DOUBLING ZETA INTEGRALS AND LOCAL FACTORS FOR METAPLECTIC GROUPS

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In memory of Professor Hiroshi Saito

1. Introduction

Some twenty five years ago, Piatetski-Shapiro and Rallis [GPSR] discovered a Rankin-Selberg integral representation for the standard L-function for $G \times \mathrm{GL}_1$, where G is a classical group. This Rankin-Selberg integral is called the doubling zeta integral. The associated local zeta integral can be used, à la Tate, to define the local γ-factor $\gamma(s, \pi \times \chi, \psi)$ for an irreducible representation $\pi \boxtimes \chi$ of $G \times \mathrm{GL}_1$. This was sketched in [PSR], but the definitive treatment only appeared in the recent work [LR] of Lapid-Rallis. The paper [LR] treated the case of symplectic, orthogonal and unitary groups. There are however two classes of classical groups to which the theory of doubling zeta integrals applies but which were not treated in [LR]. The first of these is the class of metaplectic groups Mp_{2n} which are the unique two-fold covers of the symplectic groups Sp_{2n}. The other is the class of quaternion unitary groups, which are inner forms of symplectic and orthogonal groups.

The purpose of this paper is to treat the theory of the doubling zeta integral for the metaplectic group Mp_{2n}, following the footsteps of Lapid-Rallis [LR]. In particular, given an irreducible genuine representation $\sigma \boxtimes \chi$ of $\mathrm{Mp}_{2n} \times \mathrm{GL}_1$, we define the local γ-factor $\gamma(s, \sigma \times \chi, \psi)$, the local L-factor $L(s, \sigma, \chi, \psi)$ and the ϵ-factor $\epsilon(s, \sigma \times \chi, \psi)$. Further, we verify that the local γ-factors satisfy the analog of the Ten Commandments in [LR, Thm. 4], which characterize them uniquely. This theory of local factors is used crucially in our paper [GS] with Savin, where we study the genuine representation theory of Mp_{2n}.

We highlight a couple of key differences between the metaplectic case and the linear case:

- in the metaplectic case, the local L-factor $L(s, \sigma \times \chi, \psi)$ depends on an additive character ψ of the local field in question, whereas in the linear case, it does not. This reflects the fact that, in the local Langlands correspondence for Mp_{2n} (as developed in [GS]), the L-parameter of a representation may change when one changes ψ, whereas in the linear case, the L-parameter does not change; only the internal parametrization within an L-packet changes.
- in the symplectic or orthogonal case, the local factors do not change when one replaces the representation of G by its contragredient, whereas in the metaplectic case, they do. More precisely, one has

$$\gamma(s, \sigma^\vee \times \chi, \psi) = \gamma(s, \sigma \times \chi, \psi_{-1}) = \gamma(s, \sigma \times \chi_{-1}\chi, \psi).$$

This reflects the fact that, in the symplectic or orthogonal case, the process of taking contragredient preserves the L-packets, whereas in the metaplectic case, it does not.

We should mention that an important prior work regarding the doubling zeta integral of Mp_{2n} is the unpublished preprint [Sw] of J. Sweet. There, he computed the normalizing factor for an intertwining operator on a degenerate principal series representation. We have benefitted greatly from the results in the preprint [Sw] (see Proposition 4.1 below), though at the end of the day, it is possible to establish the results of this paper without resort to [Sw]. Another recent work concerning the doubling zeta integral of Mp_{2n} is that of Zorn [Z], though it concerns the evaluation of the zeta integrals on specific test vectors. Finally, an alternative definition of the local γ-factors for generic representations of Mp_{2n} is given in the thesis of D. Szpruch via the Langlands-Shahidi method.

Acknowledgments: The author first met Professor Hiroshi Saito in January 2003 when he visited University of Kyoto for a conference. Professor Saito's warm hospitality will be fondly remembered.

The author's work is partially supported by NSF grant DMS-0801071. This paper is written while during the author's visit at the National University of Singapore (NUS) in the Fall of 2010. The author thanks NUS for providing an excellent working environment. Thanks are also due to Erez Lapid for several helpful email exchanges.

2. Metaplectic Groups

Let k be a local field of characteristic zero and residue characteristic $p > 0$.

2.1. **Symplectic Group.** Let W be a $2n$-dimensional vector space over k equipped with a nondegenerate skew-symmetric form $\langle -,-\rangle_W$ and let $\mathrm{Sp}(W)$ be the associated symplectic group. We may fix a Witt basis of W, consisting of vectors

$$e_1, ..., e_n, e_n^*, ..., e_1^*$$

satisfying

$$\langle e_i, e_j\rangle_W = \langle e_i^*, e_j^*\rangle_W = 0 \quad \text{and} \quad \langle e_i, e_j^*\rangle_W = \delta_{ij}.$$

For any $1 \le k \le n$, let

$$X_k = \mathrm{Span}(e_1, ..., e_k) \quad \text{and} \quad X_k^* = \mathrm{Span}(e_1^*, ..., e_k^*),$$

so that $W = X_n \oplus X_n^*$. We also set

$$W_{n-k} = \mathrm{Span}(e_{k+1}, ..., e_n, e_n^*, ..., e_{k+1}^*)$$

so that

$$W = X_k \oplus W_{n-k} \oplus X_k^*.$$

2.2. **Parabolic Subgroups.** We now describe the parabolic subgroups of $\mathrm{Sp}(W)$ up to conjugacy. Consider the flag of isotropic subspaces

$$X_{k_1} \subset X_{k_1+k_2} \subset ... \subset X_{k_1+...+k_r} \subset W.$$

The stabilizer of such a flag is a parabolic subgroup P whose Levi factor M is given by

$$M \cong \mathrm{GL}(k_1) \times .....\mathrm{GL}(k_r) \times \mathrm{Sp}(W_{n-k_1-...-k_r}),$$

where $\mathrm{GL}(k_i)$ is the group of invertible linear maps on $\mathrm{Span}(e_{k_{i-1}+1}, ...e_{k_{i-1}+k_i})$. In particular, the maximal parabolic subgroups of $\mathrm{Sp}(W)$ are simply the stabilizers $P(X_k)$ of the isotropic spaces X_k $(1 \leq k \leq n)$. For a given k, a Levi subgroup of $P(X_k)$ is given by

$$M(X_k) = \mathrm{GL}(X_k) \times \mathrm{Sp}(W_{n-k})$$

and its unipotent radical $N(X_k)$ sits in a short exact sequence

$$1 \longrightarrow Z(X_k) \longrightarrow N(X_k) \longrightarrow \mathrm{Hom}(W_{n-k}, X_k) \longrightarrow 1$$

where $Z(X_k) \cong \mathrm{Sym}^2 X_k$ is isomorphic to the space of symmetric bilinear form on Y_k. When $k = n$, $N(X_k) = Z(X_k)$ is abelian and $P(X_n)$ is called the Siegel parabolic subgroup.

2.3. **Metaplectic Group.** The group $\mathrm{Sp}(W)$ has a unique two-fold cover $\mathrm{Mp}(W)$. As a set, we may write

$$\mathrm{Mp}(W) = \mathrm{Sp}(W) \times \{\pm 1\}$$

with group law given by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot c(g_1, g_2))$$

for some 2-cocycle c on $\mathrm{Sp}(W)$ valued in $\{\pm 1\}$. Without describing c explicitly, let us describe the restriction of this double cover over a maximal parabolic subgroup $P(X_k)$ of $\mathrm{Sp}(W)$.

The covering splits uniquely over the unipotent radical $N(X_k)$ of $P(X_k)$. Thus, we may regard $N(X_k)$ canonically as a subgroup of $\mathrm{Mp}(W)$ and one has a Levi decomposition

$$\widetilde{P}(X_k) = \widetilde{M}(X_k) \cdot N(X_k)$$

We need to describe the covering over $M(X_k) \cong \mathrm{GL}(X_k) \times \mathrm{Sp}(W_{n-k})$.

Not surprisingly, the restriction of the covering to $\mathrm{Sp}(W_{n-k})$ is nothing but the unique two-fold cover $\mathrm{Mp}(W_{n-k})$ of $\mathrm{Sp}(W_{n-k})$. The covering over $\mathrm{GL}(X_k)$ can be described as follows. Consider the set

$$\mathrm{GL}(X_k) \times \{\pm 1\}$$

with multiplication law

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot (\det g_1, \det g_2))$$

where $(\det g_1, \det g_2)$ denotes the Hilbert symbol. Then $\widetilde{\mathrm{GL}}(X_k)$ is precisely this double cover of $\mathrm{GL}(X_k)$.

