

# GLOBAL ENDOSCOPIC LIFTS FROM $PGL_3$ TO $G_2$

WEE TECK GAN AND GORDAN SAVIN

## 1. Introduction

This paper is part of a project devoted to the construction of endoscopic  $L$ -packets for the split exceptional group of type  $G_2$  over a number field  $F$ . Recall that the dual group of  $G_2$  is  $G_2(\mathbb{C})$  and consider the natural inclusion

$$\gamma : SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

Let  $\pi$  be a cuspidal automorphic representation of  $PGL_3(\mathbb{A})$  (where  $\mathbb{A}$  is the adèle ring of  $F$ ). The functoriality conjecture of Langlands predicts that, via the inclusion  $\gamma$  of dual groups, there is a global  $L$ -packet  $\mathcal{L}(\pi)$  of  $G_2$  associated to  $\pi$ . Indeed, for each place  $v$  of  $F$ , there *should* be a local  $L$ -packet  $\mathcal{L}(\pi_v)$  of  $G_2(F_v)$ . It is expected that  $\mathcal{L}(\pi_v)$  contains a unique generic representation  $\sigma(\pi_v)$ . Hence, the global  $L$ -packet should contain a distinguished element  $\sigma(\pi) = \otimes_v \sigma(\pi_v)$ . It is the unique (abstractly) generic representation in  $\mathcal{L}(\pi)$  and the conjecture of Langlands predicts that it is cuspidal automorphic. The lifting  $\pi \mapsto \sigma(\pi)$  is the endoscopic lift of the title. In order to prove the existence of this lift, two major problems need to be solved. Firstly, the distinguished element  $\sigma(\pi_v)$  of the local  $L$ -packet  $\mathcal{L}(\pi_v)$  needs to be defined, and secondly the occurrence of the representation  $\sigma(\pi)$  in the space of cusp forms on  $G_2$  needs to be proved.

Our approach to the lifting  $\pi \mapsto \sigma(\pi)$  is via the theta correspondence arising from restricting the minimal representation  $\Pi$  of  $E_6$  to the subgroup  $PGL_3 \times G_2$ . The whole task here can be also broken down into two parts. The first one is to determine the local correspondence, and to show that it is compatible with the local Langlands functoriality. The second, of course, is to prove the cuspidality and non-vanishing of the global theta lift.

For non-archimedean  $F_v$ , the first problem was solved in [GS]. The local theta correspondence arising from restricting the minimal representation  $\Pi_v$  of  $E_6$  was determined essentially completely there. In particular, if  $\pi_v$  is generic, it was shown that there is a unique generic representation  $\sigma(\pi_v)$  of  $G_2(F_v)$  such that

$$\text{Hom}_{PGL_3 \times G_2}(\Pi_v, \pi_v \otimes \sigma(\pi_v)) \neq 0,$$

in which case the dimension of this space is 1. Moreover, if  $\pi_v$  is non-supercuspidal, then this  $\sigma(\pi_v)$  is indeed equal to the functorial lift of  $\pi_v$  (generic non-supercuspidal representations of  $G_2$  have been classified by Shahidi). When  $\pi_v$  is supercuspidal, so is  $\sigma(\pi_v)$ . We have thus specified a generic representation  $\sigma(\pi_v)$  of  $G_2(F_v)$  for each finite place  $v$ , and the function  $\pi_v \mapsto \sigma(\pi_v)$  is functorial to the extent that parametrization of representations of  $G_2(F_v)$  is known.

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Next, we turn our attention to an archimedean place  $v$ . In this case, the local parametrization is known, and the map  $\pi_v \mapsto \sigma(\pi_v)$  is already well defined. Thus, we must show that it is compatible with the local theta correspondence. Unlike the case of classical theta correspondence, the analog of Howe's conjecture is not known here; so we do not know a priori that  $\pi_v$  has a unique theta lift. Nevertheless, in [HPS] (see also [L]), it was shown that the correspondence is functorial on the level of infinitesimal characters. More precisely, if  $\pi_v$  and  $\sigma_v$  are representation of  $PGL_3(F_v)$  and  $G_2(F_v)$  respectively, such that  $\pi_v \otimes \sigma_v$  is a quotient of  $\Pi_v$ , then  $d\gamma(\lambda) = \lambda'$  where  $d\gamma$  is the differential of the inclusion  $\gamma$  of the dual groups, and  $\lambda$  and  $\lambda'$  are the infinitesimal characters of  $\pi_v$  and  $\sigma_v$ , respectively. For a given  $\pi_v$ , the correspondence of infinitesimal characters severely limits what  $\sigma_v$  could be, but stops short of showing the compatibility of the two correspondences.

Our first main result of this paper is for  $F_v = \mathbb{R}$ . By controlling the minimal  $K$ -types of principal series representations in the theta correspondence, we show that the theta correspondence is functorial for spherical generic unitary representations, and for generic unitary representations with integral infinitesimal character. The failure to obtain a complete answer is due to the fact that some representations (such as non-spherical principal series) are not completely determined by its infinitesimal character and the minimal  $K$ -type.

Our next result is the non-vanishing of the global theta lifting of  $\pi$ . In particular, there exists a cuspidal automorphic representation  $\sigma$  of  $G_2(\mathbb{A})$  such that  $\sigma_v \cong \sigma(\pi_v)$  for every finite place  $v$ . At archimedean places, of course, we have the matching of infinitesimal characters. However, if we restrict ourselves to representation of real groups for which we have the compatibility of the Langlands and theta correspondences, we have the following precise result:

**Main Theorem .** *Assume that  $F$  is totally real. Let  $\pi$  be an irreducible cuspidal representation of  $PGL_3(\mathbb{A})$ . Assume that for each real place  $\infty$ ,  $\pi_\infty$  is either spherical or has integral infinitesimal character. Let  $\sigma(\pi)$  be the generic representation of  $G_2(\mathbb{A})$  defined above.*

(i) *(Existence of functorial lift) The representation  $\sigma(\pi)$  occurs in the space of globally generic cusp forms on  $G_2$ .*

(ii) *(Uniqueness of weak lifting) Let  $\sigma = \otimes_v \sigma_v$  be a globally generic cuspidal representation of  $G_2(\mathbb{A})$  which is nearly equivalent to  $\sigma(\pi)$ . Then  $\sigma$  is equal to  $\sigma(\pi)$  as a subspace of the space of globally generic cusp forms.*

The proof of the non-vanishing of the global theta lift in (i) is not difficult. However, as we point out in the remark at the end of Section 7, there is a common error at one point of the proof, which was perpetuated in many papers on this subject in the literature. The proof of (ii) will rely on a result of Ginzburg and Jiang [GJ], which says that a generic cuspidal representation  $\sigma$  of  $G_2(\mathbb{A})$  is a global theta lift if and only if its partial standard  $L$ -function has a pole at  $s = 1$ .

