(i), it is now easy to see that

$$(\Pi_{J,v})_{C_J,\chi} = C_c^\infty(\Omega'_{J,v}) \quad \text{and} \quad (\Pi_{J,v})_{C_J} = (\Pi_{J,v})_{U_J} = (\Pi_{J,v})_{N_J}$$

where

$$\Omega'_{J,v} = \{(N_J(x), x^\#, x, 1) : x \in J_v\} \cong J_v.$$

Thus

$$C_c^\infty(F_v^\times) \otimes (\Pi_{J,v})_{C_J,\chi}) = C_c^\infty(\Omega_{J,v}) \cong C_c^\infty(F_v^\times \times J_v)$$

and the action of $B_J(F_v) \subset P_J(F_v)$ is simply given by the restriction of the action of $P_J(F_v)$ on $C_c^\infty(\Omega_{J,v})$ described in (3.1).

3.5. **Archimedean case.** Now assume that $v$ is an archimedean place of $F$, so that $J_v$ and $H_{J,v}$ are split. In this case, instead of working with (admissible) $(\mathfrak{g}, K)$-modules, we shall largely work with their Casselman-Wallach globalizations which are smooth Frechet representations of moderate growth. Similarly, when we work with automorphic forms later on, we shall work with smooth automorphic forms of uniform moderate growth [W], rather than their $K$-finite versions. For a reference for this, see [W] or [C].

Let us recall the main facts we need concerning the minimal representation of $H_J(F_v)$. As in the non-archimedean case, the minimal representation $\Pi_{J,v}$ admits a unique (up to scaling) embedding

$$j_v : \Pi_{J,v} \hookrightarrow Ind_{P_J}^{H_J} \chi_J \cdot \delta_{P_J}^{s_J} \quad \text{(unnormalized induction)},$$

where $\chi_J$ is the trivial character here. Moreover, for each non-trivial character $\chi$ of $N_J(F_v)$, one has

$$\mathrm{Hom}_{N_J(F_v)}(\Pi_{J,v}, \mathbb{C}_\chi) \neq 0 \Longleftrightarrow \chi \in \Omega_J(F_v),$$

where the Hom space considered here refers to the space of continuous linear functionals. When $\chi \in \Omega_J(F_v)$, one may extend $\chi$ to a character of $R_{J,\chi}(F_v) \cdot N_J(F_v)$ by insisting that $R_{J,\chi}(F_v)$ acts trivially. Then using Bruhat theory (cf. [GRS, Thm. 6.2] for a similar proof), one can show the multiplicity-one result:

$$\dim \mathrm{Hom}_{R_{J,\chi} \cdot N_J(F_v)}(\Pi_{J,v}, \mathbb{C}_\chi) = 1.$$

We shall denote a non-zero element of this Hom space for $\chi \in \Omega_J(F_v)$ by $L^0_{\chi,v}$.

3.6. **Automorphic realization of $\Pi_J$.** Now we consider the global setting. Let $\Pi_J = \otimes_v \Pi_{J,v}$ be the global minimal representation of $H_J(\mathbb{A})$. From the local results above, there is a $H_J(\mathbb{A})$-equivariant embedding

$$j : \Pi_J \hookrightarrow I_J(s_J) := Ind_{P_J}^{H_J} \chi_J \cdot \delta_{P_J}^{s_J}.$$

Because $\Pi_J$ is self-contragredient, we have

$$j^* : I_J(1-s_J) = Ind_{P_J}^{H_J} \chi_J^{-1} \cdot \delta_{P_J}^{1-s_J} \longrightarrow \Pi_J.$$

One can consider the Eisenstein series $E(f,s,g)$ associated to the degenerate principal series $I_J(s)$. In this regard, we should mention that one can work with not-necessarily-$K$-finite Eisenstein series here, in view of a recent result of Lapid [La] which extends some standard results in the theory of Eisenstein series (such as meromorphicity and functional equation) from the $K$-finite setting to the smooth setting.

**Proposition 3.1.** *For $f \in I_J(1-s_J)$, the Eisenstein series $E(f,s,g)$ has a pole of order 1 at $s = 1 - s_J$. The map*

$$f \mapsto Res_{s=1-s_J} E(f,s,-)$$

*factors through $j^*$ and induces a non-zero embedding of $\Pi_J$ into the space $\mathcal{A}_2(H_J)$ of square-integrable automorphic forms on $H_J(\mathbb{A})$.*

This proposition was shown in an unpublished manuscript [Ru2] of Rummelhart when $J = D$. When $J = E$, it was shown in [GGJ, Prop. 5.1(ii)]. Using the functional equation satisfied by the Eisenstein series, we obtain:

**Corollary 3.2.** *If $f \in \Pi_J$, then $E(j(f),s,g)$ is holomorphic at $s = s_J$ and the map $f \mapsto E(j(f),s_J,-)$ gives a non-zero $H_J$-equivariant embedding*

$$\theta_J : \Pi_J \hookrightarrow \mathcal{A}_2(H_J).$$

*Proof.* If $\Phi_s \in I_J(s)$ is a flat section, then the functional equation for Eisenstein series gives:

$$E(M(s)\Phi, 1-s, -) = E(\Phi, s, -)$$

where

$$M(s) : I_J(s) \longrightarrow I_J(1-s)$$

is a standard intertwining operator. At $s = 1 - s_J$, both $E(\Phi,s,-)$ and $M(s)$ have a pole of order 1 (for some $\Phi$). Moreover, the residue $M^*(1-s_J)$ of $M(s)$ at $s = 1-s_J$ is (up to scalars) the map $j \circ j^*$, so that $M^*(1-s_J)(\Phi) = j(j^*(\Phi)) \in I_J(s_J)$. Taking residues at $s = 1-s_J$ on both sides of the functional equation and using the proposition, we deduce the corollary. ∎

3.7. **Fourier coefficients of $\Pi_J$.** Let us fix a non-trivial unitary character $\psi$ of $F\backslash\mathbb{A}$. Then the set of unitary characters of $N_J(\mathbb{A})$ which is trivial on $N_J(F)$ can be identified with $\bar{V}_J(F)$ using $\psi$, the Killing form and the exponential map. In particular, it makes sense to say whether $\chi$ belongs to the minimal orbit $\Omega_J(F)$.

Suppose that $\chi$ is a character of $N_J(\mathbb{A})$ which is trivial on $N_J(F)$. Then for each $f \in \Pi_J$, we may consider the Fourier coefficient $\theta_J(f)_{N_J,\chi}$ of $\theta_J(f)$ with respect to $\chi$. It is a function on $H_J(\mathbb{A})$ given by

$$\theta_J(f)_{N_J,\chi}(h) = \int_{N_J(F)\backslash N_J(\mathbb{A})} \overline{\chi(n)} \cdot \theta_J(f)(nh)\, dn.$$

The continuous linear functional on $\Pi_J$ given by

$$L_\chi : f \mapsto \theta_J(f)_{N_J,\chi}(1)$$

is clearly an element of the space $\mathrm{Hom}_{N_J(\mathbb{A})}(\Pi_J, \mathbb{C}_\chi)$. Thus, in view of the local results of §3.1, the linear functional $L_\chi$ is nonzero iff $\chi \in \Omega_J(F)$ or $\chi$ is trivial.

Suppose then that $\chi \in \Omega_J(F)$. In this case, by the discussion at the end of §3.1, $L_\chi$ is necessarily fixed by the finite-adelic group $R_{J,\chi}(\mathbb{A}_{fin})$. Since $L_\chi$ is clearly fixed by $R_{J,\chi}(F)$, it follows by the weak approximation theorem that $L_\chi$ is fixed by the adelic group $R_{J,\chi}(\mathbb{A})$. In view of the local uniqueness results at the end of §3.1 and §3.5, we deduce that $L_\chi$ is Eulerian, i.e. for a decomposable vector $f = \otimes_v f_v \in \Pi_J$, one has

$$L_\chi(f) = \prod_v L^0_{\chi,v}(f_v)$$

where the $L^0_{\chi,v}$'s are the corresponding local functionals. In particular, we have:

$$\theta_J(f)_{N_J,\chi}(h) = \prod_v L^0_{\chi,v}(h_v \cdot f_v).$$

3.8. **Representations of $G_2$.** We introduce some notations for representations of $G_2$. If $\chi_1$ and $\chi_2$ are two characters of $F_v^\times$, we shall let $\pi(\chi_1, \chi_2)$ denote the principal series of $GL_2$ unitarily induced from the character $\chi_1 \times \chi_2$ of the diagonal torus. We also let $St$ denote the Steinberg representation.