Hence, we have

$$\widetilde{M}(X_k) = \left(\widetilde{\mathrm{GL}}(X_k) \times \mathrm{Mp}(W_{n-k})\right) / \Delta\mu_2.$$

More generally, for any parabolic subgroup P, one has the Levi decomposition

$$\widetilde{P} = \widetilde{M} \cdot N$$

with

$$\widetilde{M} \cong \widetilde{\mathrm{GL}}(k_1) \times_{\mu_2} ... \times_{\mu_2} \widetilde{\mathrm{GL}}(k_r) \times_{\mu_2} \mathrm{Mp}(W_{n-k_1-...-k_r}).$$

2.4. **Representations of $\widetilde{\mathrm{GL}}(X_k)$.** The (genuine) representation theory of $\widetilde{\mathrm{GL}}(X_k)$ can be easily related to the representation theory of $\mathrm{GL}(X_k)$. Indeed, the determinant map

$$\det : \mathrm{GL}(X_k) \longrightarrow \mathrm{GL}(1)$$

has a natural lifiting

$$\widetilde{\det} : \widetilde{\mathrm{GL}}(X_k) \longrightarrow \widetilde{\mathrm{GL}}(1)$$

given by

$$\widetilde{\det}(g,\epsilon) = (\det g, \epsilon).$$

On the other hand, if we fix an additive character ψ of k, then there is a natural genuine character of $\widetilde{\mathrm{GL}}(1)$ defined by:

$$(a,\epsilon) \mapsto \epsilon \cdot \gamma(a,\psi)^{-1}$$

with

$$\gamma(a,\psi) = \gamma(\psi_a)/\gamma(\psi)$$

and the Weil index $\gamma(\psi)$ is an 8-th root of unity associated to ψ by Weil. Composing this genuine character by $\widetilde{\det}$ gives a genuine character χ_ψ of $\widetilde{\mathrm{GL}}(X_k)$, which satisfies

$$\chi_\psi(g,\epsilon)^2 = (\det g, -1).$$

Using the genuine character χ_ψ, one obtains a bijection between $\mathrm{Irr}(\mathrm{GL}(X_k))$ and the set $\mathrm{Irr}(\widetilde{\mathrm{GL}}(X_k))$ of genuine irreducible representations of $\widetilde{\mathrm{GL}}(X_k)$, via:

$$\tau \mapsto \tilde{\tau}_\psi = \tau \otimes \chi_\psi.$$

We stress that this bijection depends on the choice of the additive character ψ.

2.5. **Parabolic Induction.** After the above discussion, one sees that given an irreducible representation τ of $\mathrm{GL}(X_k)$ and an irreducible representation π of $\mathrm{Mp}(W_{n-k})$, one has an irreducible representation $\tilde{\tau}_\psi \boxtimes \pi$ of $\widetilde{M}(X_k)$. Thus, one may consider the parabolically induced representation

$$I_{P(X_k),\psi}(\tau,\pi) = \mathrm{Ind}_{\widetilde{P}(X_k)}^{\mathrm{Mp}(W)} \tilde{\tau}_\psi \boxtimes \pi \quad \text{(normalized induction)}.$$

More generally, for any parabolic subgroup $P = M \cdot N$ and irreducible representation τ_i of $\mathrm{GL}(k_i)$ and π of $\mathrm{Mp}(W_{n-k_1-\ldots-k_r})$, one has the induced representation

$$I_{P,\psi}(\tau_1, ..., \tau_r, \pi).$$

A particular case of this is when $P = B$ is the Borel subgroup, so that each $k_i = 1$. In that case, given characters χ_1,...,χ_n, one has the principal series representations

$$I_{B,\psi}(\chi_1, ..., \chi_n).$$

If the χ_i's are unramified, we shall call such a representation an unramified principal series representation; note that this notion of "unramified representations" depends on the choice of ψ.

2.6. **Central signs.** We could restrict the genuine character χ_ψ of $\tilde{\mathrm{GL}}(X_n)$ to the center $\widetilde{Z}$ (the preimage in $\mathrm{Mp}(W)$ of the center Z of $\mathrm{Sp}(W)$) of $\mathrm{Mp}(W)$. We denote this character of $\widetilde{Z}$ by χ_ψ as well. This character allows one to define a central sign for irreducible genuine representations σ of $\mathrm{Mp}(W)$.

Indeed, for an irreducible genuine representation σ of $\mathrm{Mp}(W)$, one may consider its central character ω_σ , which is a genuine character of $\tilde{Z}$. We define the central sign $z_\psi(\sigma)$ of σ by

$$z_\psi(\sigma) = \omega_\sigma(-1)/\chi_\psi(-1) \in \{\pm 1\},$$

where we note that the quotient above is independent of the choice of the preimage in $\tilde{Z}$ of $-1 \in Z$.

3. **Doubling Zeta Integrals.**

We now recall the basic definition of the doubling zeta integral. Set $\mathbb{W} = W + (-W)$, so that there is a natural map

$$\mathrm{Mp}(W) \times_{\mu_2} \mathrm{Mp}(W) \hookrightarrow \mathrm{Mp}(\mathbb{W}).$$

Consider the diagonally embedded W^Δ which is a maximal isotropic subspace of $\mathbb{W}$. The maximal parabolic subgroup of $\mathrm{Sp}(\mathbb{W})$ stabilizing W^Δ is

$$P(W^\Delta) = \mathrm{GL}(W^\Delta) \cdot N(W^\Delta).$$

Going to covering groups, one has

$$\tilde{P}(W^\Delta) = \tilde{\mathrm{GL}}(W^\Delta) \cdot N(W^\Delta) \subset \mathrm{Mp}(\mathbb{W}).$$

Fix a non-trivial additive character ψ of F. Then ψ determines a genuine character χ_ψ of $\tilde{\mathrm{GL}}(W^\Delta)$ as we explained in the previous section. For any character χ of $F^\times$, one can define the degenerate principal series representation

$$I_{\tilde{P}(W^\Delta),\psi}(s,\chi) := \mathrm{Ind}_{\tilde{P}(W^\Delta}^{\mathrm{Mp}(\mathbb{W})} \chi_\psi \cdot (\chi \circ \det)|\det|^s.$$

Now let σ be an irreducible genuine representation of $\mathrm{Mp}(W)$ with contragredient $\sigma^\vee$. One has a natural $\mathrm{Mp}(W)$-invariant pairing $\langle -,- \rangle$ on $\sigma^\vee \otimes \sigma$. For $v \in \sigma$, $v^\vee \in \sigma^\vee$ and a holomorphic section $f_s \in I_{\tilde{P}(W^\Delta,\psi}(s,\chi)$, the doubling zeta integral is defined for $Re(s) >> 0$, by

$$Z_{W,\psi}(s,\chi)(f,v^\vee,v) = \int_{\mathrm{Mp}(W)} f_s(g,1) \cdot \langle gv^\vee, v \rangle \, dg$$

for a fixed Haar measure dg on $\mathrm{Mp}(W)$. Note here that we have used the matrix coefficient $\langle gv^\vee, v\rangle$ of $\sigma^\vee$ rather than the matrix coefficient $\langle gv, v^\vee \rangle$ of σ. Thus, $Z_{W,\psi}(s,\chi)$ defines a $\mathrm{Mp}(W) \times \mathrm{Mp}(W)$-invariant linear form

$$Z_{W,\psi}(s,\chi) : I_{\tilde{P}(W^\Delta,\psi}(s,\chi) \otimes \sigma^\vee \otimes \sigma \longrightarrow \mathbb{C}.$$

The analytic properties of this zeta integral, as a function of s, are as summarized by:

Theorem 3.1. *(i) There exists a constant c such that whenever $Re(s) > c$, the integral $Z_{W,\psi}(s,\chi)(f,v^\vee,v)$ converges for all data f, v and $v^\vee$. If π is tempered, then we may take $c = -1/2$.*

(ii) If f_s is a holomorphic section of $I_{\tilde{P}(W^\Delta),\psi}(s,\chi)$, then the function $Z_{W,\psi}(s,\chi)(f,v^\vee,v)$ admits meromorphic continuation to $\mathbf{C}$. *When k is p-adic, $Z_{W,\psi}(s,\chi)(f,v,v^\vee)$ is a rational function in q^{-s} (where q is the cardinality of the residue field of k).*

(iii) For each s_0, there exists data f, v and $v^\vee$ such that $Z_{W,\psi}(s,\chi)(f,v^\vee,v)$ is finite and nonzero.