Of course, we would like to construct the whole local packet  $\mathcal{L}(\pi_v)$  and to show that the elements of the global packet occur in the cuspidal spectrum with the multiplicities predicted by Arthur. As remarked in [GS], the local theta correspondence for  $PGL_3 \times G_2$  is not sufficient for constructing the whole local packet; one needs to consider the dual pair  $PD^\times \times G_2$  with  $D$  a degree 3 division algebra. This dual pair was studied in [S] but the results there need to be

sharpened. The global question requires a detailed study of the global theta correspondence which is usually very difficult. We hope to return to these questions in the future.

## 2. Real correspondence

In this section we shall state our main result on the theta correspondence in the real case. First, however, we need to describe the minimal representation of the split exceptional group  $H(\mathbb{R})$  of type  $E_6$ . Let  $\mathfrak{h}$  be the complexified Lie algebra of  $H(\mathbb{R})$ , and  $K \cong Sp_8/\langle \pm 1 \rangle$  the complexified maximal compact subgroup of  $H(\mathbb{R})$ .

**Proposition 1.** *Let  $\Pi_\infty$  be the  $(\mathfrak{h}, K)$ -module of the minimal representation of  $H(\mathbb{R})$ . Then*

$$\Pi_\infty|_K = \bigoplus_{n=0}^{\infty} V_{n\omega_4}$$

where  $\omega_4$  is the fourth fundamental weight for  $Sp_8$ , equal to  $(1, 1, 1, 1)$  using the standard realization of the root system  $C_4$ , as in Bourbaki [Bo].

So far there has been practically no result on the continuous spectrum of exceptional dual pair correspondences. The best result (in that direction) is the so-called matching of infinitesimal characters. More precisely, if  $\pi$  and  $\sigma$  are two representations of  $PGL_3(\mathbb{R})$  and  $G_2(\mathbb{R})$  such that  $\pi \otimes \sigma$  is a quotient of  $\Pi_\infty$ , then we know that the infinitesimal character of  $\pi$  determines the infinitesimal character of  $\sigma$  as follows.

Recall that the root systems of  $SL_3$  and  $G_2$  can be realized in standard coordinates  $(t_1, t_2, t_3)$  with  $t_1 + t_2 + t_3 = 0$ . Let  $\alpha_1 = (1, -1, 0)$  and  $\alpha_2 = (0, 1, -1)$  be the simple roots for  $SL(3)$ , and let  $\beta_1 = (1, -1, 0)$  and  $\beta_2 = (-1, 2, -1)$  be the simple roots for  $G_2$ . This gives an identification  $\gamma$  of the root lattice  $\Lambda$  of  $SL_3$  with the root lattice  $\Lambda'$  of  $G_2$ .

The infinitesimal character of  $\pi$  can be identified with a point  $\lambda$  in  $\Lambda \otimes \mathbb{C}$ . Let  $\lambda' = \gamma(\lambda)$  be the corresponding point in  $\Lambda' \otimes \mathbb{C}$ . The matching theorem ([HPS] and [Li]) says that the infinitesimal character of  $\sigma$  is  $\lambda'$ . Of course, the matching of infinitesimal characters implies that for a given  $\pi$ , there could only be finitely many irreducible  $\sigma$  such that  $\pi \otimes \sigma$  is a quotient of  $\Pi_\infty$ , but it stops well short of determining  $\sigma$  in terms of  $\pi$ . However, we have the following important refinement:

**Theorem 2.** *Let  $\pi$  and  $\sigma$  be generic unitary representations of  $PGL_3(\mathbb{R})$  and  $G_2(\mathbb{R})$  respectively, such that  $\pi \otimes \sigma$  is a quotient of  $\Pi_\infty$ . Then (using the notation for representations introduced in the final section), we have:*

$\pi$	$\pi(\lambda, 1)$	$\pi_0(s, t)$	$\pi(k, t)$	$\pi(\lambda, \xi)$	$\pi_1(s, t), t \neq 0$	$\pi_1(s, 0)$
$\sigma$	$\sigma(\lambda', 1)$	$\sigma_0(s, t)$	$\sigma(k, t)$	$\sigma(\lambda', \xi')$	$\sigma_1(s, t), t \neq 0$	?

where  $\lambda' = \gamma(\lambda)$ .

The first two columns deal with spherical unitary representations: tempered principal series and complementary series respectively. The third column contains tempered generalized principal series. In particular, we have proven that the theta correspondence of generic unitary representations is compatible with Langlands functoriality in these three cases.

The fourth and fifth columns deal with non-spherical representations: tempered principal series and complementary series respectively. If  $\pi = \pi(\lambda, \xi)$ , then  $\sigma = \sigma(\lambda', \xi')$  for some non-trivial  $\xi'$ . Finally, if  $\pi = \pi_1(s, t)$  and  $t \neq 0$ , then  $\sigma = \sigma_1(s, t)$ , and we have proven functoriality in this case as well. However, if  $t = 0$ , then  $\sigma = \sigma_1(s, 0)$  or possibly one other non-spherical complementary series representation with the same infinitesimal character.

To summarize, we can show that the correspondence for generic unitarizable representations is functorial except for non-spherical tempered representations, where we have not determined how  $\xi'$  depends on  $\xi$ , and for non-spherical complementary series in the case  $t = 0$ . We highlight an important special case:

**Corollary 3.** *Let  $\pi$  and  $\sigma$  be generic unitarizable representations of  $PGL_3(\mathbb{R})$  and  $G_2(\mathbb{R})$  respectively, such that  $\pi \otimes \sigma$  is a quotient of  $\Pi_\infty$ . Then  $\pi$  is spherical if and only if  $\sigma$  is so. Moreover, if  $\lambda$  and  $\lambda'$  are the corresponding infinitesimal characters, then  $\lambda' = \gamma(\lambda)$ .*

The proof of Theorem 2 is based on the matching of infinitesimal characters and the analysis of minimal  $K$ -types. The failure to completely describe the correspondence is due to the fact that a representation might not be determined by its infinitesimal character and a minimal  $K$ -type.

In the last section, we list all generic unitarizable representations of  $PGL_3(\mathbb{R})$  according to their infinitesimal characters, and then all generic unitary representations of  $G_2(\mathbb{R})$  with the matching infinitesimal characters. The matching of infinitesimal characters alone implies the third columns of Theorem 2, and shows that if  $\pi$  is a tempered principal series (resp. a complementary series) then  $\sigma$  must also be a tempered principal series (resp. a complementary series). Thus it remains to show that  $\pi$  is spherical if and only if  $\sigma$  is, which will be accomplished by exploiting minimal  $K$ -types.