Recall that the Levi subgroups $M$ and $L$ of the two parabolic subgroups are both isomorphic to $GL_2$. We fix these isomorphisms so that

$$\delta_P = |det|^3 \quad \text{and} \quad \delta_Q = |det|^5.$$

Now if $\tau$ is a representation of $GL_2$ and $R = P$ or $Q$, then we set

$$I_R(\tau, s) = Ind_R^{G_2} \delta_R^{1/2} \cdot (\tau \otimes |det|^s)).$$

If $I_R(\tau, s)$ has a unique Langlands quotient, we denote this quotient by $J_R(\tau, s)$. If $\tau$ is the trivial representation $\mathbf{1}$, we shall simply write $I_R(s)$ for $I_R(\mathbf{1}, s)$. We shall also denote by $R_P$ or $R_Q$ the normalized Jacquet functor with respect to $P$ and $Q$ respectively. Thus, for example,

$$R_P(\pi) = \delta_P^{-1/2} \cdot \pi_N.$$

Similarly, if $B = P \cap Q$ is the Borel subgroup of $G_2$ and $\chi$ is a character of $T = M \cap L$, we write $I_B(\chi)$ for the normalized induction of $\chi$.

We shall be especially interested in the degenerate princiapl series $I_P(1/2)$. The following lemma describes the structure of this representation (cf. [M] for non-archimedean cases and [G2, Lemma 4.8 and §4.10] for archimedean cases).

**Lemma 3.3.** *(i) If $v$ is $p$-adic, then $I_{P,v}(1/2)$ has a filtration*

$$0 \subset I_{0,v} \subset I_{1,v} \subset I_{2,v} = I_{P,v}(1/2)$$

*such that*

$$\begin{cases} I_{0,v} \cong \Sigma_v \\ I_{1,v}/I_{0,v} \cong J_{Q,v}(St, 1/2) \\ I_{2,v}/I_{1,v} \cong J_{Q,v}(\pi(1,1), 1). \end{cases}$$

*Moreover, $\Sigma_v$ is the unique irreducible submodule and is a discrete series representation with 1-dimensional space of Iwahori-fixed vectors. The representation $J_{Q,v}(\pi(1,1),1)$ is the unique irreducible quotient.*

*(ii) If $v$ is real, then there is a non-split exact sequence*

$$0 \longrightarrow \pi_\kappa \longrightarrow I_{P,v}(1/2) \longrightarrow J_{Q,v}(\pi(1,1),1) \longrightarrow 0.$$

*(iii) If $v$ is complex, $I_{P,v}(\frac{1}{2}) = J_{Q,v}(\pi(1,1),1)$ is irreducible.*

*(iv) In each case, the unique irreducible quotient $J_{Q,v}(\pi(1,1),1)$ is spherical.*

3.9. **Distinguished representations.** Using the fixed additive character $\psi_v$ and the Killing form, we may identify $\bar{V}$ with the set of unitary characters of $N(F_v)$. As we noted above, the $M(F_v)$-orbits on $\bar{V}$ are indexed by cubic $F_v$-algebras. If $B_v$ is an étale cubic algebra, let $\psi_{B_v}$ be a character of $N(F_v)$ in the orbit indexed by $B_v$. If $\pi_v$ is a representation of $G_2$, we say that $\pi_v$ is $B_v$*-distinguished* if

$$\mathrm{Hom}_N(\pi_v, \mathbb{C}_{\psi_{B'_v}}) = \begin{cases} 0, \text{ if } B'_v \neq B_v; \\ \text{non-zero, if } B'_v = B_v. \end{cases}$$

For example, it is known that the representation $J_{Q,v}(\pi(1,1),1)$ is $F_v^3$-distinguished (cf. [GGJ]).

3.10. **The representation $\Sigma_v$.** We conclude this preliminary section with a closer look at the representation $\Sigma_v$. The representation $\Sigma_v$ is a discrete series representation, but the key fact we need is that $\Sigma_v$ has a 1-dimensional space of Iwahori-fixed vectors.

Let $K_v = G_2(\mathcal{O}_{F_v})$ be a hyperspecial maximal compact subgroup of $G_2(F_v)$, and let $K_1$ be its first principal congruence subgroup so that $K_v/K_1 \cong G_2(\mathbb{F}_q)$. Because $\Sigma_v$ has a 1-dimensional space of Iwahori-fixed vectors, $\Sigma_v^{K_1}$ is an *irreducible* representation of $G_2(\mathbb{F}_q)$ with a 1-dimensional space of Borel-fixed vectors. In particular, $\Sigma_v^{K_1}$ occurs with multiplicity one in

$$Ind_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(1) = I_{B,v}(\chi)^{K_1}$$

for any unramified character $\chi$ of the Borel subgroup $B(F_v)$.

We shall need the following proposition later when we study the effect of the standard intertwining operators on $\Sigma_v$.

**Proposition 3.4.** *(i) Consider the natural $Q(\mathbb{F}_q)$-equivariant map*

$$\Sigma_v^{K_1} \hookrightarrow Ind_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)} 1 \longrightarrow Ind_{(B\cap L)(\mathbb{F}_q)}^{L(\mathbb{F}_q)}(1) \cong \mathbf{1} \oplus St_L,$$

*where the second map is restriction of functions and $St_L$ is the Steinberg module of $L(\mathbb{F}_q)$. The image of this map is equal to the Steinberg module $St_L$.*

*(ii) Similarly, the image of the $P(\mathbb{F}_q)$-equivariant map*

$$\Sigma_v^{K_1} \hookrightarrow Ind_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)} 1 \longrightarrow Ind_{(B\cap M)(\mathbb{F}_q)}^{M(\mathbb{F}_q)}(1) \cong \mathbf{1} \oplus St_M,$$

*is contained in the trivial submodule $\mathbf{1}$.*

*Proof.* (i) Clearly, this map is non-zero. If the image includes the trivial representation, then we would have a non-zero $Q(\mathbb{F}_q)$-equivariant projection $\Sigma_v \longrightarrow \mathbf{1}$. This would imply that $\Sigma_v$ has a non-zero $Q(\mathbb{F}_q)$-fixed vector $f$. This vector $f$ must be the unique (up to scaling) $B(\mathbb{F}_q)$-fixed vector. Now because

$$\Sigma_v^{K_1} \hookrightarrow I_{P,v}(1/2)^{K_1} = Ind_{P(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)}(1),$$

we see that the $B(\mathbb{F}_q)$-fixed vector in $\Sigma_v^{K_1}$ is in fact fixed by $P(\mathbb{F}_q)$. Since $P(\mathbb{F}_q)$ and $Q(\mathbb{F}_q)$ generate $G_2(\mathbb{F}_q)$, the vector $f$ would be fixed by $G_2(\mathbb{F}_q)$, contradicting the fact that $\Sigma_v^{K_1}$ is irreducible and non-trivial.

(ii) This follows immediately from the fact that

$$\Sigma_v^{K_1} \hookrightarrow Ind_{P(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)} 1,$$

□

## 4. **Local Theta Lifts**

In this section, we recall various results concerning local theta correspondence for the two dual pairs of interest in this paper. The key result to take note here is Proposition 4.5.