(iv) There is a non-negative integer k (depending on s_0) such that $(s-s_0)^k \cdot Z_{W,\psi}(s,\chi)(f,v^\vee,v)$ is holomorphic at $s=s_0$ and is nonzero there for some choice of data.

(v) Let $Z^*_{W,\psi}(s_0,\chi)$ denote the leading term in the Laurent expansion at $s=s_0$ of $Z_{W,\psi}(s,\chi)$ as a linear form, so that

$$Z^*_{W,\psi}(s_0,\chi)(f,v^\vee,v) = \Big((s-s_0)^k \cdot Z_{W,\psi}(s,\chi)(f,v^\vee,v)\Big)\,|_{s=s_0}.$$

Then $Z^*_{W,\psi}(s_0,\chi)$ is a nonzero element of $\mathrm{Hom}_{\mathrm{Mp}(W)\times\mathrm{Mp}(W)}(I_{\tilde{P}(W^\Delta),\psi}(s_0,\chi)\otimes\sigma^\vee\otimes\sigma,\mathbb{C})$.

4. Normalization of intertwining operator.

We continue with the notation of the previous section. One has an intertwining operator

$$M(\chi,s) : I_{\tilde{P}(W^\Delta),\psi}(s,\chi) \longrightarrow I_{\tilde{P}(W^\Delta),\psi}(-s,\chi^{-1})$$

defined by

$$M(\chi,s)(f)(g) = \int_{N(W^\Delta)} f(wng)\,dn$$

where

$$w = (1_W, -1_W) \in \mathrm{Mp}(W) \times_{\mu_2} \mathrm{Mp}(W).$$

This integral converges for $Re(s)$ sufficiently large and has a meromorphic continuation to all of $\mathbb{C}$. Note that this intertwining operator depends on the choice of the Haar measure dn on $N(W^\Delta)$. In [Sw] and [Z], a normalization of $M(\chi,s)$ is given as in [LR], and we recall its definition briefly.

We fix a nondegenerate character on the abelian group $N(W^\nabla) \cong \mathrm{Sym}^2 W^\nabla$ as follows. To every element

$$A \in N(W^\Delta) \subset \mathrm{Hom}(W^\nabla, W^\Delta),$$

one obtains a character ψ_A of $N(W^\nabla)$ defined by

$$\psi_A(X) = \mathrm{Tr}_{W^\nabla}(X \circ A).$$

If one identifies $W^\nabla \cong W$ via $(w,-w) \mapsto 2w$ and $W^\Delta \cong W$ via $(w,w) \mapsto w$ (following [LR]), then $A : W \longrightarrow W$, and so one may consider $\det A$.

Henceforth, we assume that A corresponds to the split nondegenerate quadratic form on the even dimensional space W^Δ, in which case we may assume $\det A \in (-1)^n \cdot F^{\times 2}$.

Now one can show (cf. the appendix of [Z]) that, for all s,

$$\dim \mathrm{Hom}_{N(W^\nabla)}(I_{\tilde{P}(W^\Delta,\psi}(s,\chi), \psi_A^{-1}) = 1.$$

Moreover, a nonzero element of this 1-dimensional space can be given as a generalized Jacquet integral:

$$\mathcal{W}_{\psi,A}(\chi,s)(f)=\int_{N(W^{\nabla})} f(u)\cdot\psi_A(u)\,du.$$

This integral converges for $Re(s)$ sufficiently large and has an analytic continuation to $\mathbb{C}$. Note that $\mathcal{W}_{\psi,A}(\chi,s)$ depends on the choice of the Haar measure du.

By the multiplicity one result above, there is a meromorphic function $c_{\psi,A}(\chi,s)$ such that

$$\mathcal{W}_{\psi,A}(\chi^{-1},-s)\circ M(\chi,s)=c_{\psi,A}(\chi,s)\cdot\mathcal{W}_{\psi,A}(\chi,s).$$

The explicit determination of the function $c_{\psi,A}(\chi,s)$ has been given in [Sw] (see also [Z]). We shall recall the answer in the unramified case below. In general, the following consequence of Sweet's results is sufficient for our purpose:

Proposition 4.1. *We have*

$$c_{\psi,A}(s,\chi\chi_a)=\chi_a(-1)^n\cdot c_{\psi,A}(s,\chi)$$

for any quadratic character χ_a (associated to $a\in F^{\times}$),

$$c_{\psi_a,A}(s,\chi)=c_{\psi}(s,\chi)\cdot\chi(a)^{-2n}\cdot|a|^{-2ns}\cdot\chi_a(-1)^n,$$

and

$$c_{\psi,a\cdot A}(s,\chi)=c_{\psi,A}(s,\chi)\cdot\chi(a)^{-2n}|a|^{-2ns}.$$

Indeed, we shall give an independent proof of this proposition later on.

Now one may normalize the intertwining operator $M(\chi,s)$ by setting

$$M^*_{\psi,A}(\chi,s)=c_{\psi,A}(\chi,s)^{-1}\cdot M(\chi,s),$$

so that

$$\mathcal{W}_{\psi,A}(\chi^{-1},-s)\circ M^*_{\psi,A}(\chi,s)=\mathcal{W}_{\psi,A}(\chi,s).$$

Then this normalized operator is independent of the choice of the Haar measures dn or du. Moreover, it satisfies

$$M^*_{\psi,A}(\chi^{-1},-s)\circ M^*_{\psi,A}(\chi,s)=1.$$

The above proposition implies:

Corollary 4.2. *As linear maps from $I_{\tilde{P}(W^{\Delta},\psi}(s,\chi)$ to $I_{\tilde{P}(W^{\Delta}),\psi}(-s,\chi^{-1})$, one has:*

$$M^*_{\psi,a\cdot A}(\chi,s)=\chi(a)^{2n}\cdot|a|^{2ns}\cdot M^*_{\psi,A}(\chi,s).$$

5. Definition of Local Factors.

It can be shown that for generic $s \in \mathbb{C}$,

$$\dim \mathrm{Hom}_{\mathrm{Mp}(W) \times \mathrm{Mp}(W)}(I_{\tilde{P}(W^{\Delta}, \psi}(s, \chi) \otimes \sigma^{\vee} \otimes \sigma, \mathbb{C}) = 1.$$

Thus, one obtains a functional equation

$$\begin{aligned} & Z_{W,\psi}(-s, \chi^{-1}) \circ (M^*_{\psi,A}(\chi, s) \otimes 1_{\sigma} \otimes 1_{\sigma^{\vee}}) \\ = \; & z_{\psi}(\sigma) \cdot \chi(\det A) \cdot |\det(A)|^s \cdot \gamma_W(s + \frac{1}{2}, \sigma \times \chi, A, \psi) \cdot Z_{W,\psi}(s, \chi), \end{aligned}$$

for some meromorphic function $\gamma_W(s + \frac{1}{2}, \sigma \times \chi, A, \psi)$. It follows from Corollary 4.2 that $\gamma_W(s, \sigma \times \chi, A, \psi)$ is independent of the choice of A, and so we may denote it by $\gamma_W(s, \sigma \times \chi, \psi)$.

Definition:

(i) The function $\gamma_W(s, \sigma \times \chi, \psi)$ is called the standard γ-factor of $\sigma \boxtimes \chi$.

(ii) If σ is tempered, we may write

$$\gamma_W(s, \sigma \times \chi, \psi) = \epsilon(s, \sigma \times \chi, \psi) \cdot \frac{L(1 - s, \sigma^{\vee} \times \chi^{-1}, \psi)}{L(s, \sigma \times \chi, \psi)}$$

where $\epsilon(s, \sigma \times \chi, \psi)$ is a monomial function of q^{-s} and $L(s, \sigma \times \chi, \psi)^{-1}$ is the numerator of the rational function $\gamma(s, \sigma \times \chi, \psi)$, normalized so that it is a polynomial in q^{-s} with constant term 1. The function $\epsilon(s, \sigma \times \chi, \psi)$ is called the standard epsilon factor of $\sigma \boxtimes \chi$ and $L(s, \sigma \times \chi, \psi)$ is the standard L-factor of $\sigma \boxtimes \chi$ relative to the choice of ψ.

(iii) If σ is non-tempered, we realize σ as a Langlands quotient of a standard module and define $\epsilon(s, \sigma \times \chi, \psi)$ and $L(s, \sigma \times \chi, \psi)$ by multiplicativity, following Shahidi.