Recall that the (complexified) maximal compact subgroups of  $PGL_3(\mathbb{R})$  and  $G_2(\mathbb{R})$  are  $K_1 = SO(3)$  and  $K_2 = SL_{2,l} \times SL_{2,s} / \langle \pm 1 \rangle$  respectively. Here,  $SL_{2,l}$  (resp.  $SL_{2,s}$ ) is generated by a pair of long (resp. short) roots. Let  $V_k$  be the irreducible representation of  $SL_2(\mathbb{C})$  of highest weight  $k$  (and dimension  $k + 1$ ). The minimal  $K_1$ -type of a non-spherical principal series of  $PGL_3(\mathbb{R})$  is  $V_2$ , and the minimal  $K_2$ -type of a non-spherical principal series of  $G_2(\mathbb{R})$  is  $V_0 \otimes V_2$ . Clearly, to prove the theorem, it suffices to show that for every  $n$ , the restriction of  $V_{n\omega_4}$  to  $K_1 \times K_2$  does not contain  $V_0 \otimes (V_0 \otimes V_2)$  and  $V_2 \otimes (V_0 \otimes V_0)$ .

This sort of branching result is very difficult, and we start by remarking that in both cases the type is trivial on  $SL_{2,l}$ . In particular our first task is to compute the  $SL_{2,l}$ -invariants in  $V_{n\omega_4}$ .

The inclusion of  $K_1 \times K_2$  into  $K$  is given by a combination of

$$\begin{cases} SO(3) \times SL_{2,s} \subseteq Sp_6 \\ Sp_6 \times SL_{2,l} \subseteq Sp_8 \end{cases}$$

where the first is the usual Howe dual pair, and the second is given by decomposing an 8 dimensional symplectic space as a direct sum of two nondegenerate subspaces of dimension 6 and 2. We have the following special case of the branching from  $Sp(2N)$  to  $Sp(2N-2) \times Sp(2)$ ; see [C] and [Z].

**Proposition 4.**

$$V_{n\omega_4}|_{Sp_6 \times SL_{2,l}} = \bigoplus_{k=0}^n V_{(n-k)\omega_2 + k\omega_3} \otimes V_k$$

where  $V_k$  is the irreducible representation of  $SL_2$  of highest weight  $k$ , and  $\omega_2 = (1, 1, 0)$  and  $\omega_3 = (1, 1, 1)$  are the second and third fundamental weight for  $Sp_6$  respectively.

The proof of Theorem now reduces to the following proposition which will be proved in the next section.

**Proposition 5.** *Let  $\omega_2$  be the second fundamental weight for  $Sp_6$ .*

- *Let  $X_n = \text{Hom}_{SO(3)}(V_0, V_{n\omega_2})$ . Then the three-dimensional representation  $V_2$  of  $SL_{2,s}$  does not appear in the  $SL_{2,s}$ -module  $X_n$ .*
- *Let  $Y_n = \text{Hom}_{SO(3)}(V_2, V_{n\omega_2})$ . Then the trivial representation  $V_0$  of  $SL_{2,s}$  does not appear in the  $SL_{2,s}$ -module  $Y_n$ .*

### 3. Branching rules

The purpose of this section is to prove Proposition 5 which is not easy at all. Let  $GL_3$  be the subgroup of  $Sp_6$  containing  $SO(3)$ , and such that the center of  $GL_3$  is equal to a maximal split torus of  $SL_2$ . We shall first calculate the restriction of  $V_{n\omega_2}$  to  $GL_3$ , and then further restrict down to  $SO(3)$  by exploiting the following fact:

**Proposition 6.** *Let  $\mu = (x, y, z)$ , with  $x \geq y \geq z$  integers, be the typical highest weight for  $GL_3$ , and let  $V_\mu$  denote the corresponding irreducible representation. Assume that  $x + y + z$  is even. The restriction of  $V_\mu$  to  $SO(3)$  either contains  $V_0$  or  $V_2$  (but not both) with multiplicity one. It contains  $V_0$  if and only if all integers  $x$ ,  $y$ , and  $z$  are even. Otherwise, it contains  $V_2$ , and this happens when only one of the three integers is even.*

*Proof.* This is a simple reinterpretation of the following well known branching result:

**Lemma 7.** *Let  $V_{p,q}$  be the irreducible representation of  $SL_3$  with highest weight  $p\omega_1 + q\omega_2$ , where  $\omega_1$  and  $\omega_2$  are the two fundamental weights of  $SL_3$ . Let  $m_{p,q}(k)$  denote the multiplicity of  $V_k$  (the representation of  $SO(3)$  with highest weight  $k$ ) in  $V_{p,q}$ . Then*

$$m_{p,q}(0) + m_{p,q}(2) = 1$$

and  $m_{p,q}(0) = 1$  iff  $p$  and  $q$  are both even.

□

The strategy of the proof of Proposition 5 will be based on the following simple observation.

**Lemma 8.** *Let  $V$  be a finite dimensional representation of  $SL_2$ . Let  $V(k) \subseteq V$  be the weight  $k$  subspace. Then  $V_n$  does not appear in  $V$  iff  $V(n) = V(n+2)$ .*

Thus to prove the first part of Proposition 5, for example, we need to show that  $\dim X_n(2)$  is equal to  $\dim X_n(4)$  or equivalently to  $\dim X_n(-4)$ .

How does one compute  $\dim X_n(2k)$  in general? Recall that the center of  $GL_3$  is a maximal split torus of  $SL_{2,s}$ . Therefore, as a  $GL_3$ -module,  $V_{n\omega_2}(2k)$  consists of representations  $V_\mu$  with

$x + y + z = 2k$ . From proposition 6, these contribute to  $X_n(2k)$  if and only if all three integers  $x, y$  and  $z$  are even. Let us denote by  $X(2k)$  the set of all highest weights  $\mu = (x, y, z)$  for  $GL_3$  such that  $x + y + z = 2k$ , with  $x, y$  and  $z$  all even, and write  $m_n(\mu)$  for the multiplicity of  $V_\mu$  in  $V_{n\omega_2}$ . Then

$$\dim X_n(2k) = \sum_{\mu \in X(2k)} m_n(\mu).$$

Now for  $\mu \in X(2)$ , let  $\mu_- = (x - 2, y - 2, z - 2)$ , which is an element of  $X(-4)$ . The map  $\mu \mapsto \mu_-$  defines a bijection of  $X(2)$  with  $X(-4)$ . We will show that the multiplicities  $m_n(\mu)$  and  $m_n(\mu_-)$  coincide. This implies that  $\dim X_n(2) = \dim X_n(-4)$ , and the first part of Proposition 5 follows. A similar, albeit more complicated, trick works for  $Y_n$ . We shall explain it later.

Thus, we need to calculate the multiplicity  $m_n(\mu)$  of  $V_\mu$  in  $V_{n\omega_2}$ . Since  $V_{n\omega_2}$  is self dual, and  $V_\mu^* \cong V_\lambda$ , where  $\lambda = (-z, -y, -x)$ , we have  $m_n(\mu) = m_n(\lambda)$ . So, from now on, we shall assume that  $y \geq 0$ . The multiplicity  $m_n(\mu)$  will be calculated using the Weyl character formula.