4.1. **Lifting from $S_E$ to $G_2$.** Consider first the dual pair $S_E \times G_2$. For each place $v$ of $F$, we may write:

$$\Pi_{E,v} = \bigoplus_{\eta_v} \eta_v \otimes \theta_{E_v}(\eta_v)$$

for some representations $\theta_{E_v}(\eta_v)$ of $G_2(F_v)$, as $\eta_v$ ranges over the irreducible characters of $S_E(F_v)$. By results of Huang-Magaard-Savin [HMS, Prop. 6.1] and Vogan [V, Thm. 18.10], we have:

**Proposition 4.1.** *(i) Each $\theta_{E_v}(\eta_v)$ is non-zero and irreducible.*

*(ii) If $E_v$ is split, then*

$$\theta_{E_v}(\mathbf{1_v}) \cong J_{Q,v}(\pi(1,1),1).$$

*(iii) If $E_v$ is a field (so that $v$ is finite) and $\{\chi_{E,v}, \chi_{E,v}^{-1}\}$ the two cubic characters of $S_E(F_v) = Gal(E_v/F_v)$, then*

$$\theta_{E_v}(\mathbf{1_v}) \cong J_{Q,v}(\pi(\chi_{E,v}, \chi_{E,v}^{-1}), 1).$$

*Further, $\theta_{E_v}(\chi_{E,v})$ and $\theta_{E_v}(\chi_{E,v}^{-1})$ are supercuspidal and contragredient to each other. When $E_v$ is unramified, $\theta_{E_v}(\chi_{E,v})$ and $\theta_{E_v}(\chi_{E,v}^{-1})$ are obtained by compact induction:*

$$\theta_{E_v}(\chi_{E,v}) \cong ind_{G_2(\mathcal{O}_{F_v})}^{G_2(F_v)} \pi(\chi_{E,v}) \quad \text{and} \quad \theta_{E_v}(\chi_{E,v}^{-1}) \cong ind_{G_2(\mathcal{O}_{F_v})}^{G_2(F_v)} \pi(\chi_{E,v}^{-1})$$

*where $\pi(\chi_{E,v})$ and $\pi(\chi_{E,v}^{-1})$ are two unipotent supercuspidal representations of $G_2(\mathbb{F}_q)$ and are the minimal $K_v$-types of $\theta_{E_v}(\chi_{E,v})$ and $\theta_{E_v}(\chi_{E_v}^{-1})$ respectively.*

**Remarks:** The results of [HMS] were established under the assumption that the residual characteristic of $F_v$ is not equal to 2. This is because the authors made use of a result of Moeglin-Waldspurger relating the leading term of the local character expansion of a representation to the twisted Jacquet modules supported by the representation. In fact, one does not need to appeal to the local character expansion to obtain control of the possible twisted Jacquet modules supported by the minimal representation, as explained in [GS2]. With this knowledge of the twisted Jacquet modules of the minimal representation, the results of [HMS] can be shown regardless of residual characteristic.

In view of this proposition, we have the global decomposition

$$\Pi_E = \bigoplus_{\eta} \eta \otimes \theta_E(\eta)$$

as $\eta = \otimes_v \eta_v$ runs over the irreducible characters of $S_E(\mathbb{A})$, and

$$\theta_E(\eta) = \bigotimes_v \theta_{E,v}(\eta_v).$$

If we realize $\Pi_E$ as a submodule in $\mathcal{A}_2(H_E)$ using $\theta_E$, we may consider the restriction of $f \in \Pi_E$ to the subgroup $G_2$. The following proposition was shown in [GGJ, Thm. 8.2(i)]:

**Proposition 4.2.** *The restriction of functions from $H_E(\mathbb{A})$ to $G_2(\mathbb{A})$ gives a $G_2(\mathbb{A})$-equivariant map*

$$\Pi_E \longrightarrow \mathcal{A}_2(G_2).$$

*Moreover, the image is isomorphic to*

$$\bigoplus_{\eta} \eta^{S_E(F)} \otimes \theta_E(\eta).$$

*In particular, the $\chi_E^D$-isotypic part $\Pi_E[\chi_E^D]$ of $\Pi_E$ is an irreducible $G_2$-module and admits an embedding into $\mathcal{A}_2(G_2)$.*

4.2. **Lifting from $PGL_3$ to $G_2$.** Suppose that $D_v$ is split, so that we are looking at the dual pair

$$PGL_3 \times G_2 \subset H_{D_v}.$$

If $v$ is non-archimedean, the theta correspondence for this dual pair was essentially completely determined in [GS1]. For archimedean $v$'s, the results are somewhat fragmentary. Hence, we shall focus on the non-archimedean case here.

For each irreducible representation $\pi_v$ of $PGL_3$, the maximal $\pi_v$-isotypic quotient of $\Pi_{D_v}$ has the form $\pi_v \otimes \Theta_{D_v}(\pi_v)$ for some smooth representation $\Theta_{D_v}(\pi_v)$ of $G_2(F_v)$. We let $\theta_{D_v}(\pi_v)$ be the maximal semisimple quotient of $\Theta_{D_v}(\pi_v)$. We are interested in the cases when $\pi_v$ is trivial or is a cubic character $\chi_{E_v}$. The following proposition was shown in [GS1]:

**Proposition 4.3.** *Let $v$ be a finite place of $F$.*

*(i) If $\pi_v = \mathbf{1_v}$, then*

$$\Theta_{D_v}(\mathbf{1_v}) = I_{P,v}(1/2)$$

*and thus*

$$\theta_{D_v}(\mathbf{1_v}) = J_{Q,v}(\pi(1,1),1)$$

*is the unique irreducible quotient of $\Theta_{D_v}(\mathbf{1_v})$.*

*(ii) If $\pi_v = \chi_{E,v}$ is a cubic character, then*

$$\theta_{D_v}(\chi_{E,v}) = J_{Q,v}(\pi(\chi_{E,v}, \chi_{E,v}^{-1}), 1).$$

4.3. **Remarks:** Indeed, if we realize $\Pi_{D_v}$ as a submodule of $I_D(2/11)$, then by restriction of functions, we have a natural map

$$Res_v : \Pi_{D_v} \hookrightarrow I_D(2/11) \longrightarrow I_{P,v}(1/2).$$

This map is clearly $PD_v$-invariant and $G_2$-equivariant. Moreover, it is surjective, since the spherical vector of $I_D(2/11)$ restricts to that of $I_{P,v}(1/2)$ and the latter is generated by its spherical vector. In view of Prop. 4.3(i), this natural map is precisely the projection of $\Pi_{D_v}$ onto its maximal $\mathbf{1_v}$-isotypic quotient. When $v$ is archimedean, we still have this natural map. However, we don't know if $I_{P,v}(1/2)$ is the maximal $\mathbf{1_v}$-isotypic quotient or not.

4.4. **Lifting from $PD_v^\times$ to $G_2$.** Now assume that $D_v$ is a division algebra, so that $v$ is finite. In this case, since $PD_v^\times$ is compact, we may write:

$$\Pi_{D_v} = \bigoplus_{\pi_v} \pi_v \otimes \theta_{D_v}(\pi_v)$$

as $\pi_v$ runs over the irreducible representations of $PD_v^\times$. The theta correspondence here has been investigated by Savin [Sa], who showed:

**Proposition 4.4.** *Assume that $D_v$ is a division algebra. Then*

$$\theta_{D_v}(\mathbf{1_v}) = \Sigma_v$$

*is the unique irreducible submodule of $I_{P,v}(1/2)$. Further, if $\chi_{E,v}$ is a cubic character associated to the cyclic cubic field $E_v$, then $\theta_{D_v}(\chi_{E,v})$ and $\theta_{D_v}(\chi_{E,v}^{-1})$ are irreducible supercuspidal representations which are contragredient to each other. Moreover, when $E_v$ is unramified, we have:*

$$\theta_{D_v}(\chi_{E,v}) \cong ind_{G_2(\mathcal{O}_{F_v})}^{G_2(F_v)} \pi(\chi_{E,v}^{D_v}) \quad \text{and} \quad \theta_{D_v}(\chi_{E,v}^{-1}) \cong ind_{G_2(\mathcal{O}_{F_v})}^{G_2(F_v)} \pi((\chi_{E,v}^{D_v})^{-1})$$

*where $\pi(\chi_{E,v})$ and $\pi(\chi_{E,v}^{-1})$ are the two unipotent supercuspidal representations of $G_2(\mathbb{F}_q)$ encountered in Prop. 4.1(ii).*

4.5. **Remarks:** As above, the projection $\Pi_{D_v} \longrightarrow \Sigma_v$ can be realized by

$$Res_v : \Pi_{D_v} \hookrightarrow I_{P_D,v} \longrightarrow I_{P,v}(1/2).$$

This time, however, the map is not surjective: its image is the unique irreducible submodule $\Sigma_v$.