It follows from definition that

$$\gamma_W(s, \sigma \times \chi| - |^t, \psi) = \gamma_W(s + t, \sigma \times \chi, \psi).$$

The rest of this paper is devoted to verifying other less obvious properties of $\gamma_W(s, \sigma \times \chi, \psi)$. The first order of business is to understand the unramified case thoroughly.

6. Unramified Case.

Assume that the residue characteristic p of F is odd and all the data involved (i.e. σ, χ, ψ and A) are unramified. Then one has

$$\sigma \subset I_{\tilde{B},\psi}(\chi_1, ..., \chi_n)$$

for some unramified characters χ_i of $F^{\times}$. In this case, one would like to check that

$$L(s, \sigma \times \chi, \psi) = L^*(s, \sigma \times \chi, \psi) := \prod_{i=1}^{n} L(s, \chi_i \chi) \cdot L(s, \chi_i^{-1} \chi).$$

However, this is not immediate from the definition. One can show this desired equality by an explicit calculation of the various quantities used in the definition of the γ-factor. The results are summarized in the following proposition.

Proposition 6.1. *Suppose that p is odd and σ, χ, ψ and A are unramified. Let f_0, v_0 and $v_0^\vee$ be the spherical vectors in $I_{\tilde{P}(W^\Delta),\psi}(s,\chi)$, σ and $\sigma^\vee$ respectively.*

(i) One has:

$$Z_{W,\psi}(s,\chi)(f_0,v_0^\vee,v_0) = \frac{L^*(s,\sigma\times\chi,\psi)}{d(\chi,s)}$$

where

$$d(\chi,s) = \prod_{k=1}^{n} L(2s,+2k,\chi^2).$$

(ii) One has

$$\mathcal{W}_{\psi,A}(\chi,s)(f_0) = \frac{1}{d(\chi,s)}.$$

(iii) One has

$$M(\chi,s)(f_0) = \prod_{j=1}^{n} \frac{L(2s-2j+1,\chi^2)}{L(2s+2j,\chi^2)}.$$

(iv) One has

$$c_{\psi,A}(\chi,s) = \prod_{j=1}^{n} \gamma(2s-(2j-1),\chi^2,\psi)^{-1},$$

where the γ-factors on the right are those of Tate. In particular, one has

$$M^*_{\psi,A}(\chi,s)(f_0) = \frac{d(\chi^{-1},-s)}{d(\chi,s)}\cdot f_0.$$

Proof. The statement (i) was when by J. S Li [Li, Prop. 4.6]. (ii) can be shown in the same way as in [LR, Prop. 4]. (iii) was shown by Zorn [Z, §4]. Finally, (iv) was shown by Sweet [Sw] though it also follows from (ii) and (iii). □

Corollary 6.2. *Assume the same hypotheses as the above proposition. Then one has*

$$\gamma_W(s,\sigma\times\chi,\psi) = \frac{L^*(1-s,\sigma\times\chi^{-1},\psi)}{L^*(s,\sigma\times\chi,\psi)}.$$

In particular, one has

$$L(s,\sigma\times\chi,\psi) = L^*(s,\sigma\times\chi,\psi).$$

Proof. The first assertion follows from the above proposition and the definition of $\gamma_W(s,\sigma\times\chi,\psi)$, on noting that $z_\psi(\sigma)=1$. For the second assertion, suppose that

$$\sigma \subset I_{\tilde{B},\psi}(\chi_1,...,\chi_n).$$

If σ is tempered and χ is unitary, then χ_i is unitary for each i and hence there can be no cancellation of factors in $L^*(1-s,\sigma\times\chi^{-1},\psi)$ with those of $L^*(s,\sigma\times\chi,\psi)$. This implies the desired equality in the tempered case. The non-tempered case follows by definition and the fact that every non-tempered unramified σ is the unique Langlands quotient of an unramified standard module. □

7. Doubling Zeta Integral for $\tilde{\mathrm{GL}}(X)$.

In order to establish the crucial property of multiplicativity of the γ-factors defined above, it is necessary to consider the doubling zeta integral associated to a representation of a Levi subgroup M of $\mathrm{Mp}(W)$. If $X \subset W$ is an isotropic subspace, with $W = X + W_0 + X^*$ and, $P(X) = M(X) \cdot N(X)$ is the associated maximal parabolic, then $M(X) \cong \tilde{\mathrm{GL}}(X) \times_{\mu_2} \mathrm{Mp}(W_0)$. Thus, we need to consider the doubling zeta integral associated to a genuine irreducible representation of $\tilde{\mathrm{GL}}(X)$. The nonlinear group $\tilde{\mathrm{GL}}(X)$ and its genuine representations were described in Section 2.

Let X be a vector space and set $\mathbb{X} = X \oplus X$. Let

$$X^{\Delta} = \{(x,x) : x \in X\} \subset \mathbb{X} \quad \text{and} \quad X^{\nabla} = \{(x,-x) : x \in X\},$$

so that

$$\mathbb{X} = X^{\Delta} \oplus X^{\nabla}.$$

Let $P(X^{\Delta})$ be the parabolic subgroup of $\mathrm{GL}(\mathbb{X})$ stabilizing X^{Δ}. The Levi subgroup of $P(X^{\Delta})$ is

$$M(X^{\Delta}) = \mathrm{GL}(X^{\Delta}) \times \mathrm{GL}(X^{\nabla}),$$

and its unipotent radical is

$$N(X^{\Delta}) = \mathrm{Hom}(X^{\nabla}, X^{\Delta}).$$

For ease of notation, we shall simply write det for the rational character $\det_{X^{\Delta}} \times \det_{X^{\nabla}}^{-1}$ of $M(X^{\Delta})$. Then one has the degenerate principal series representation

$$I_{\tilde{P}(X^{\Delta}),\psi}(\chi, s) := \mathrm{Ind}_{\tilde{P}(X^{\Delta})} \chi_{\psi} \cdot (\chi \circ \det) \cdot |\det|^s$$

of $\tilde{\mathrm{GL}}(\mathbb{X})$.

We may now repeat the construction of the doubling zeta integral. Let τ be an irreducible representation of $\mathrm{GL}(X)$, and set

$$\tau_{\psi} := \chi_{\psi} \otimes \tau$$

so that τ_{ψ} is a genuine representation of $\tilde{\mathrm{GL}}(X)$. Then one sets

$$\tilde{Z}_{X,\psi}(s,\chi)(\tilde{f}, v^{\vee}, v) = \int_{\tilde{\mathrm{GL}}(X)} \tilde{f}_s(g,1) \cdot \langle \tau_{\psi}^{\vee}(g) v^{\vee}, v\rangle \cdot dg$$

for $\tilde{f}_s \in I_{\tilde{P}(X^{\Delta}),\psi}(\chi, s)$, $v \in \tau_{\psi}$ and $v^{\vee} \in \tau_{\psi}^{\vee}$. This is the doubling zeta integral for the representation $\tau_{\psi} \boxtimes \chi$ of $\tilde{\mathrm{GL}}(X) \times \mathrm{GL}_1$, and it defines a $\tilde{\mathrm{GL}}(X) \times \tilde{\mathrm{GL}}(X)$- invariant form

$$\tilde{Z}_{X,\psi}(s,\chi) : I_{\tilde{P}(X^{\Delta}),\psi}(\chi, s) \otimes \tau_{\psi}^{\vee} \otimes \tau_{\psi} \longrightarrow \mathbb{C}.$$

As before, one has a standard intertwining operator

$$\tilde{M}(\chi, s) : I_{\tilde{P}(X^{\Delta}),\psi}(\chi, s) \longrightarrow I_{\tilde{P}(X^{\Delta}),\psi}(\chi^{-1}, -s),$$

and one may define an normalized intertwining operator $\tilde{M}_{\psi}^{*}(\chi, s)$ as above (cf. also [LR]). Namely, one fixes the element $B \in N(X^{\Delta})$ corresponding to an isomorphism $X^{\nabla} \to X^{\Delta}$.