**3.1. Partition function.** Let  $e_1, e_2$  and  $e_3$  be the standard basis of  $\mathbb{R}^3$ . Then  $e_i + e_j$  (we allow  $i = j$ ) are positive roots of  $Sp_6$  which are not in  $GL_3$ . Define the partition function  $p(a, b, c)$  to be the number of ways the vector  $(a, b, c)$  in  $\mathbb{Z}^3$  can be written as a linear combination of  $e_i + e_j$  with non-negative integer coefficients. We shall now gather several elementary facts about this partition function.

**Lemma 9.** (i)  $p(a, b, c) = 0$  unless  $a, b, c$  are non-negative and  $a + b + c$  is even.

(ii) If  $a + b + c$  is even, then  $(a, b, c)$  is a combination of  $e_i + e_j$  (with  $i \neq j$ ) with non-negative integral coefficients iff

$$\begin{cases} a \leq b + c \\ b \leq c + a \\ c \leq a + b \end{cases}$$

in which case the combination is unique, since  $e_i + e_j$  with  $(i \neq j)$  are linearly independent.

Define  $q(a, b, c)$  to be 1 or 0 depending on whether  $(a, b, c)$  is a combination of  $e_i + e_j$  ( $i \neq j$ ) with non-negative integer coefficients or not. Clearly, we have the following simple lemma.

**Lemma 10.**

$$p(a, b, c) = \sum q(a - i, b - j, c - k)$$

where the sum is taken over all even, non-negative integers  $i, j$  and  $k$ .

**3.2. Branching law continued.** Let  $\mu = (x, y, z)$  with  $y \geq 0$ . The Weyl character formula implies that the multiplicity  $m_n(\mu)$  of  $V_\mu$  in  $V_{n\omega_2}$  is

$$m_n(\mu) = \sum_{w \in W} e(w) p(w(n\omega_2 + \rho) - \rho - \mu)$$

where the sum is over the Weyl group for  $Sp_6$ , and  $\rho = (3, 2, 1)$ . Since  $p = 0$  unless the argument has non-negative entries, one easily checks, due to  $y \geq 0$  and the specific nature of  $\omega_2$ , that the sum reduces to

$$p((n, n, 0) - \mu) - p((n, n, -2) - \mu) - p((n-1, n+1, 0) - \mu) + p((n-1, n+1, -2) - \mu).$$

Moreover, none of these terms appear if  $z > 0$  or  $x + y + z$  is odd. So we shall assume that  $z \leq 0$  and  $x + y + z$  is even.

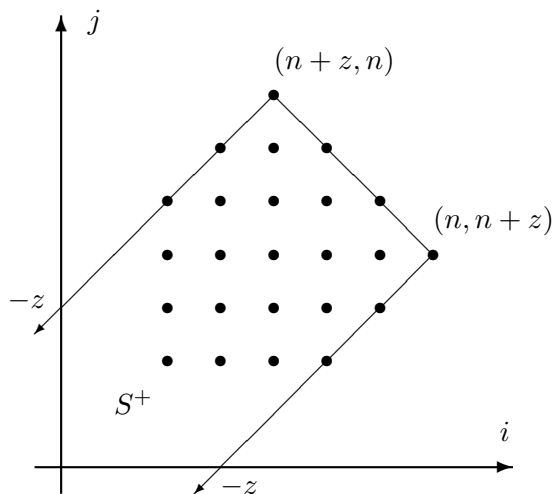
After a slight change of variables, Lemma 10 implies that

$$p(n-x, n-y, -z) - p(n-x, n-y, -z-2) = \sum q(n-i, n-j, -z)$$

where the sum is taken over all  $(i, j) \equiv (x, y) \pmod{2}$  such that  $x \leq i$  and  $y \leq j$ . From the second part of Lemma 9, we see that  $q(n-i, n-j, -z) = 0$  unless  $(i, j)$  satisfies the equations

$$\begin{cases} i + j \leq 2n + z \\ i \leq j - z \\ j \leq i - z. \end{cases}$$

This is a half-strip  $S^+$  with vertices at  $(n+z, n)$  and  $(n, n+z)$ , and in the direction  $(-1, -1)$ . We illustrate it by the following figure, modeled after  $\mu = (4, 4, -6)$ , and  $n = 14$ .



The black dots are the points  $(i, j)$  in  $S^+$  such that  $x \leq i$ ,  $y \leq j$ , and  $(i, j) \equiv (x, y) \pmod{2}$ . It follows that the difference of the first two terms in the formula for  $m_n(\mu)$  counts the black points in the above picture.

Similarly, the difference

$$p(n-x, n-y, -z) - p(n-x, n-y, -z-2)$$

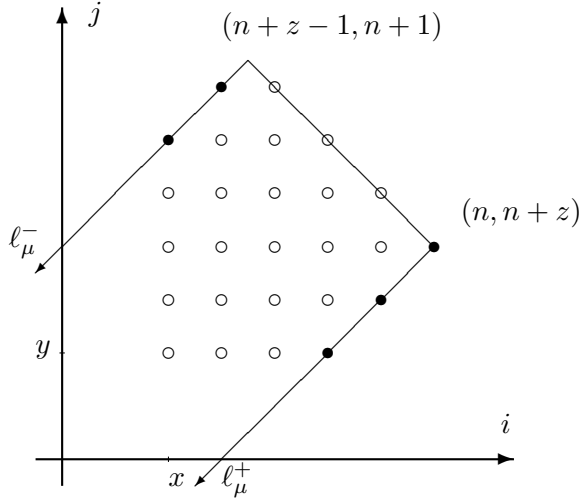
leads us to a half strip  $S^-$ , which is a translate of  $S^+$  by  $(-1, 1)$ . Its vertices are at  $(n+z-1, n+1)$  and  $(n-1, n+z+1)$ . In particular, we have obtained the following:

**Theorem 11.** *Let  $\mu = (x, y, z)$  with  $y \geq 0$ . Then  $m_n(\mu)$  (the multiplicity of  $V_\mu$  in  $V_{n\omega_2}$ ) is zero unless  $z \leq 0$  and  $x + y + z$  even. When these hold, let  $\ell_\mu^+$  be the ray  $(n, n + z) - t(1, 1)$  with  $t \geq 0$ , and  $\ell_\mu^-$  be the ray  $(n + z - 1, n + 1) - t(1, 1)$  with  $t \geq 0$ .*

*Let  $n^+(\mu)$  and  $n^-(\mu)$  be the number of points  $(i, j) \equiv (x, y) \pmod{2}$  with  $x \leq i$  and  $y \leq j$  on  $\ell_\mu^+$  and  $\ell_\mu^-$  respectively. Then*

$$m_n(\mu) = n^+(\mu) - n^-(\mu).$$

We can illustrate this theorem with the following figure, again modeled after  $\mu = (4, 4, -6)$ , and  $n = 14$ . In particular, it follows that the multiplicity of  $V_{(4,4,-6)}$  in  $V_{14\omega_2}$  is  $3 - 2 = 1$ .