4.6. **Identity of local theta lifts.** In view of Propositions 4.1, 4.3 and 4.4, we have the following crucial proposition:

**Proposition 4.5.** *Assume that*

(i) *$v$ is a finite place;*
(ii) *if $D_v$ is ramified, then $E_v$ is the unramified cubic field extension of $F_v$.*

*Then we have:*

$$\theta_{D_v}(\chi_{E,v}) \cong \theta_{E_v}(\chi_{E,v}^{inv(D_v)})$$

*as representations of $G_2(F_v)$.*

This proposition is the local analog of the global Theorem 1.2. Ideally, we would like to remove the two conditions in the proposition, since it will shorten the proof of Theorem 1.2 considerably. Unfortunately, the local theta correspondence at archimedean places is not very well understood. However, we shall be able to finesse this local difficulty by using some interplay of local and global considerations. As for removing the second condition, we only have the following result to offer:

**Proposition 4.6.** *Assume that $D_v$ is a division algebra (so that $v$ is finite). Then we have*

$$\theta_{D_v}(\chi_{E,v}) \cong \theta_{E_v}(\chi_{E,v}^{inv(D_v)})$$

*as representations of $Q(F_v)$.*

4.7. **An explicit isomorphism.** We shall prove Prop. 4.6 by writing down an explicit isomorphism using the unitary model of $\Pi_{J,v}$ described in (3.3). Recall that we have

$$\widehat{\Pi_{D,v}} = L^2(F_v^\times \times F_v \times D_v) \quad \text{and} \quad \widehat{\Pi_{E,v}} = L^2(F_v^\times \times F_v \times E_v).$$

Let us focus on the $\chi_{E,v}$-isotypic space in $\widehat{\Pi_{D,v}}$. The action of $PD_v^\times$ on $L^2(F_v^\times \times F_v \times D_v)$ is via its adjoint action on $D_v$. Now a Schwarz function on $D_v$ which is a $\chi_{E,v}$-eigenform for $PD_v^\times$ is completely determined by its restrictions to $E_v \hookrightarrow D_v$. Thus under restriction of functions, we have

$$S(D_v)[\chi_{E,v}] \cong S(E_v)[\chi_{E_v}^{D_v}].$$

Now we can define a map

$$\Xi : \Pi_{D,v}[\chi_{E,v}] \longrightarrow \Pi_{E,v}[\chi_{E_v}^{D_v}]$$

by

$$\Xi(f)(t,a,x) = |N_E(x_0)|_v \cdot f(t,a,x).$$

Here

$$x_0 = x - \frac{1}{3}T_E(x)$$

is the trace zero part of $x \in E_v$. Then we have

**Proposition 4.7.** *Assume that $D_v$ is a division algebra. The map $\Xi$ is bijective and $Q(F_v)$-equivariant. Thus it gives an explicit isormorphism*

$$\theta_{D_v}(\chi_{E,v}) \cong \theta_{E_v}(\chi_{E,v}^{inv(D_v)})$$

*of $Q(F_v)$-modules.*

*Proof.* As we explained in (3.3), [Ru1, Props. 43 and 47] give the formulas for the action of a set of generators of $G_2(F_v)$ on both minimal representations. Using these formulas, it is straightforward to check the $Q(F_v)$-equivariance. We omit the details since the result is not used in the rest of the paper. □

4.8. **Remarks:**

(i) By composing $\Xi$ with the projection map from $\Pi_{D,v}$ to its $\chi_{E,v}$-isotypic subspace, we have a map

$$\Delta : \Pi_{D,v} \longrightarrow \Pi_{D,v}[\chi_{E,v}] \xrightarrow{\ \Xi\ } \Pi_{E,v}[\chi_{E_v}^{D_v}].$$

This is given by:

$$\Delta(f)(t,a,x) = |N_E(x_0)| \cdot \int_{E_v^\times \backslash D_v^\times} f(t,a,h^{-1}xh) \cdot \overline{\chi_{E,v}(N_{D_v}(h))}\, dh.$$

Compare this with the map in Snitz's thesis [S] described after Theorem 1.1.

(ii) What prevents one from showing that $\Xi$ is $G_2$-equivariant? The only remaining thing to check is that $\Xi$ is equivariant for the action of the Weyl group element $w_\alpha$. According to [Ru1, Prop. 43], the action of $w_\alpha$ on $\Pi_{J,v}$ is essentially given by the Fourier transform over $F_v \times J_v$. Thus the identity

$$\Xi(w_\alpha \cdot f) = w_\alpha \cdot \Xi(f)$$

reduces to an equality of the type:

(Fourier transform of $f$ along $D_v$)$|_{E_v}$ = Fourier transform of $f_v|_{E_v}$ along $E_v$.

In Snitz's thesis [S], one has the quadratic analog of this identity, with $D_v$ a quaternion algebra [S, Prop. 46]. The verification of this identity in [S] is one of the most non-trivial computation there (see [S, Pg. 41-47]). Unfortunately, because of our unfamiliarity with cubic division algebras, we are not able to verify the desired identity in the cubic case by a direct computation.

## 5. **Siegel-Weil Formula**

In this section, we consider the dual pair

$$PD^{\times} \times G_2 \subset H_D$$

and proves the honest Siegel-Weil formula which identifies the theta lift of the trivial representation of $PD^{\times}$.

Recall that Corollary 3.2 gives us an embedding

$$\theta_D : \Pi_D \hookrightarrow \mathcal{A}_2(H_D).$$

We may thus consider the theta integral

$$I(f)(g) = \int_{G_D(F)\backslash G_D(\mathbb{A})} \theta_D(f)(gh)\, dh$$

which converges absolutely and defines an element of $\mathcal{A}(G_2)$. Indeed, $I$ defines a $G_D(\mathbb{A})$-invariant and $G_2(\mathbb{A})$-equivariant map from $\Pi_D$ to $\mathcal{A}(G_2)$. Here, the Haar measure $dh$ on $G_D(\mathbb{A})$ is chosen so that

$$\int_{G_D(F)\backslash G_D(\mathbb{A})} dh = 1.$$

On the other hand, we have another such map from $\Pi_D$ to $\mathcal{A}(G_2)$ given as follows. There is a natural map

$$Res : \Pi_D \hookrightarrow I_D(2/11) \longrightarrow I_P(1/2)$$

given by restriction of functions. The map $Res : \Pi_D \longrightarrow I_P(1/2)$ is non-zero, $G_D(\mathbb{A})$-invariant and $G_2(\mathbb{A})$-equivariant.

The main theorem of this section is:

**Theorem 5.1.** *For any $f \in \Pi_D$, the Eisenstein series $E(Res(f), s, g)$ is holomorphic at $s = \frac{1}{2}$ and $E(Res(f), 1/2, -)$ is nonzero if and only if $Res(f)$ is nonzero. Moreover,*

$$I(f)(g) = E(Res(f), 1/2, g).$$

The rest of the section is devoted to the proof of this theorem.