Then $\psi_B(X) = \psi(\mathrm{Tr}_{X^\nabla}(XB))$ is a unitary character of $N(X^\nabla)$, and one has the corresponding generalized Jacquet integral $\tilde{\mathcal{W}}_{\psi,B}(\chi,s)$ on $I_{\tilde{P}(X^\Delta),\psi}(\chi,s)$. By a multiplicity one result, one has

$$\tilde{\mathcal{W}}_{\psi,B}(\chi^{-1},-s)\circ \tilde{M}(\chi,s) = \tilde{c}_{\psi,B}(\chi,s)\cdot \tilde{\mathcal{W}}_{\psi,B}(\chi,s),$$

for some meromorphic function $\tilde{c}_{\psi,B}(\chi,s)$. Then the normalized intertwining operator is defined by:

$$\tilde{M}^*_{\psi,B}(\chi,s) = \tilde{c}_{\psi,B}(\chi,s)^{-1}\cdot \tilde{M}(\chi,s).$$

Finally, one defines a standard γ-factor $\tilde{\gamma}(s,\tau\times\chi,\psi)$ by the local functional equation

$$\tilde{Z}_X(-s,\chi^{-1},\tilde{M}^*_\psi(\chi,s)(\tilde{f}),v,v^\vee) =$$
$$\tilde{\gamma}(s+1/2,\tau\times\chi,\psi)\cdot(\tau\otimes\chi)(-1_{X^\Delta})\cdot\chi(\det B)\cdot|\det B|^{2s}\tilde{Z}_X(s,\chi,\tilde{f},v,v^\vee),$$

where $\det B$ is defined in the same way as in the symplectic or metaplectic case. As before, one has the analog of Corollary 4.2 which ensures that the γ-factor here is independent of the scaling of B.

7.1. **Relation to linear case.** Now one has the linear version of the above construction, as detailed in [LR]. Thus, for the given representation τ of $\mathrm{GL}(X)$, one has the zeta integral $Z_X(s,\chi)(f,v^\vee,v)$, the normalized intertwining operator $M^*_{\psi,B}(\chi,s)$ and the γ-factor $\gamma_X(s,\tau\times\chi,\psi)$ defined by

$$Z_X(-s,\chi^{-1})(M^*_{\psi,B}(\chi,s)f,v^\vee,v) =$$
$$\gamma_X(s+1/2,\tau\times\chi,\psi)\cdot(\tau\otimes\chi)(-1_{X^\Delta})\cdot\chi(\det B)\cdot|\det B|^{2s}\cdot Z_X(s,\chi,f,v^\vee,v).$$

We note here a typo in [LR, equation (25)], where on the right hand side of the above equation, they had $\tau(-1)$ instead of $(\tau\otimes\chi)(-1)$. Moreover, it was shown in [LR] that

$$\gamma_X(s,\tau\times\chi,\psi) = \gamma^{GJ}(s,\tau\times\chi,\psi)\cdot\gamma^{GJ}(s,\tau^\vee\times\chi,\psi),$$

where γ^{GJ} refers to the γ-factors of Godement-Jacquet. If one examines the proof of this identity in [LR], one sees that it is the definition using $(\tau\otimes\chi)(-1)$ which leads to this desired relation.

The goal of this subsection is to relate the nonlinear doubling zeta integral $\tilde{Z}_X$ and the γ-factor $\tilde{\gamma}_X(s,\tau\times\chi,\psi)$ defined in the previous subsection to the linear versions.

We first note that there is a natural $\tilde{\mathrm{GL}}(\mathbb{X})$-equivariant isomorphism

$$\iota(\chi,s): I_{P(X^\Delta)}(\chi,s)\otimes\chi_\psi \cong I_{\tilde{P}(X^\Delta),\psi}(\chi,s)$$

given by

$$f\mapsto \tilde{f} = f\cdot\chi_\psi.$$

The following lemma is easy to check.

Lemma 7.1. *One has:*

$$\tilde{\mathcal{W}}_{\psi,B}(\chi,s)\circ\iota(\chi,s) = \mathcal{W}_{\psi,B}(\chi,s)$$

and

$$\tilde{M}(\chi,s)\circ\iota(\chi,s) = \chi_\psi(-1_X)\cdot\iota(\chi^{-1},-s)\circ M(\chi,s).$$

Thus,

$$\tilde{c}_{\psi,B}(\chi,s) = \chi_\psi(-1_X) \cdot c_{\psi,B}(\chi,s),$$

so that

$$\tilde{M}^*_{\psi,B}(\chi,s) \circ \iota(\chi,s) = \iota(\chi^{-1},-s) \circ M^*_{\psi,B}(\chi,s).$$

Moreover,

$$\tilde{Z}_X(s,\chi,,\iota(\chi,s)(f),v,v^\vee) = 2 \cdot Z_X(s,\chi,f,v,v^\vee).$$

The lemma immediately implies:

Proposition 7.2. *One has*

$$\tilde{\gamma}_X(s,\tau\times\chi,\psi) = \gamma_X(s,\tau\times\chi,\psi).$$

In particular, $\tilde{\gamma}_X(s,\tau\times\chi,\psi)$ is independent of the choice of the isomorphism B, and not just indpendent of the scaling of B.

8. **The Main Theorem.**

Now we come to the main theorem of this paper.

Theorem 8.1. *One has:*

(i) (Multiplicativity) Suppose that

$$\sigma \subset \mathrm{Ind}_{\tilde{P}(X),\psi}^{\mathrm{Mp}(W)} \tau \boxtimes \sigma_0,$$

where $\tilde{P}(X)$ *is a maximal parabolic subgroup with Levi factor* $\mathrm{GL}(X) \times_{\mu_2} \mathrm{Mp}(W_0)$*,* τ *is an irreducible representation of* $\mathrm{GL}(X)$ *and* σ_0 *is an irreducible genuine representation of* $\mathrm{Mp}(W_0)$*, then*

$$\gamma_W(s,\sigma\times\chi,\psi) = \gamma_X(s,\tau\times\chi,\psi)\cdot\gamma_{W_0}(s,\sigma_0\times\chi,\psi).$$

(ii) (Variation of ψ*) If* $a \in F^\times$ *and* $\psi_a(x) = \psi(ax)$*, then*

$$\gamma_W(s,\sigma\times\chi,\psi_a) = \chi(a)^{2n}\cdot|a|^{2n(s-1/2)}\cdot\gamma_W(s,\sigma\times\chi_a\chi,\psi).$$

(iii) (Outer automorphism) Let σ^a *denote the twist of* σ *by an outer automorphism of* $\mathrm{Mp}(W)$ *given by conjugation by an element in the similitude group* $\mathrm{GSp}(W)$ *of similitude* a*. Then one has*

$$\gamma_W(s,\sigma^a\times\chi,\psi) = \gamma_W(s,\sigma\times\chi_a\chi,\psi).$$

(iv) (Duality) One has

$$\gamma_W(s,\sigma^\vee\times\chi,\psi) = \gamma_W(s,\sigma\times\chi,\psi_{-1}) = \gamma_W(s,\sigma\times\chi_{-1}\chi,\psi).$$

(v) (Functional equation) One has

$$\gamma_W(s,\sigma\times\chi,\psi)\cdot\gamma_W(1-s,\sigma^\vee\times\chi^{-1},\psi_{-1}) = 1.$$

(vi) (Archimedean case) Suppose that F is archimedean so that (by the analog of the subrepresentation theorem), σ is contained in a principal series representation $I_{\tilde{B},\psi}(\chi_1,...,\chi_n)$, then

$$\gamma_W(s,\sigma\times\chi,\psi)=\prod_{i=1}^{n}\gamma(s,\chi_i\chi,\psi)\cdot\gamma(s,\chi_i^{-1}\chi,\psi),$$

where the γ-factors on the right are those of Tate.

(vii) (Global property) Suppose that k is a number field with ring of adeles $\mathbb{A}$ and W is a symplectic space over k. Let σ be a cuspidal representation of $\mathrm{Mp}(W)_{\mathbb{A}}$, χ a Hecke character of $\mathbb{A}^\times$ and $\psi=\otimes_v\psi_v$ a nontrivial additive character of $k\backslash\mathbb{A}$. Let S be a finite set of places outside of which all data involved are non-archimedean and unramified, then one has the global functional equation

$$L^S(s,\sigma\times\chi,\psi)=\prod_{v\in S}\gamma_W(s,\sigma_v\times\chi_v,\psi_v)\cdot L^S(1-s,\sigma^\vee\times\chi^{-1},\psi),$$

where

$$L^S(s,\sigma\times\chi,\psi)=\prod_{v\notin S}L(s,\sigma_v\times\chi_v,\psi_v)$$

(when $Re(s)$ is sufficiently large) is the partial L-function of $\sigma\times\chi$ with respect to ψ.

Moreover, the functions $\gamma_W(s,\sigma,\chi,\psi)$ are characterized by properties (i), (ii) and (vii).

9. **Proof of Main Theorem.**

The rest of the paper is devoted to the proof of the theorem.