Now we can finish the proof of the first part of Proposition 5. Let  $\mu = (x, y, z)$  such that  $x + y + z = 2$ . Assume first that  $y > 0$ , so  $y \geq 2$ , since  $y$  is even. Thus both multiplicities  $m_n(\mu)$  and  $m_n(\mu_-)$  can be calculated using the above theorem. The rays  $\ell_{\mu_-}^+$  and  $\ell_{\mu_-}^-$  are obtained by translating the rays  $\ell_\mu^+$  and  $\ell_\mu^-$  down and left by 2 respectively. Next, note that  $x + y + z = 2$  and  $y \geq 2$  imply that the point  $(x, y)$  is strictly between the rays  $\ell_\mu^+$  and  $\ell_\mu^-$ . It follows that

$$\begin{cases} n^+(\mu) = n^+(\mu_-) \\ n^-(\mu) = n^-(\mu_-) \end{cases}$$

so that  $m_n(\mu) = m_n(\mu_-)$ . If  $y \leq 0$ , then we can apply similar considerations to the dual representations  $V_\mu^*$  and  $V_{\mu_-}^*$ . The first part of Proposition 5 is proved.

Now we consider the second part of Proposition 5. We need to show that  $\dim Y_n(0) = \dim Y_n(2)$ . Recall that for an even integer  $2k$ , we have defined  $X(2k)$  to be the set of all highest weights  $(x, y, z)$  for  $GL_3$  such that  $x + y + z = 2k$ , with  $x, y, z$  all even. Let  $Y(2k)$  be similarly defined, except that only one of the three integers is even. Now, if  $(x, y, z)$  is in  $Y(2k)$ , then we can add (or subtract) 1 to each of the two odd entries to obtain, after permuting the entries if necessary, an element in  $X(2k + 2)$  (or in  $X(2k - 2)$ ). This gives us a partition of  $Y(2k)$  into subsets  $Y_\mu(2k)$  parametrized by elements of  $X(2k + 2)$  (or  $X(2k - 2)$ ).

Then it is not difficult to see that

$$\begin{cases} \dim Y_n(0) = \sum_{\mu \in X(-2)} \sum_{\lambda \in Y_\mu(0)} m_n(\lambda) \\ \dim Y_n(2) = \sum_{\mu \in X(4)} \sum_{\lambda \in Y_\mu(2)} m_n(\lambda). \end{cases}$$

Now if  $\mu = (x, y, z) \in X(4)$ , then  $\mu_- = (x - 2, y - 2, z - 2) \in X(-2)$  and this sets up a bijection of  $X(4)$  with  $X(-2)$ . Thus the second part of Proposition 5 will follow after we show:

**Lemma 12.** *For each  $\mu = (x, y, z)$  in  $X(4)$ ,*

$$\sum_{\lambda \in Y_\mu(2)} m_n(\lambda) = \sum_{\lambda \in Y_{\mu_-}(0)} m_n(\lambda)$$

*Proof.* Assume first that  $y > 0$ , so that  $y \geq 2$  since  $y$  is even. Since  $x + y + z = 4$ , it follows that  $z \leq 0$ . Since the case  $x = y$  can be similarly dealt with, we shall assume that  $x > y$  as well. Then

$$Y_\mu(2) = \{\lambda_1 = (x - 1, y - 1, z), \lambda_2 = (x, y - 1, z - 1), \lambda_3 = (x - 1, y, z - 1)\}$$

and

$$Y_{\mu_-}(0) = \{\lambda'_1 = (x - 1, y - 1, z - 2), \lambda'_2 = (x - 2, y - 1, z - 1), \lambda'_3 = (x - 1, y - 2, z - 1)\}.$$

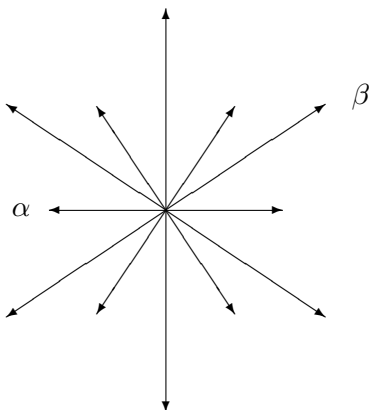
Now one easily checks that we have the following sequence of equalities

$$\begin{cases} n^+(\lambda_1) = n^+(\lambda'_3) \\ n^-(\lambda'_3) = n^-(\lambda_3) \\ n^+(\lambda_3) = n^+(\lambda'_1) \\ n^-(\lambda'_1) = n^-(\lambda_2) \\ n^+(\lambda_2) = n^+(\lambda'_2) \\ n^-(\lambda'_2) = n^-(\lambda_1) \end{cases}$$

Summing up these equalities gives the desired conclusion. Of course, if  $y \leq 0$ , then we can compute the multiplicities by passing to dual representations. We leave the details to the reader.  $\square$

#### 4. Structure of Groups

In this section, we establish some notations to be used in the rest of the paper. Let  $F$  be any field of characteristic zero. We first describe the maximal parabolic subgroups of  $G_2$ . Fix a maximal split torus  $M$  of  $G_2$  contained in a Borel subgroup  $P = MN$ . This establishes a system of simple roots  $\{\alpha, \beta\}$  for  $G_2$  with  $\alpha$  short:



Let  $P_1 = M_1N_1$  and  $P_2 = M_2N_2$  be the two non-conjugate maximal parabolic subgroups of  $G_2$  so that  $P = P_1 \cap P_2$  and  $M = M_1 \cap M_2$ . Both  $M_1$  and  $M_2$  are isomorphic to  $GL_2$ . Further, we stipulate that  $P_2$  is the Heisenberg parabolic so that  $N_2$  is a 5-dimensional Heisenberg group. Then  $N_1$  is a 3-step nilpotent group.

Let  $\bar{P}_2 = M_2\bar{N}_2$  be the opposite parabolic of  $P_2$ . Then using the Killing form and the exponential map,  $Hom(N_2, \mathbb{G}_a)$  can be identified with  $V_2 = Lie(\bar{N}_2^{ab})$  where  $\bar{N}_2^{ab} = \bar{N}_2/[\bar{N}_2, \bar{N}_2]$ . Fix a Chevalley basis of  $Lie(G_2)$ . For every root  $\delta$ , this gives an identification of the corresponding root subgroup  $N_\delta$  with the additive group  $\mathbb{G}_a$ . The Lie algebra of  $N_\delta$  is the line spanned by the corresponding Chevalley basis element and will be denoted by  $F_\delta$ . We now have

$$V_2 = F_{-\beta} \oplus F_{-\alpha-\beta} \oplus F_{-2\alpha-\beta} \oplus F_{-3\alpha-\beta}.$$

Moreover, for an appropriate identification of  $M_2$  with  $GL_2$ , the representation of  $M_2$  on  $V_2$  is isomorphic to  $Sym^3(F^2) \otimes det^{-1}$ .