5.1. **Constant terms of $\Pi_D$.** We begin with the left hand side of the Siegel-Weil formula, i.e. the theta integral. For this, we need to recall some properties of the automorphic realization $\theta_D$ of the minimal representation. In particular, we need to know the constant terms of $\theta(\Pi_D)$ along the unipotent radicals of the two maximal parabolic subgroups of $H_D$. The following proposition is proved in the course of constructing the automorphic realization $\theta_D$ of $\Pi_D$:

**Proposition 5.2.** *For $f \in \Pi_D \subset I_D(2/11)$,*

$$\theta_D(f)_{N_D}(g) = f(g).$$

*When pulled back to a function on $GL_2 \times (SL_1(D) \times_{\mu_3} SL_1(D))$, $\theta(f)_{U_D}$ is the product of an Eisenstein series of $GL_2$ and a constant function on $SL_1(D) \times_{\mu_3} SL_1(D)$.*

5.2. **The representation** $Res(\Pi_D)$**.** Now we come to the Eisenstein series side of the Siegel-Weil formula. The following lemma describes the image of the restriction map $Res$ and is a consequence of the remarks in (4.3) and (4.5).

**Lemma 5.3.** *Let $S_D$ be the set of places $v$ where $D_v$ is non-split. Then the image of $Res$ is:*

$$Res(\Pi_D) = \left(\bigotimes_{v\in S_D} \Sigma_v\right) \otimes \left(\bigotimes_{v\notin S_D} I_{P,v}(\frac{1}{2})\right).$$

We note that $\#S_D \geq 2$.

5.3. **The Eisenstein series** $E(Res(f), s, g)$**.** Let us consider the Eisenstein series $E(\Phi, s, g)$ associated to a standard section $\Phi_s \in I_P(s)$. It was shown by Ginzburg-Jiang [GJ] that $E(\Phi, s, g)$ has at most a double pole at $s = 1/2$ and this double pole is attained, for example, by the spherical section. However, in view of Lemma 5.3, we are only interested in those standard sections which pass through $Res(\Pi)$. For these sections, the leading term of the Laurent expansion vanishes and so we are led to study the higher order terms in the Laurent expansion

$$E(\Phi, s, g) = \frac{A_{-2}(\Phi)}{(s-\frac{1}{2})^2} + \frac{A_{-1}(\Phi)(g)}{s-\frac{1}{2}} + ...$$

**Proposition 5.4.** *(i) The maps $A_{-2}$ and $A_{-1}$ are both zero on $Res(\Pi_D)$, but $A_0$ is non-zero on $Res(\Pi_D)$.*

*(ii) For $f \in \Pi_D$, the constant term of $A_0(Res(f))$ along $N$ is given by:*

$$A_0(Res(f))_N(g) = Res(f)(g).$$

*(iii) The map $A_0 : Res(\Pi_D) \longrightarrow \mathcal{A}(G_2)$ is $G_2$-equivariant.*

*Proof.* The proof of (i) and (ii) is a fairly standard exercise in the theory of Eisenstein series, albeit a non-trivial and somewhat intricate one.

To simplify notation, let us write $S$ for $S_D$. We shall work with a distinguished standard section $\Phi_s = \otimes_v \Phi_{v,s}$ defined as follows. For $v \in S$, let $\Phi_{v,s}$ be the standard section associated to a non-zero vector in the 1-dimensional space of Iwahori-fixed vectors. For $v \notin S$, we let $\Phi_{v,s}$ be the spherical section. By Lemma 5.3, the representation $Res(\Pi_D)$ is generated by the distinguished vector $\Phi_{1/2}$. Thus it suffices to show $E(\Phi, s, g)$ is holomorphic at $s = 1/2$ with constant term $\Phi_{1/2}(g)$.

The constant term of $E(\Phi, s, g)$ along the unipotent radical $N_0$ of the Borel subgroup $B$ is given by [MW, Pg. 91-92, Prop. II 1.7]:

$$E_{N_0}(\Phi, s, g) = \Phi_s(g) + M(w_\beta, s)(\Phi)(g) + M(w_\alpha w_\beta, s)(\Phi)(g) +$$
$$+ M(w_\beta w_\alpha w_\beta, s)(\Phi)(g) + M(w_\alpha w_\beta w_\alpha w_\beta, s)(\Phi)(g) + M(w_\beta w_\alpha w_\beta w_\alpha w_\beta, s)(\Phi)(g).$$

Here, $w_\alpha$ and $w_\beta$ are representatives in $G_2(\mathbb{Z})$ of the simple reflections in the Weyl group of $G_2$ associated to the simple roots $\alpha$ and $\beta$. Moreover, $M(w, s)$ is the standard intertwining operator associated to an element $w$ in the Weyl group and we are interested in its analytic behaviour at $s = 1/2$. Indeed, since

$$I_P(1/2) \hookrightarrow I_B(\chi_0)$$

where

$$\chi_0 = \alpha + \beta,$$

$M(w, 1/2)$ is the restriction to $I_P(1/2)$ of the standard intertwining operator

$$M(w, \chi_0) : I_B(\chi_0) \longrightarrow I_B(w \cdot \chi_0).$$

These intertwining operators factor as:

$$M(w,s)(\Phi)(g) = M_S(w,s)(\Phi_S)(g_S) \cdot M^S(w,s)(\Phi^S)(g^S)$$

where

$$M_S(w,s)(\Phi_S)(g_S) = \prod_{v \in S} M_v(w,s)(\Phi_v)(g_v)$$

and

$$M^S(w,s)(\Phi^S)(g^S) = \prod_{v \notin S} M_v(w,s)(\Phi_v)(g_v).$$

As discovered by Langlands in his classic monograph "Euler Products" [L], the value of $M^S(w,s)(\Phi^S)(1)$ can be calculated by the Gindikin-Karpelevich formula, yielding a product of local zeta functions. These unramified calculations were carried out by Ginzburg-Jiang in [GJ]. We tabulate the results below, describing the behaviour of $M^S(w,s)(\Phi^S)(1)$ at $s = 1/2$.

| $w$ | $M^S(w, \frac{1}{2})(\Phi^S)(1)$ | Behaviour at $s = \frac{1}{2}$ |
|---|---|---|
| 1 | 1 | holomorphic and non-zero |
| $w_\beta$ | $\frac{\zeta^S(1)}{\zeta^S(2)}$ | pole of order 1 |
| $w_\alpha w_\beta$ | $\frac{\zeta^S(1)}{\zeta^S(2)} \cdot \frac{\zeta^S(2)}{\zeta^S(3)}$ | pole of order 1 |
| $w_\beta w_\alpha w_\beta$ | $\frac{\zeta^S(1)}{\zeta^S(2)} \cdot \frac{\zeta^S(2)}{\zeta^S(3)} \cdot \frac{\zeta^S(1)}{\zeta^S(2)}$ | pole of order 2 |
| $w_\alpha w_\beta w_\alpha w_\beta$ | $\frac{\zeta^S(1)}{\zeta^S(2)} \cdot \frac{\zeta^S(2)}{\zeta^S(3)} \cdot \frac{\zeta^S(1)}{\zeta^S(2)} \cdot \frac{\zeta^S(1)}{\zeta^S(2)}$ | pole of order 3 |
| $w_\beta w_\alpha w_\beta w_\alpha w_\beta$ | $\frac{\zeta^S(1)}{\zeta^S(2)} \cdot \frac{\zeta^S(2)}{\zeta^S(3)} \cdot \frac{\zeta^S(1)}{\zeta^S(2)} \cdot \frac{\zeta^S(1)}{\zeta^S(2)} \cdot \frac{\zeta^S(0)}{\zeta^S(1)}$ | zero of order $\#S - 3$ |

Observe that if $S$ is empty and $\Phi$ is the spherical section, the last 2 terms in the table have poles of order 3. However, because the residues of the complete zeta function at $s = 0$ and $s = 1$ sum to zero, the Eisenstein series only has a pole of order at most 2 at $s = 1/2$.