9.1. **Multiplicativity.** The proof of multiplicativity is by the same argument as that in [LR]. Namely, one may apply [LR, Prop 2 and Lemma 9] and relate the doubling zeta integral for σ on $\mathrm{Mp}(W)$ to the doubling zeta integral for σ_0 on $\mathrm{Mp}(W_0)$ and the doubling zeta integral for $\tau_\psi=\chi_\psi\cdot\tau$ on $\tilde{\mathrm{GL}}(X)$. In particular, by [LR, Prop. 2], one has:

$$Z_{W,\psi}(s,\chi)(f,\alpha^\vee,\alpha)= \int_{\tilde{P}(X)\times\tilde{P}(X)\backslash\mathrm{Mp}(W)\times\mathrm{Mp}(W)} Z_{W_0\times X,\psi}(s,\chi)\left(\Psi(\chi,s)(f)(g),\alpha^\vee(g),\alpha(g)\right)\,dg. \tag{9.1}$$

Here, we have set

$$Z_{W_0\times X,\psi}(s,\chi)=Z_{W_0,\psi}(s,\chi)\otimes\tilde{Z}_{X,\psi}(s,\chi),$$

and we have regarded $\alpha\in\sigma\subset\mathrm{Ind}_{\tilde{P}(X),\psi}^{\mathrm{Mp}(W)}\tau\boxtimes\sigma_0$ as a function on $\mathrm{Mp}(W)$ taking values in $\tau\boxtimes\sigma_0$. Moreover, $\Psi(\chi,s)$ is an intertwining operator

$$\Psi(\chi,s):I_{\tilde{P}(W^\triangle),\psi}(\chi,s)\longrightarrow I_{\tilde{P}(\mathbb{X}),\psi}\left(I_{\tilde{P}(W_0^\triangle),\psi}(\chi,s)\boxtimes I_{\tilde{P}(X^\triangle),\psi}(\chi,s)\right)$$

as defined in [LR].

Furthermore, in the normalization of the intertwining operator for $\mathrm{Mp}(\mathbb{W})$, let us choose the element $A \in \mathrm{Hom}(W^{\nabla}, W^{\Delta})$ so that $A(X^{\nabla}) = X^{\Delta}$, in which case we also have $A(W_0^{\nabla}) \subset W_0^{\Delta}$. Thus the restriction of A to X^{∇} determines an element

$$B \in \mathrm{Hom}(X^{\nabla}, X^{\Delta}) = N(X^{\Delta}).$$

Similarly, the restriction of A to W_0^{∇} determines an element

$$A_0 \in \mathrm{Hom}(W_0^{\nabla}, W_0^{\Delta}) = N(W_0^{\Delta}).$$

By [LR, Lemma 9], one has

$$\Psi(\chi^{-1}, -s) \circ M^*_{W,\psi,A}(\chi, s) = \mathrm{Ind}_{\tilde{P}(\mathbb{X})}^{\mathrm{Mp}(\mathbb{W})} \left(M^*_{W_0 \times X, \psi, A_0 \otimes B}(\chi, s) \right) \circ \Psi(\chi, s).$$

Hence, we have:

$$Z_{W,\psi}(-s, \chi^{-1})(M^*_{W,\psi,A}(\chi, s)(f), \alpha^{\vee}, \alpha) \tag{9.2}$$

$$= \int_{\tilde{P}(X) \times \tilde{P}(X) \backslash \mathrm{Mp}(W) \times \mathrm{Mp}(W)} Z_{W_0 \times X}(s, \chi) \left(M^*_{W_0 \times X, \psi, A_0 \otimes B}(\chi, s)(\Psi(\chi, s)(f)(g)), \alpha^{\vee}(g), \alpha(g) \right) dg.$$

Now apply the local functional equations of the two zeta integrals in equation (9.2) and compare with equation (9.1). Noting that

$$z_{\psi}(\sigma) = z_{\psi}(\sigma_0) \cdot \tau(-1),$$

and

$$\det A = \det A_0 \cdot \det B \cdot \det(-B) = (-1)^{\dim X} \det(A_0) \cdot (\det B)^2.$$

one deduces that

$$\gamma_W(s, \sigma \times \chi, \psi) = \gamma_{W_0}(s, \sigma_0 \times \chi, \psi) \cdot \tilde{\gamma}_X(s, \tau \times \chi, \psi).$$

The result then follows by Proposition 7.2.

9.2. **Archimedean case.** We note that statement (vi) of the main theorem follows immediately from statement (i), as a consequence of the subrepresentation theorem.

9.3. **Variation of ψ.** Now we are ready to prove statement (ii) of the main theorem. We note that

$$\begin{aligned}
&I_{\tilde{P}(W^{\Delta}),\psi_a}(\chi, s) = I_{\tilde{P}(W^{\Delta}),\psi}(\chi\chi_a, s), \\
&Z_{W,\psi_a}(s, \chi) = Z_{W,\psi}(s, \chi\chi_a) \\
&M_{\psi_a}(\chi, s) = M_{\psi}(\chi\chi_a, s) \\
&z_{\psi_a}(\sigma) = z_{\psi}(\sigma) \cdot \chi_a(-1)^n.
\end{aligned}$$

Thus,

$$M^*_{\psi_a,A}(\chi, s) = \frac{c_{\psi,A}(s, \chi\chi_a)}{c_{\psi_a,A}(s, \chi)} \cdot M^*_{\psi,A}(\chi\chi_a, s).$$

On comparing the local functional equations for $Z_{W,\psi_a}(s,\chi)$ and $Z_{W,\psi}(s,\chi\chi_a)$, one concludes that

$$\gamma_W(s+1/2,\sigma\times\chi,\psi_a)/\gamma_W(s+1/2,\sigma\times\chi_a\chi,\psi)=\frac{c_{\psi,A}(s,\chi\chi_a)}{c_{\psi_a,A}(s,\chi)}.$$

Thus statement (ii) holds (for any given σ) if and only if:

$$c_{\psi_a,A}(s,\chi)=|a|^{-2ns}\cdot\chi(a)^{-2n}\cdot c_{\psi,A}(s,\chi\chi_a).$$

This is the first and second identities in Proposition 4.1. However, we can prove this identity without resorting to Proposition 4.1 (which was taken as a blackbox from the unpublished work of Sweet [Sw]). Indeed, it follows from the above that if the identity in (ii) holds for one particular σ, then it holds for arbitrary σ! Now, by multiplicativity (i.e. statement (i)), it is easy to see that the identity in (ii) holds for principal series representations induced from the Borel subgroup $\tilde{B}$ of $\mathrm{Mp}(W)$. Hence we are done with (ii).

9.4. **Outer automorphism of** $\mathrm{Mp}(\mathbb{W})$**.** To address statement (iii) of the main theorem, we need to recall some facts about outer automorphisms of $\mathrm{Mp}(W)$. We follow the treatment of Szpruch [Sz]. Consider the element g_a of $\mathrm{GSp}(\mathbb{W})$ which acts as identity on W^{Δ} and by the scalar $a\in F^{\times}$ on W^{∇}. Then g_a has similitude a. Conjugation by g_a defines an (outer) automorphism of $\mathrm{Sp}(\mathbb{W})$, which preserves the maximal parabolic $P(W^{\Delta})$. It is a basic fact that this outer automorphsim has a unique lift to $\mathrm{Mp}(\mathbb{W})$; we shall denote this lifted automorphism of $\mathrm{Mp}(\mathbb{W})$ by ν_a. Thus,

$$\nu_a(h,\epsilon)=(g_a h g_a^{-1},\epsilon_a(h)\cdot\epsilon),$$

for some $\epsilon_a(h)=\pm1$. The following lemma records some facts about the function ϵ_a.

Lemma 9.3. *(i) If $h\in N(W^{\Delta})$, then $\epsilon_a(h)=1$.*

(ii) If $h\in M(W^{\Delta})=\mathrm{GL}(W^{\Delta})$, then $\epsilon_a(h)=\chi_a(\det_{W^{\Delta}}h)$.

(iii) If $h=(1,-1)\in\mathrm{Sp}(W)\times\mathrm{Sp}(-W)\in\mathrm{Sp}(\mathbb{W})$, then $\epsilon_a(h)=\chi_a(-1)^n$.