If  $F$  is a local field and  $\psi$  is a unitary character of  $F$ , then composition with  $\psi$  gives an identification of  $V_2$  with the group of unitary characters of  $N_2(F)$ . Hence a unitary character of  $N_2(F)$  can be represented by a 4-tuple  $(a, b, c, d) \in V_2(F)$ . Similarly, if  $F$  is a global field and  $\psi$  a unitary character of  $F \backslash \mathbb{A}$ , then composition with  $\psi$  gives an identification of  $V_2(F)$  with the group of characters of  $N_2(\mathbb{A})$  trivial on  $N_2(F)$ .

Now let  $H$  be the split adjoint group of type  $E_6$ . Then  $PGL_3 \times G_2$  forms a dual pair in  $H$  (cf. [MS] and [GRS2]). If we consider the adjoint action of the maximal torus  $M \subset G_2$  on  $Lie(H)$ , the zero weight space of course contains  $Lie(PGL_3)$ , whereas the non-zero weights form a root system of type  $G_2$  (cf. [GS2]). While the long root spaces are still of dimension 1, the short root spaces have dimension 9 and can naturally be given the structure of a  $3 \times 3$  matrix algebra  $J$ . Hence, morally speaking,  $H \cong G_2(J)$ .

In any case, for each root  $\delta$  of  $G_2$ , we write  $\mathbf{N}_\delta$  for the corresponding root subgroup of  $H$ , so that

$$\mathbf{N}_\delta = \begin{cases} \mathbb{G}_a & \text{if } \delta \text{ is long;} \\ J & \text{if } \delta \text{ is short,} \end{cases}$$

and the inclusion of  $N_\delta$  into  $\mathbf{N}_\delta$  is given by the inclusion of scalars. The action of  $PGL_3$  on  $\mathbf{N}_\delta$  is trivial if  $\delta$  is long and is given by conjugation of  $3 \times 3$  matrices if  $\delta$  is short.

Using this restricted root system of type  $G_2$ , we can define a maximal parabolic subgroup  $\mathbf{P}_2 = \mathbf{M}_2 \mathbf{N}_2$  of  $H$  such that its unipotent radical  $\mathbf{N}_2$  is a Heisenberg group which is supported on the same root subgroups as  $N_2$  (using  $\mathbf{N}_\delta$  in place of  $N_\delta$ ). In particular,

$$\mathbf{P}_2 \cap (PGL_3 \times G_2) = PGL_3 \times P_2.$$

As before, using the Killing form and the exponential map, we can identify  $Hom(\mathbf{N}_2, \mathbb{G}_a)$  with  $\mathbf{V}_2 = Lie(\overline{\mathbf{N}}_2^{ab})$ . Here,

$$\mathbf{V}_2 = F_{-\beta} \oplus J_{-\alpha-\beta} \oplus J_{-2\alpha-\beta} \oplus F_{-3\alpha-\beta}$$

and we shall represent an element in  $\mathbf{V}_2$  as  $(a, x, y, d)$  with  $a, d \in F$  and  $x, y \in J$ . As before, elements of  $\mathbf{V}_2(F)$  can be identified with a suitable group of unitary characters when  $F$  is local or global.

The Levi subgroup  $\mathbf{M}_2$  is isomorphic to  $\mathbb{G}_m \times_{\mu_2} (SL_6/\mu_3)$ . The center of  $\mathbf{M}_2$  is equal to the center of  $M_2$ , and thus is isomorphic to  $\mathbb{G}_m$  by the coroot  $(3\alpha + 2\beta)^\vee$ . The quotient of  $\mathbf{M}_2$  by its center is isomorphic to  $PGL_6$ . Moreover, there is a basis element of  $Hom_F(\mathbf{M}_2, \mathbb{G}_m)$  whose restriction to the maximal split torus  $M$  of  $G_2$  is equal to the highest root  $\alpha_0 = 3\alpha + 2\beta$ . We denote this rational character by  $\alpha_0$  as well.

The group  $\mathbf{M}_2 \cap (PGL_3 \times G_2) = PGL_3 \times M_2$  acts naturally on  $\mathbf{V}_2$ . The action of  $PGL_3$  is clear from our description above. For later purposes, it will be necessary to know the action of  $\mathbf{N}_\alpha \subset \mathbf{M}_2$  more explicitly. It is easiest to describe this action on the level of Lie algebras; one then uses the exponential map to obtain the action of a group element. More generally, for  $z \in Lie(\mathbf{N}_\alpha)$ , we have:

$$z : (a, x, y, d) \mapsto (Tr(xz), y \times z, dz, 0).$$

Here,  $Tr$  is the trace form on  $J$  and  $y \times z = (y + z)^\# - y^\# - z^\#$  where  $y \mapsto y^\#$  is the quadratic map such that  $y \cdot y^\# = det(y)$ .

The inclusion  $N_2 \hookrightarrow \mathbf{N}_2$  induces a projection map

$$Hom(\mathbf{N}_2, \mathbb{G}_a) \rightarrow Hom(N_2, \mathbb{G}_a)$$

which is  $PGL_3 \times M_2$ -equivariant. The resulting projection  $\mathbf{V}_2 \rightarrow V_2$  is given by

$$(a, x, y, d) \mapsto (a, Tr(x), Tr(y), d).$$

The restricted root system of type  $G_2$  also allows us to define another maximal parabolic subgroup  $\mathbf{P}_1$  whose unipotent radical is supported on the same root subgroups as  $N_1$ , using  $\mathbf{N}_\delta$  instead of  $N_\delta$ . In particular,  $\mathbf{M}_2 \cap \mathbf{P}_1$  is a maximal parabolic subgroup of  $\mathbf{M}_2$ . Its unipotent radical is equal to  $\mathbf{N}_\alpha$  and a Levi subgroup is  $\mathbf{M} = \mathbf{M}_1 \cap \mathbf{M}_2$ .

### 5. Global correspondence

Now we consider the global setting, so that  $F$  is a number field. Let  $\Pi = \otimes_v \Pi_v$  be the global minimal representation of  $H(\mathbb{A})$ . In [GRS1], an  $H(\mathbb{A})$ -equivariant embedding

$$\Pi \hookrightarrow \mathcal{A}_2(H)$$

of  $\Pi$  into the space of square-integrable automorphic forms on  $H$  was constructed. We shall identify  $\Pi$  with its image in  $\mathcal{A}_2(H)$ . There are several important properties of this automorphic theta module, especially concerning its Fourier expansion along  $\mathbf{N}_2$ , which will be discussed at the end of this section.