At the places in $S$, the local intertwining operators can be analyzed by decomposing them as a composition of operators associated to the simple reflections $w_\alpha$ and $w_\beta$, and then appealing to results about intertwining operators for $SL_2$. We tabulate the results below before explaining how they are proved.

| $w$ | Behaviour of $M_S(w, \frac{1}{2})(\Phi_S)$ at $s = \frac{1}{2}$ |
|---|---|
| 1 | holomorphic and non-zero |
| $w_\beta$ | zero of order $\#S$ |
| $w_\alpha w_\beta$ | zero of order of order $\#S$ |
| $w_\beta w_\alpha w_\beta$ | zero of order $2 \cdot \#S$ |
| $w_\alpha w_\beta w_\alpha w_\beta$ | zero of order $2 \cdot \#S$ |
| $w_\beta w_\alpha w_\beta w_\alpha w_\beta$ | zero of order $\#S$ |

By the above two tables, and using the fact that $\#S \geq 2$, one sees easily that if $w \neq 1$, $M(w,s)(\Phi)(g)$ vanishes at $s = 1/2$. This proves (i) and (ii) of the proposition. To deduce (iii), one considers the difference

$$F(h) = A_0(Res(g \cdot f))(h) - A_0(Res(f))(hg)$$

as a function of $h$ (with $g$ fixed). It is an automorphic form whose cuspidal support is the Borel subgroup. However, by (ii), its constant term along $N_0$ is

$$(g \cdot f)(h) - f(hg) = 0.$$

Thus $F$ is zero and (iii) is proved. Hence, the proposition is proved as long as we can justify the second table above.

This justification is the main difficulty of the proof. The key inputs are Prop. 3.4 and the following simple observation.

**Observation:** Let $w_\gamma$ be a reflection in a simple root $\gamma$ and let $G_\gamma$ be the associated $SL_2$. Suppose that $f \in I_B(\chi)$ and consider $M(w_\gamma, \chi) : I_B(\chi) \longrightarrow I_B(w_\gamma \chi)$. Let $M^*(w_\gamma, \chi)$ be the leading term in the Laurent expansion of $M(w_\gamma, \chi)$ at $\chi$, so that $M^*(w_\gamma, \chi)$ is a $G_2$-equivariant map. Then $M_v^*(w_\gamma, \chi)(f) = 0$ if and only if for all $f'$ in the $K_v$-span of $f$, the restriction of $f'$ to $G_\gamma$ is in the kernel of the standard intertwining operator on $G_\gamma \cong SL_2$. (Here, we recall from §3.10 that $K_v = G_2(\mathcal{O}_v)$ is a hyperspecial maximal compact subgroup of $G_2(F_v)$).

Let us fix $v \in S$. We shall exploit the above observation by taking $f = \Phi_v$ to be the Iwahori-fixed vector in $\Sigma_v$, so that the $K_v$-span of $\Phi_v$ is $\Sigma_v^{K_1}$. We consider each Weyl group element in turn.

- $w = w_\beta$. Consider the simple reflection $w_\beta$. The restriction of $\Phi_v$ to $G_\beta$ lies in the principal series of $SL_2$ which has the trivial representation as the quotient and the Steinberg representation as the submodule. Thus $M_v(w_\beta, s)(\Phi_v)$ is holomorphic at $s = 1/2$, and is either non-zero or has a zero of order 1 there. We need to show that it has a zero of order 1. By the observation, we need to show that the restriction of $\Sigma_v^{K_1}$ to $G_\beta$ lies in the Steinberg submodule. But this is precisely what Prop. 3.4 tells us. We have thus proved the desired result when $w = w_\beta$.

  Consider the derivative $M_v'(w_\beta, 1/2)$ of $M_w(w_\beta, s)$ at $s = 1/2$. It gives a map

  $$M_v'(w_\beta, 1/2) : I_B(\chi_0) \longrightarrow I_B(w_\beta \chi_0).$$

  This map is the second term in the Laurent expansion of $M_v(w_\beta, s)$ at $s = 1/2$ and so it is not $G_2(F_v)$-intertwining. However, it is still $K_v$-intertwining and is non-zero when restricted to $\Sigma_v$. Thus we have a $K_v$-intertwining map

  $$M_v'(w_\beta, 1/2) : I_B(\chi_0)^{K_1} \longrightarrow I_B(w_\beta \chi_0)^{K_1}.$$

  For any unramified character $\chi$ of $B$, we have a canonical identification

  $$I_B(\chi)^{K_1} = Ind_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)} 1.$$

  Thus we get a $G_2(\mathbb{F}_q)$-equivariant map

  $$M_v'(w_\beta, 1/2) : Ind_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)} 1 \longrightarrow Ind_{B(\mathbb{F}_q)}^{G_2(\mathbb{F}_q)} 1$$

  A key point here is that since $\Sigma_v^{K_1}$ occurs with multiplicity one in $I_B(\chi)^{K_1}$, this map acts by a non-zero scalar on $\Sigma_v$.

- $w = w_\alpha w_\beta$. Now consider $M_v(w_\alpha w_\beta, s)(\Phi_v)$. We may factor $M_v(w_\alpha w_\beta, \chi_0)$ as $M_v(w_\alpha, w_\beta\chi_0) \circ M_v(w_\beta, \chi_0)$. Because $M'(w_\beta 1/2)$ acts as a non-zero scalar on $\Sigma_v^{K_1}$, we now need to know the effect of $M_v(w_\alpha, w_\beta\chi_0)$ on $\Sigma_v^{K_1} \subset I_B(w_\beta\chi_0)^{K_1}$.

  When one restricts the elements of $I_B(w_\beta(\chi_0))$ to $G_\alpha$, one gets an irreducible principal series at which the standard intertwining operator of $SL_2$ is holomorphic. Thus $M_v(w_\alpha, w_\beta\chi_0)$ is holomorphic and non-zero. Hence $M_v(w_\alpha w_\beta, s)(\Phi_v)$ also has a zero of order 1 at $s = 1/2$.

- $w = w_\beta w_\alpha w_\beta$. Next we consider the operator $M_v(w_\beta, w_\alpha w_\beta\chi_0)$. The restriction of the elements of $I_B(w_\alpha w_\beta\chi_0)$ to $G_\beta$ again lies in the reducible principal series with the trivial representation as a quotient. By the observation above and Prop. 3.4 again, we see that this operator has a zero of order 1 when restricted to the space $\Sigma_v^{K_1} \subset I_B(w_\alpha w_\beta\chi_0)$. Thus $M_v(w_\beta w_\alpha w_\beta, s)(\Phi_v)$ has a zero of order 2 at $s = 1/2$.

- $w = w_\alpha w_\beta w_\alpha w_\beta$. Consider the operator $M_v(w_\alpha, w_\beta w_\alpha w_\beta\chi_0)$. The restriction of the elements of $I_B(w_\beta w_\alpha w_\beta\chi_0)$ to $G_\alpha$ lies in the reducible principal series with the trivial representation as a quotient. On the other hand, by Prop. 3.4, the restrictions of the elements in $\Sigma_v^{K_1}$ are $K_\alpha$-spherical. Thus this operator is holomorphic and non-zero when restricted to the space $\Sigma_v^{K_1}$ and $M_v(w_\alpha w_\beta w_\alpha w_\beta, s)(\Phi_v)$ has a zero of order 2 at $s = 1/2$.

- $w = w_\beta w_\alpha w_\beta w_\alpha w_\beta$. Finally, we consider $M_v(w_\beta, w_\alpha w_\beta w_\alpha w_\beta\chi_0)$. The restriction of the elements of $I_B(w_\alpha w_\beta w_\alpha w_\beta\chi_0)$ to $G_\beta$ lies in the unitary principal series (induced from the trivial character of the torus). The corresponding $SL_2$-standard intertwining operator has a pole of order 1. Thus, $M_v(w_\beta w_\alpha w_\beta w_\alpha w_\beta, s)(\Phi_v)$ has a zero of order 1.