It follows from the lemma that the automorphism ν_a induces a natural map

$$\nu_a: I_{\tilde{P}(W^{\Delta}),\psi}(\chi,s)\longrightarrow I_{\tilde{P}(W^{\Delta}),\psi}(\chi_a\cdot\chi,s)$$

given by

$$\nu_a(f)(h)=f(\nu_a(h)).$$

9.5. **Some computations.** Now we can compute the effect of the change of ψ on various objects used in the definition of the γ-factors. Recall that

$$I_{\tilde{P}(W^{\Delta}),\psi_a}(\chi,s)=I_{\tilde{P}(W^{\Delta}),\psi}(\chi\cdot\chi_a,s).$$

The following lemma relates the generalized Jacquet integrals $\mathcal{W}_{\psi_a,A}(\chi,s)$ and $\mathcal{W}_{\psi,A}(\chi,s)$.

Lemma 9.4. *As linear functionals on $I_{\tilde{P}(W^{\Delta}),\psi_a}(\chi,s)$, one has*

$$\mathcal{W}_{\psi_a,A}(\chi,s)=|a|^{-n(2n+1)}\cdot(\mathcal{W}_{\psi,A}(\chi,s)\circ\nu_{a^{-1}})$$

Proof. For $f \in I_{\tilde{P}(W^\Delta),\psi_a}(\chi, s)$, with $Re(s) >> 0$, one has

$$\begin{aligned}\mathcal{W}_{\psi_a,A}(\chi,s)(f) &= \int_{N(W^\nabla)} f(n)\cdot\psi_{a\cdot A}(n)^{-1}\,dn\\ &= \int_{N(W^\nabla)} f(n)\cdot\psi_A(\nu_a(n))^{-1}\,dn\\ &= \int_{N(W^\nabla)} f(\nu_{a^{-1}}(n))\cdot\psi_A(n)^{-1}\cdot|a|^{-n\cdot(2n+1)}\,dn\end{aligned}$$

□

The following lemma establishes the relation between ν_a and the standard intertwining operator $M(\chi, s)$.

Lemma 9.5. *As operators from $I_{\tilde{P}(W^\Delta),\psi}(\chi, s)$ to $I_{\tilde{P}(W^\Delta),\psi}(\chi_a\chi^{-1}, -s)$, one has*

$$\nu_a \circ M_\psi(\chi,s) = \chi_{-1}(a)^n\cdot\chi(a)^{2n}\cdot|a|^{2ns}\cdot M_{\psi_a}(\chi,s)\circ\nu_a.$$

Proof. This is given by a similar computation as in the proof of the previous lemma:

$$\begin{aligned}\nu_a(M_\psi(\chi,s)f)(h) &= M(\chi,s)(f)(\nu_a(h))\\ &= \int_{N(W^\Delta)} f(wn\cdot\nu_a(h))\,dn\\ &= \int_{N(W^\Delta)} f(\nu_a(\nu_{a^{-1}}(wn)\cdot h))\,dn\end{aligned}$$

Now we note that

$$\nu_{a^{-1}}(w)\cdot w^{-1} = (a\cdot 1_{W^\Delta}, \chi_a(-1)^n) \in \tilde{\mathrm{GL}}(W^\Delta) \subset \mathrm{Mp}(\mathbb{W}).$$

Hence, we have:

$$\begin{aligned}\nu_a(M_\psi(\chi,s)f)(h) &= \int_{N(W^\Delta)} \chi_{-1}(a)^n\cdot\chi(a)^{2n}\cdot|a|^{2ns}\cdot f(\nu_a(wnh))\,dn\\ &= \chi_{-1}(a)^n\cdot\chi(a)^{2n}\cdot|a|^{2ns}\cdot M_{\psi_a}(\chi,s)\circ\nu_a.\end{aligned}$$

□

Corollary 9.6. *One has*

$$c_{\psi_a,A}(\chi,s) = \chi_{-1}(a)^n\cdot\chi(a)^{-2n}\cdot|a|^{-2ns}\cdot c_{\psi,A}(\chi,s)$$

and

$$\nu_a\circ M^*_{\psi_a,A}(\chi,s) = M^*_{\psi,A}(\chi,s)\circ\nu_a.$$

In particular, we have given an independent proof of the second identity in Proposition 4.1. Together with the results of §9.3, we have thus proven the first two identities of Proposition 4.1, independently of [Sw]. The third identity in Proposition 4.1 can be checked by computations analogous to those in the two lemmas above.

9.6. **Outer automorphism.** Let us write

$$W = X \oplus X^*$$

with X maximal isotropic. Then one has an outer automorphism of $\mathrm{Sp}(W)$ given by conjugation by the element which acts as 1 on X and a on X^*. The lifting of this to $\mathrm{Mp}(W)$ is denoted by μ_a. Moreover, the outer automorphism $\mu_a \times \mu_a$ on $\mathrm{Mp}(W) \times_{\mu_2} \mathrm{Mp}(W)$ is the restriction of the outer automorphism of $\mathrm{Mp}(\mathbb{W})$ given by the conjugation of the element of $\mathrm{GSp}(\mathbb{W})$ which acts as 1 on $\mathbb{X} = X^\Delta + X^\nabla$ and as a on $\mathbb{X}^* = (X^*)^\Delta + (X^*)^\nabla$. We denote this outer automorphism of $\mathrm{Mp}(\mathbb{W})$ by μ_a as well.

For any irreducible representation σ of $\mathrm{Mp}(W)$, the twisted representation σ^a is realized on the same space as σ, but the action of $\mathrm{Mp}(W)$ is defined by:

$$\sigma^a(g) = \sigma(\mu_a^{-1}(g)).$$

Now consider the local zeta integral for σ^a:

$$\begin{aligned} Z_{W,\psi}(s,\chi,\sigma^a)(f,v^\vee,v) &= \int_{\mathrm{Mp}(W)} f(g,1)\cdot\langle(\sigma^a)^\vee(g)v^\vee, v\rangle\, dg. \\ &= \int_{\mathrm{Mp}(W)} f(g,1)\cdot\langle\sigma^\vee(\mu_a^{-1}(g))(v^\vee), v\rangle\, dg \\ &= \int_{\mathrm{Mp}(W)} f(\mu_a(g,1))\cdot\langle\sigma^\vee(g)(v^\vee), v\rangle\, dg \\ &= \int_{\mathrm{Mp}(W)} f(\mu_a\nu_a^{-1}\nu_a(g,1)))\cdot\langle\sigma^\vee(g)(v^\vee), v\rangle\, dg. \end{aligned}$$

Now it is easy to see that $\mu_a\nu_a^{-1}$ is the inner automorphism given by the element m_a in the Levi subgroup $\mathrm{GL}(W^\Delta) = M(W^\Delta)$ of $P(W^\Delta)$ which acts as 1 on X^Δ and as a on $(X^*)^\Delta$. Thus we have

$$Z_{W,\psi}(s,\chi,\sigma^a)(f,v^\vee,v) = \int_{\mathrm{Mp}(W)} f(m_a\nu_a(g,1))m_a^{-1})\cdot\langle\sigma^\vee(g)(v^\vee), v\rangle\, dg$$

$$= \delta_{P(W^\Delta)}(m_a)^{1/2}\cdot\chi_\psi(m_a)\cdot|a|^{ns}\cdot\chi(a)^n\cdot Z_{W,\psi_a}(s,\chi,\sigma)(\nu_a(m_a^{-1}f), v^\vee, v).$$

We have thus shown that

Lemma 9.7.

$$Z_{W,\psi}(s,\chi,\sigma^a) = \delta_{P(W^\Delta)}(m_a)^{1/2}\cdot\chi_\psi(m_a)\cdot|a|^{ns}\cdot\chi(a)^n\cdot Z_{W,\psi_a}(s,\chi,\sigma)\circ(\nu_a\circ m_a^{-1}\otimes 1_{\sigma^\vee}\otimes 1_\sigma).$$

Similarly, one has

$$Z_{W,\psi}(-s,\chi^{-1},\sigma^a)\circ M^*_{\psi,A}(s,\chi)$$

$$= \delta_{P(W^\Delta)}(m_a)^{1/2}\cdot\chi_\psi(m_a)\cdot|a|^{-ns}\cdot\chi(a)^{-n}\cdot Z_{W,\psi_a}(-s,\chi^{-1},\sigma)\circ(\nu_a\circ m_a^{-1}\circ M^*_{\psi,A}(s,\chi))$$

$$= \delta_{P(W^\Delta)}(m_a)^{1/2}\cdot\chi_\psi(m_a)\cdot|a|^{-ns}\cdot\chi(a)^{-n}\cdot Z_{W,\psi_a}(-s,\chi^{-1},\sigma)\circ(M^*_{\psi_a,A}(s,\chi)\circ\nu_a\circ m_a^{-1})$$

where the last equality follows by Corollary 9.6.