Now let  $\pi = \otimes_v \pi_v$  be an irreducible cuspidal automorphic representation of  $PGL_3(\mathbb{A})$ . By the multiplicity one theorem, we know that  $\pi$  has a canonical realization in the space of cusp forms and we shall identify  $\pi$  with this realization. For  $f \in \pi$  and  $\varphi \in \Pi$ , define an automorphic form on  $G_2(\mathbb{A})$  by:

$$\Theta(\varphi, f)(g) = \int_{PGL_3(F) \backslash PGL_3(\mathbb{A})} \varphi(gh) \cdot f(h) dh.$$

We shall denote the linear span of all such automorphic forms on  $G_2$  by  $\Theta(\pi)$ ; it is a  $G_2(\mathbb{A})$ -submodule of  $\mathcal{A}(G_2)$ .

Given an automorphic form on  $G_2$ , one may define its global Whittaker function (associated to a given generic character of  $N(F) \backslash N(\mathbb{A})$ ). This is defined by the usual integral. We say that an automorphic form is *globally degenerate* if its global Whittaker function is zero (as a function on  $G_2(\mathbb{A})$ ). Let  $\mathcal{A}_{cusp,d}$  be the subspace of degenerate cusp forms, and let  $\mathcal{A}_{cusp,gen}$  be its orthogonal complement in the space of cusp forms. An irreducible summand of  $\mathcal{A}_{cusp,gen}$  is said to be *globally generic*.

The main result of this section is:

**Theorem 13.** (i)  $\Theta(\pi)$  is contained in the space of cusp forms on  $G_2$ .

(ii) Let  $\Theta_{gen}(\pi)$  be the projection of  $\Theta(\pi)$  onto  $\mathcal{A}_{cusp,gen}$ . Then  $\Theta_{gen}(\pi)$  is non-zero.

The proof of this theorem is largely computational and will be given in the following two sections. It is based on the expansion of  $\varphi$  along  $\mathbf{N}_2$ . More precisely, let  $\mathbf{Z}$  be the center of  $\mathbf{N}_2$ , and consider the constant term of  $\varphi$  along

$$\varphi_{\mathbf{Z}}(g) = \int_{\mathbf{Z}(F) \backslash \mathbf{Z}(\mathbb{A})} \varphi(zg) dz.$$

Recall from the previous section that any character of  $\mathbf{N}_2(F) \backslash \mathbf{N}_2(\mathbb{A})$  is  $\psi_X$  for some  $X$  in  $\mathbf{V}_2(F)$ . Let  $\Omega \subseteq \mathbf{V}_2$  be the unique non-trivial  $\mathbf{M}_2$ -orbit in  $\mathbf{V}_2$ . It is the orbit of the highest weight vector. Assume that  $X \neq 0$ . By [MS], for any finite place  $v$ , we know that

$$\dim \operatorname{Hom}_{\mathbf{N}_2(F_v)}(\Pi_v, \psi_X) = \begin{cases} 1 & \text{if } x \in \Omega; \\ 0 & \text{otherwise.} \end{cases}$$

Now let  $\varphi_X$  be the Fourier coefficient of  $\varphi$  defined by the character  $\psi_X$ . We have:

**Proposition 14.** *For any  $\varphi \in \Pi$ , one has the Fourier expansion*

$$\varphi_{\mathbf{Z}}(g) = \varphi_{\mathbf{N}_2}(g) + \sum_{X \in \Omega(F)} \varphi_X(g).$$

Moreover, it follows by the weak approximation theorem that for any  $m \in [\mathbf{M}_2(\mathbb{A}), \mathbf{M}_2(\mathbb{A})]$  which fixes  $X$ , we have

$$\varphi_X(mg) = \varphi_X(g).$$

We also need some facts about the constant term  $\varphi_{\mathbf{N}_2}$ . From the construction of the embedding  $\Pi \hookrightarrow \mathcal{A}_2(H)$  in [GRS], one can show that the subspace of  $\mathcal{A}(\mathbf{M}_2)$  consisting of the functions  $\varphi_{\mathbf{N}_2}|_{\mathbf{M}_2}$  is isomorphic to  $|\alpha_0|^2 \oplus |\alpha_0|^{3/2} \cdot \Pi(PGL_6)$ . Here,  $\Pi(PGL_6)$  is the irreducible representation of  $PGL_6(\mathbb{A})$  unitarily induced from the trivial character of the maximal parabolic subgroup of  $PGL_6$  with Levi factor  $GL_5$ . The embedding of  $\Pi(PGL_6)$  into  $\mathcal{A}(\mathbf{M}_2)$  is by the formation of Eisenstein series.

In some sense,  $\Pi(PGL_6)$  can be considered the minimal representation of  $\mathbf{M}_2(\mathbb{A})$ . More precisely, if we consider the Fourier expansion of  $\varphi \in \Pi(PGL_6)$  along  $\mathbf{N}_\alpha \cong J$  (the unipotent radical of  $\mathbf{P}_1 \cap \mathbf{P}_2 = \mathbf{M} \cdot \mathbf{N}_\alpha$ ), then we have the following analog of Prop. 14:

**Proposition 15.** *For any  $\varphi \in \Pi(PGL_6)$ , one has the Fourier expansion*

$$\varphi(g) = \varphi_{\mathbf{N}_\alpha}(g) + \sum_{X \in \Omega(F)} \varphi_X(g), \quad g \in \mathbf{M}_2(\mathbb{A})$$

where  $\Omega(F)$  is the set of  $X \in \mathbf{N}_\alpha(F) \cong J(F)$  which has rank one. Moreover, if  $m \in [\mathbf{M}(\mathbb{A}), \mathbf{M}(\mathbb{A})]$  fixes  $X$ , then  $\varphi_X(mg) = \varphi_X(g)$ .

Finally, we need the following lemma concerning the constant term  $\varphi_{\mathbf{N}_\alpha}$ .

**Lemma 16.** *For any  $\varphi \in \Pi(PGL_6)$ , the restriction of  $\varphi_{\mathbf{N}_\alpha}$  to the subgroup  $PGL_3 \subset \mathbf{M}$  is an automorphic form with cuspidal support along the Borel subgroup of  $PGL_3$ . In particular,  $\varphi_{\mathbf{N}_\alpha}|_{PGL_3}$  is orthogonal to the space of cusp forms.*

## 6. Cuspidality

In this section, we prove Theorem 13(i). We need to show that the constant terms of  $\Theta(\varphi, f)$  along  $N_1$  and  $N_2$  are both identically zero.

We first compute the constant term  $\Theta(\varphi, f)_{N_2}$  of  $\Theta(\varphi, f)$  along  $N_2$ . Since  $\mathbf{Z}$  is also the center of  $N_2$ , we can replace  $\varphi$  by  $\varphi_{\mathbf{Z}}$  in  $\Theta(\varphi, f)_{N_2}$ . Using the Fourier expansion of  $\varphi_{\mathbf{Z}}$  along  $\mathbf{N}_2$ , we deduce that

$$\Theta(\varphi, f)_{N_2}(1) = \int_{PGL_3(F) \backslash PGL_3(\mathbb{A})} [\varphi_{\mathbf{N}_2}(h) + \sum_{X \in \Omega_0} \varphi_X(h)] f(h) dh$$

where  $\Omega_0$  is the subset of  $\Omega$  consisting of all  $X$  such that the restriction of  $\varphi_X$  to  $N_2$  is trivial.