This completes our justification of the second table above and proves the proposition. □

5.4. **Proof of Theorem 5.1.** We can now give the proof of Theorem 5.1. Writing $I(f)$ as the sum of its Eisenstein and cuspidal components:

$$I(f) = I(f)_{Eis} + I(f)_{cusp},$$

we first show that $I(f)_{Eis} = A_0(Res(f))$. For this, it suffices to show that

$$\begin{cases} I(f)_N = A_0(Res(f))_N \\ I(f)_U = A_0(Res(f))_U. \end{cases}$$

It is not difficult to see that

$$I(f)_N(g) = \int_{G_D(F)\backslash G_D(\mathbb{A})} \theta(f)_{N_D}(gh)\, dh$$

and

$$I(f)_U(g) = \int_{G_D(F)\backslash G_D(\mathbb{A})} \theta(f)_{U_D}(gh)\, dh.$$

Now the equality

$$I(f)_N(g) = A_0(Res(f))_N(g)$$

is an immediate consequence of Propositions 5.4 and 5.2, since both sides are equal to $f(g)$. This implies that $I(f)_U - A_0(Res(f))_U$ is a cusp form on $L$. But Prop. 5.2 shows that $I(f)_U$ is orthogonal to cusp forms and clearly so is $A_0(Res(f))_U$. Thus, we deduce that $I(f)_U - A_0(Res(f))_U = 0$ and $I(f)_{Eis} = A_0(Res(f))$.

It remains to show that $I(f)_{cusp} = 0$, i.e. the projection of $\{I(f) : f \in \Pi_D\}$ onto $\mathcal{A}_{cusp}(G_2)$ is zero. If not, let $\pi \cong \otimes_v \pi_v$ be an irreducible summand of this projection. By our knowledge of the local theta correspondence, we know that

$$\pi_v = \begin{cases} J_{Q,v}(\pi(1,1),1), & \text{for almost all } v; \\ \Sigma_v, & \text{if } v \in S_D. \end{cases}$$

Now we can appeal to the results of [G2], which classifies the cuspidal representations of $G_2$ which are nearly equivalent to $J_Q(\pi(1,1),1)$; this near equivalence class is one of the cubic unipotent Arthur packets. Indeed, [G2, Main Theorem (ii)] gives:

**Proposition 5.5.** *Suppose that $\pi = \otimes_v \pi_v$ occurs as a submodule in $\mathcal{A}(G_2)$, and $\pi_v \cong J_{Q,v}(\pi(1,1),1)$ for almost all $v$, then $\pi_v$ cannot be isomorphic to $\Sigma_v$ for any $v$.*

From the proposition, we conclude that $I(f)_{cusp} = 0$, and Theorem 5.1 is proven.

## 6. Liftings of Automorphic Characters

Now we come to the proof of Theorem 1.2 in the introduction. To be more precise, let us define two submodules of $\mathcal{A}_{cusp}(G_2)$ by:

$$V_D(\chi_E) = \langle \theta_D(\chi_E, f) : f \in \Pi_D \rangle \quad \text{and} \quad V_E(\chi_E^D) = \langle \theta_E(\chi_E^D, f') : f' \in \Pi_E \rangle,$$

where

$$\theta_D(\chi_E, f)(g) = \int_{G_D(F)\backslash G_D(\mathbb{A})} \theta_D(f)(gh) \cdot \overline{\chi_E(h)} \, dh$$

and

$$\theta_E(\chi_E^D, f')(g) = \int_{S_E(F)\backslash S_E(\mathbb{A})} \theta_E(f')(gs) \cdot \overline{\chi_E^D(s)} \, ds.$$

For the convenience of the reader, we restate the theorem to be proved:

**Theorem 6.1.** *Assume that $E_v$ is the unramified cubic extension for all $v \in S_D$. For each $f \in \Pi_D$, there exists a unique $f' \in \Pi_E[\chi_E^D]$ (the $\chi_E^D$-isotypic submodule of $\Pi_E$) such that*

$$\theta_D(\chi_E, f) = \theta_E(\chi_E^D, f').$$

Let us mention at the onset that if Prop. 4.5 is known for all places $v$, then Theorem 6.1 can be proved relatively quickly. Indeed, a special case of the main theorem of [G2] says that $V_E(\chi_E^D)$ occurs with multiplicity one in $\mathcal{A}(G_2)$. Thus Theorem 6.1 would follow if we can show that as abstract representations,

$$V_D(\chi_E) \cong V_E(\chi_E^D).$$

But Prop. 4.5 for *all* places would imply this isomorphism. Unfortunately, since we do not know Prop. 4.5 for all places, we shall resort to a more roundabout argument, which involves a comparison of Fourier coefficients and exploits the Siegel-Weil formula proved in the previous section.

6.1. **Fourier coefficients of $\theta_D(\chi_E, f)$.** Let us consider the Fourier coefficients along $N$ of $\theta_D(\chi_E, f)$ for $f \in \Pi_D$. Having fixed a non-trivial additive character $\psi = \prod_v \psi_v$ of $F\backslash\mathbb{A}$, the set $\overline{V}(F)$ can be identified with the set of unitary characters of $N(\mathbb{A})$ trivial on $N(F)$, using $\psi$, the Killing form and the exponential map. Thus, the $M(F)$-orbits of Fourier coefficients along $N$ are indexed by cubic $F$-algebras.

Given a character $\Psi$ of $N(F)\backslash N(\mathbb{A})$, we denote the $\Psi$-coefficient of $\theta_D(\chi_E, f)$ by $\theta_D(\chi_E, f)_{N,\Psi}$. It is a function on $G_2(\mathbb{A})$ given by:

$$\theta_D(\chi_E, f)_{N,\Psi}(g) = \int_{N(F)\backslash N(\mathbb{A})} \overline{\Psi(n)} \cdot \theta_D(\chi_E, f)(ng)\, dn.$$

We set

$$\mathcal{W}_\Psi(V_D(\chi_E)) = \langle \theta_D(\chi_E, f)_{N,\Psi} : f \in \Pi_D \rangle.$$

This is a $G_2$-submodule of the space of smooth functions on $G_2(\mathbb{A})$.

Since $\theta_D(\chi_E, f)$ is a non-generic cusp form, all its degenerate Fourier coefficients (which corresponds to non-étale cubic algebras) vanish [G1, Prop. 15.1]. On the other hand, since the representation $\theta_{D_v}(\chi_{E,v})$ is $E_v$-distinguished for all $v \notin S$, the only non-zero generic Fourier coefficients are those corresponding to the field $E$. If $\Psi_E$ is a character of $N(\mathbb{A})$ corresponding to $E$, then the map

$$f \mapsto \theta_D(\chi_E, f)_{N,\Psi_E}$$

defines an *bijective* $G_2$-equivariant map

$$\iota_E : V_D(\chi_E) \longrightarrow \mathcal{W}_{\Psi_E}(V_D(\chi_E)).$$

The injectivity of $\iota_E$ follows from the fact that every non-zero non-generic cusp form must have a non-zero generic Fourier coefficient along $N$ (cf. [G1, Prop. 15.1]).

The following proposition follows by a standard computation (cf. [GGJ, Pg. 253] for a similar computation), and so we omit its proof.