By dividing this identity by that in the above lemma, and noting that

$$z_\psi(\sigma^a) = z_{\psi_a}(\sigma),$$

we deduce that

$$\gamma_W(s, \sigma^a \times \chi, \psi) = |a|^{-2ns} \cdot \chi(a)^{-2n} \cdot \gamma_W(s, \sigma \times \chi, \psi_a),$$

which gives the identity in (iii) in view of (ii).

9.7. **Duality.** Statement (iv) of the main theorem follows from statement (iii), on noting that the contragredient of σ is simply the twist of σ by an outer automorphism associated to an element of $\mathrm{GSp}(W)$ of similitude -1 (see [MVW] and [Sun]).

9.8. **Global property.** We shall now deal with the global functional equation, and we use the notation in statement (vi). For this, we need to introduce the global zeta integral associated to a cuspidal representation $\sigma = \otimes_v \sigma_v$ of $\mathrm{Mp}(W)_{\mathbb{A}}$. Consider the global degenerate principal series representation $I_{\tilde{P}(W^\Delta),\psi}(\chi, s)$ of $\mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ and its associated Eisenstein series map

$$E_\psi(s, \chi) : I_{\tilde{P}(W^\Delta),\psi}(\chi, s) \to \mathcal{A}(\mathrm{Mp}(\mathbb{W})),$$

which is a meromorphic function of s. Here $\mathcal{A}(\mathrm{Mp}(\mathbb{W}))$ is the space of automorphic forms on $\mathrm{Mp}(W)_{\mathbb{A}}$. Then the global zeta integral

$$Z_{W,\psi}(s, \chi) : I_{\tilde{P}(W^\Delta),\psi}(\chi, s) \otimes \sigma^\vee \otimes \sigma \longrightarrow \mathbb{C},$$

is defined by:

$$Z_{W,\psi}(s, \chi)(f, \varphi^\vee, \varphi) =$$

$$\int_{(\mathrm{Mp}(W)_k \times \mathrm{Mp}(W)_k)\backslash(\mathrm{Mp}(W)_{\mathbb{A}} \times \mathrm{Mp}(W)_{\mathbb{A}})} E_\psi(s, \chi)(f)(g_1, g_2) \cdot \varphi^\vee(g_1) \cdot \varphi(g_2)\, dg_1\, dg_2.$$

The following proposition summarizes the key properties of this global zeta integral:

Proposition 9.8. *(i) For $Re(s) >> 0$, one has a factorization:*

$$Z_{W,\psi}(s, \chi) = \otimes_v Z_{W,\psi_v}(s, \chi_v).$$

(ii) If S is a sufficiently large finite set of places of k, then one has the equality of meromorphic functions of s:

$$Z_{W,\psi}(s, \chi)(f, \varphi^\vee, \varphi) = \left(\prod_{v \in S} Z_{W,\psi_v}(s, \chi_v)(f_v, \varphi_v^\vee, \varphi_v) \right) \cdot \frac{L^S(s, \sigma \times \chi, \Psi)}{d^S(s, \chi)}$$

where $d^S(s, \chi)$ is the product of partial Hecke L-functions as defined in Prop. 6.1.

(iii) One has the global functional equation:

$$Z_{W,\psi}(-s, \chi^{-1}) \circ (M_\psi(s, \chi) \otimes 1_{\sigma^\vee} \otimes 1_\sigma) = Z_{W,\psi}(s, \chi)$$

where

$$M_\psi(s, \chi) : I_{\tilde{P}(W^\Delta),\psi}(\chi, s) \longrightarrow I_{\tilde{P}(W^\Delta),\psi}(\chi^{-1}, -s)$$

is the standard global intertwining operator.

Proof. Statement (i) is due to Piatetski-Shapiro and Rallis [GPSR]. Statement (ii) is a direct consequence of Proposition 6.1. Statement (iii) is a consequence of the functional equation of Eisenstein series [MW]. □

In addition, we need to relate the global intertwining operator $M_\psi(s,\chi)$ with the normalized local intertwining operators $M^*_{\psi_v}(s,\chi_v)$. We fix a rational element $A \in N(W^\Delta)(k)$ and thus obtain the automorphic character ψ_A of $N(W^\nabla)(\mathbb{A})$ defined as in the local case. We consider the associated Fourier coefficient map on $I_{\tilde{P}(W^\Delta),\psi}(\chi,s)$ defined by:

$$\mathcal{W}_\psi(s,\chi)(f) = \int_{N(W^\nabla)(k)\backslash N(W^\nabla)(\mathbb{A})} E_\psi(s,\chi)(f)(n)\cdot\psi_A(n)^{-1}\,dn.$$

The following proposition summarizes the properties of $\mathcal{W}_{\psi_A}(s,\chi)$.

Proposition 9.9. *(i) One has:*

$$\mathcal{W}_{\psi,A}(s,\chi)(M_\psi(s,\chi)f) = \mathcal{W}_{\psi,A}(s,\chi)(f).$$

(ii) When $Re(s) >> 0$, one has

$$\mathcal{W}_{\psi,A}(s,\chi)(f) = \prod_v \mathcal{W}_{\psi_v}(s,\chi_v)(f_v).$$

Proof. Statement (i) is a consequence of the functional equation of Eisenstein series [MW]. Statement (ii) follows by unfolding the Eisenstein series when $Re(s) >> 0$. □

Now for each place v of k, the element A allows one to define the normalized intertwining operator

$$M^*_{\psi_v,A}(s,\chi_v) = c_{\psi_v,A}(s,\chi_v)^{-1}\cdot M_{\psi_v}(s,\chi_v),$$

such that

$$\mathcal{W}_{\psi_v,A}(-s,\chi_v^{-1})\circ M^*_{\psi_v,A}(s,\chi_v) = \mathcal{W}_{\psi_v,A}(s,\chi_v).$$

Morally speaking, the above proposition says that

$$``\prod_v" c_{\psi_v,A}(s,\chi_v) = 1,$$

so that

$$M_\psi(s,\chi) = ``\prod_v" M^*_{\psi_v,A}(s,\chi_v).$$

However, the Euler products above do not converge anywhere, and has to be suitably interpreted. This is given by the following proposition, after taking into account the results of Proposition 6.1:

Corollary 9.10. *For a sufficiently large finite set S of places of k, one has*

$$\prod_{v\in S} c_{\psi_v,A}(s,\chi_v) = \prod_{j=1}^{n} \frac{L^S(2j-2s,,\chi^{-2})}{L^S(2s-2j+1,\chi^2)}.$$

Proof. Consider the identity

$$\mathcal{W}_{\psi,A}(-s,\chi^{-1})(M_\psi(s,\chi)f) = \mathcal{W}_{\psi,A}(s,\chi)(f),$$

with $f = \otimes_v f_v$ such that for all $v \notin S$, the conditions of Proposition 6.1 holds. Then Proposition 6.1 implies that

$$\left(\prod_{v\in S} \mathcal{W}_{\psi_v,A}(-s,\chi_v^{-1})(M(s,\chi_v)(f_v))\right) \cdot \frac{\prod_{j=1}^n L^S(2s-2j+1,\chi^2)}{d^S(s,\chi)\cdot d^S(-s,\chi^{-1})}$$

$$= \left(\prod_{v\in S} \mathcal{W}_{\psi_v,A}(s,\chi_v)(f_v)\right) \cdot \frac{1}{d^S(s,\chi)}.$$

Thus, we have

$$\left(\prod_{v\in S} c_{\psi_v,A}(s,\chi_v)\cdot \mathcal{W}_{\psi_v,A}(s,\chi_v)(f_v)\right) \cdot \frac{\prod_{j=1}^n L^S(2s-2j+1,\chi^2)}{d^S(-s,\chi^{-1})} = \prod_{v\in S} \mathcal{W}_{\psi_v,A}(s,\chi_v^{)}(f_v).$$

Hence,

$$\prod_{v\in S} c_{\psi_v,A}(s,\chi_v) = \prod_{j=1}^n \frac{L^S(2j-2s,,\chi^{-2})}{L^S(2s-2j+1,\chi^2)}.$$

□

Combining this with the global functional equation in Proposition 9.8(iii), the results of the above corollary and Proposition 6.1, and the local functional equation of the local zeta integral in 5, one deduces the desired result as in [LR].

9.9. **Uniqueness.** The uniqueness of the γ-factors considered in the main theorem is proved in the same way as [LR].

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