**Proposition 6.2.** *With suitable normalizations of measures, we have:*

$$\theta_D(\chi_E, f)_{N,\Psi}(g) = \int_{T_E(\mathbb{A})\backslash G_D(\mathbb{A})} \theta_D(f)_{N_D,\tilde{\Psi}_E}(hg) \cdot \overline{\chi_E(h)}\, \frac{dh}{dt}.$$

*Here, $\tilde{\Psi}_E$ is a character of $N_D(\mathbb{A})$ which is equal to $\Psi_E$ when restricted to $N(\mathbb{A})$, and $T_E \cong E^\times/F^\times \subset G_D$ is the stabilizer of $\tilde{\Psi}_E$ in $G_D$.*

For a decomposable $f = \otimes_v f_v \in \Pi_D$, we know by the discussion in §3.7 that the Fourier coefficient $\theta_D(f)_{N_D,\tilde{\Psi}_E}$ is Eulerian:

$$\theta_D(f)_{N_D,\tilde{\Psi}_E}(h) = \prod_v L^0_{\tilde{\Psi}_E,v}(h_v \cdot f_v),$$

Thus, the proposition implies that we have an Eulerian expression

$$\theta_D(\chi_E, f)_{N,\Psi_E}(g) = \prod_v \int_{T_E(F_v)\backslash G_D(F_v)} L^0_{\tilde{\Psi}_E,v}(h_v g_v \cdot f_v) \cdot \overline{\chi_{E,v}(h_v)}\, \frac{dh_v}{dt_v}.$$

Let $\mathcal{W}_{\Psi_{E,v}}(\Pi_{D_v}, \chi_{E,v})$ denote the space of functions on $G_2(F_v)$ spanned by the local factors at $v$ in the above Eulerian expression. The key point to notice here is that this space only depends on the abstract representations $\Pi_{D_v}$ and $\chi_{E_v}$, and not on any automorphic realizations. In any case, we have:

**Corollary 6.3.** *As representations of $G_2(\mathbb{A})$,*

$$V_D(\chi_E) \cong \mathcal{W}_{\Psi_E}(V_D(\chi_E)) \cong \otimes_v \mathcal{W}_{\Psi_{E,v}}(\Pi_{D_v}, \chi_{E,v}).$$

6.2. **Fourier coefficients of $\theta_E(\chi_E^D, f')$.** One can do a similar analysis of the Fourier coefficients of $\theta_E(\chi_E^D, f')$ along $N$. This analysis was carried out in [GGJ] and so we shall simply state the result:

**Proposition 6.4.** *If $f' = \otimes_v f'_v \in \Pi_E$, then we have the Eulerian expression*

$$\theta_E(\chi_E^D, f')_{N,\Psi_E} = \prod_v \int_{S_E(F_v)} L^0_{\tilde{\Psi}'_E,v}(s_v g_v \cdot f'_v) \cdot \overline{\chi_{E_v}^{D_v}(s_v)}\, ds_v.$$

*Let $\mathcal{W}_{\Psi_E}(V_E(\chi_E^D))$ denote the space of functions on $G_2(\mathbb{A})$ spanned by the coefficients $\theta_E(\chi_E^D, f')_{N,\Psi_E}$, and let $\mathcal{W}_{\Psi_{E,v}}(\Pi_{E_v}, \chi_{E_v}^{D_v})$ denote the space of functions on $G_2(F_v)$ spanned by the local factor at $v$ of the Eulerian expression above. Then*

$$V_E(\chi_E^D) \cong \mathcal{W}_{\Psi_E}(V_E(\chi_E^D)) \cong \otimes_v \mathcal{W}_{\Psi_{E,v}}(\Pi_{E_v}, \chi_{E_v}^{D_v}).$$

*Again, $\mathcal{W}_{\Psi_{E,v}}(\Pi_{E_v}, \chi_{E_v}^{D_v})$ depends only on the abstract representations $\Pi_{E_v}$ and $\chi_{E_v}^{D_v}$.*

6.3. **Comparison of Fourier coefficients.** In view of Cor. 6.3 and Prop. 6.4, one can show the desired isomorphism

$$V_D(\chi_E) \cong V_E(\chi_E^D)$$

by showing that for each place $v$,

$$\mathcal{W}_{\Psi_{E,v}}(\Pi_{D_v}, \chi_{E,v}) \cong \mathcal{W}_{\Psi_{E,v}}(\Pi_{E_v}, \chi_{E_v}^{D_v}).$$

This is given by the following proposition.

**Proposition 6.5.** *Assume that $v \notin S_D$ or $v \in S_D$ but $E_v$ unramified. Then we have:*

$$\mathcal{W}_{\Psi_{E,v}}(\Pi_{D_v}, \chi_{E,v}) \cong \mathcal{W}_{\Psi_{E,v}}(\Pi_{E_v}, \chi_{E_v}^{D_v}).$$

*In fact, the two spaces are equal as spaces of functions on $G_2(F_v)$.*

*Proof.* We fix the place $v$ and consider the archimedean and finite places separately.

**(i)** $v$ **is a finite place**. We know that $\mathcal{W}_{\Psi_{E,v}}(\Pi_{D_v}, \chi_{E,v})$ is a non-zero quotient of $\theta_{D_v}(\chi_{E,v})$. Since the latter is irreducible by Prop. 4.3(ii), the two are in fact isomorphic. Similarly, we also have $\mathcal{W}_{\Psi_{E,v}}(\Pi_{E_v}, \chi_{E_v}^{D_v}) \cong \theta_{E_v}(\chi_{E_v}^{D_v})$. Thus by Prop. 4.5, we conclude the desired isomorphism of the proposition.

**(ii)** $v$ **is archimedean**. In this case, $E_v \cong F_v^3$ and $\chi_{E,v}$ is the trivial character $\mathbf{1_v}$. We know by Prop. 4.1(ii) that

$$\mathcal{W}_{\Psi_{E,v}}(\Pi_{E_v}, \mathbf{1_v}) \cong \theta_{E_v}(\mathbf{1_v}) \cong J_{Q,v}(\pi(1,1),1).$$

So it suffices to show that $\mathcal{W}_{\Psi_{E,v}}(\Pi_{D_v}, \mathbf{1_v}) \cong J_{Q,v}(\pi(1,1),1)$ also. For this, we shall need to resort to the Siegel-Weil formula proved in the previous section.

Indeed, Theorem 5.1 implies that

$$V_D(\mathbf{1}) \cong \left(\bigotimes_{w \in S_D} \Sigma_w\right) \otimes \left(\bigotimes_{w \notin S_D} I_{P,w}(1/2)\right).$$

If we compute the $\Psi_E$-th Fourier coefficient of $I(f)$ (for $f \in \Pi_D$), we obtain:

$$V_D(\mathbf{1}) \longrightarrow \mathcal{W}_{\Psi_E}(V_D(\mathbf{1})) \cong \otimes_w \mathcal{W}_{\Psi_{E,w}}(\Pi_{D_w}, \mathbf{1_w}),$$

where the first map is surjective but no longer injective. However, since $E_v = F_v^3$ is split, we know by [G2, Thm. B] that

$$\mathrm{Hom}_N(I_{P,v}(1/2), \mathbb{C}_{\Psi_{E_v}}) \cong \mathrm{Hom}_N(J_{Q,v}(\pi(1,1),1), \mathbb{C}_{\Psi_{E_v}}) \cong \mathbb{C}.$$

In other words, the unique (up-to-scaling) $(N, \Psi_{E_v})$-equivariant linear functional on $I_{P,v}(1/2)$ factors through its unique irreducible quotient $J_{Q,v}(\pi(1,1),1)$. Thus, the map $V_D(\mathbf{1}) \longrightarrow \mathcal{W}_{\Psi_E}(V_D(\mathbf{1}))$ factors through the quotient

$$J_{Q,v}(\pi(1,1),1) \otimes \left( \bigotimes_{w \in S_D} \Sigma_w \right) \otimes \left( \bigotimes_{w \notin S_D, w \neq v} I_{P,w}(1/2) \right).$$

This allows us to conclude that

$$\mathcal{W}_{\Psi_{E,v}}(\Pi_{D_v}, \mathbf{1_v}) \cong J_{Q,v}(\pi(1,1),1),$$

as desired. The proposition is proved. □

6.4. **Completion of proof of Theorem 6.1.** Since we assume that $E_v$ is unramified for all $v \in S_D$, the above proposition shows that

$$V_D(\chi_E) \cong V_E(\chi_E^D) \cong \Pi_E[\chi_E^D].$$

Thus Theorem 6.1 follows by [G2, Main Theorem (i)] which says:

$$\dim \mathrm{Hom}_{G_2}(V_E(\chi_E^D), \mathcal{A}(G_2)) = 1.$$

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