We shall now show that the integral of the sum over  $\Omega_0$  is zero. Using the projection  $\mathbf{V}_2 \rightarrow V_2$ , it follows that

$$\Omega_0 = \{(a, x, y, d) \in \Omega \mid a = \text{Tr}(x) = \text{Tr}(y) = d = 0\}.$$

The orbits of  $PGL_3(F)$  on  $\Omega_0$  are described in [MS, Prop 7.4]. More precisely, let  $X$  be in  $\Omega_0$ , and  $\mathcal{O}(X)$  its  $PGL_3(F)$ -orbit. If  $S_X$  is the stabilizer in  $PGL_3(F)$  of  $X$ , then  $S_X$  contains (as a normal subgroup) the unipotent radical  $U_X$  of a parabolic subgroup of  $PGL_3$ . It follows that

$$\int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \sum_{Y \in \mathcal{O}(X)} \varphi_Y(h) f(h) dh = \int_{S_X(F)\backslash PGL_3(\mathbb{A})} \varphi_X(h) f(h) dh.$$

Since  $\varphi_X$  is left invariant under  $U_X(\mathbb{A})$  by Prop. 14, the cuspidality of  $f$  now implies that the integral on the right hand side is zero.

It remains to show that the integral against  $\varphi_{N_2}$  is also zero. But  $PGL_3 \times M_2$  is a dual pair in  $\mathbf{M}_2$  and the integral

$$\int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \varphi_{N_2}(hg) \cdot f(h) dh, \quad \text{for } g \in M_2(\mathbb{A})$$

is simply the theta lift (using  $\Pi(PGL_6)$ ) of  $f \in \pi$  to  $M_2 \cong GL_2$ . Hence, to show that the above integral vanishes, we need to show that in the theta correspondence furnished by the dual pair  $PGL_2 \times PGL_3 \subset PGL_6$ , a cuspidal representation of  $PGL_3$  always lifts to zero.

Using Prop. 15 and Lemma 16, it is easy to show that the theta lift of  $\pi$  to  $M_2 \cong GL_2$  is always cuspidal; we omit the details here. To show that it is zero, one needs to examine a non-constant Fourier coefficient of the theta lift along  $N_\alpha \subset M_2$ . After a short computation, using Prop. 15, one finds that such a Fourier coefficient is zero if and only if the following holds:

$$\int_{GL_2(F)\backslash GL_2(\mathbb{A})} f(h) dh = 0 \quad \text{for all } f \in \pi,$$

where  $GL_2$  is the Levi subgroup of a maximal parabolic subgroup of  $PGL_3$ . But the above period integral vanishes by the results of Friedberg-Jacquet [FJ]. This shows that the constant term  $\Theta(\varphi, f)_{N_2}$  is identically zero.

We now consider the other constant term  $\Theta(\varphi, f)_{N_1}$ . Consider this as a function on  $M_1 \cong GL_2$ . Then, because of the vanishing of  $\Theta(\varphi, f)_{N_2}$ , this function on  $M_1$  is a cusp form. To show that it is zero, we examine a non-constant Fourier coefficient of  $\Theta(\varphi, f)_{N_1}$  along  $N_\beta \subset M_1$ . In other words, if  $\psi_\beta$  is a character of  $N(F)\backslash N(\mathbb{A})$  which is trivial on  $N_\alpha$  but non-trivial on  $N_\beta$ , then we want to show that  $\Theta(\varphi, f)_{N, \psi_\beta} = 0$ . Let us give this computation in detail, as a similar computation appears in the next section. Before that, we need to introduce some notations.

Without loss of generality, we may assume that the restriction of  $\psi_\beta$  to  $N_2(\mathbb{A})$  is represented by  $(1, 0, 0, 0) \in V_2$  in the notation of Section 4. Let  $\Omega_\beta$  be the set of all  $X$  in  $\Omega$  such that the restriction of  $\psi_X$  to  $N_2(\mathbb{A})$  is  $\psi_\beta$ . Then

$$\Theta(\varphi, f)_{N_2, \psi_\beta}(1) = \int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} f(h) \cdot \left[ \sum_{X \in \Omega_\beta} \varphi_X(h) \right] dh$$

To describe  $\Omega_\beta$ , we proceed as follows. By [MS, Lemma 7.5] (see also [GrS, Pg. 214]), if  $X = (a, x, y, d)$  lies in  $\Omega$  with  $a = 1$ , then

$$X = (1, x, x^\#, \det(x)),$$

for some  $x$  in  $J(F)$ . If further  $X \in \Omega_\beta$ , then  $Tr(x) = Tr(x^\#) = det(x) = 0$  and thus the characteristic polynomial of  $x$  is  $x^3 = 0$ . It follows that  $x$  is a nilpotent matrix. In particular,  $PGL_3(F)$  has three orbits on  $\Omega_\beta$ , parametrized by the rank of  $x$ . If  $x = 0$ , then the centralizer of  $x$  is  $PGL_3(F)$ , and if  $x$  has rank 1, then its centralizer contains the unipotent radical of a minimal parabolic subgroup. In particular, the cuspidality of  $f$  implies that the corresponding contributions in the above expression vanish. It remains to deal with the orbit parametrized by rank 2 nilpotent matrices. Fix a representative  $X = (1, x_0, x_0^\#, 0)$  with

$$x_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

If  $U$  denotes the group of unipotent upper triangular matrices in  $PGL_3$ , then the stabilizer of  $\psi_X$  is a codimension one subgroup  $U_0 \subset U$ . It follows that

$$\Theta(\varphi, f)_{N_2, \psi_\beta}(1) = \int_{U(F) \backslash PGL_3(\mathbb{A})} f(h) \sum_{\gamma \in U_0(F) \backslash U(F)} \varphi_X(\gamma h) dh.$$

Next, consider a homomorphism  $\mathbf{n} : U \rightarrow N_\alpha \cong \mathbb{G}_a$  defined by  $\mathbf{n}(u) = a - b$  where

$$u = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

**Lemma 17.** *Let  $X = (1, x_0, x_0^\#, 0)$  as above. Under the action of  $U \subset \mathbf{M}_2$  on  $\mathbf{V}_2$ , we have*

$$u \cdot X = \mathbf{n}(u) \cdot X$$

for every  $u \in U$ .

*Proof.* Recall the action of  $Lie(\mathbf{N}_\alpha)$  on  $\mathbf{V}_2$  given by the formula in Section 4. Exponentiating, we can obtain the corresponding action of  $\mathbf{N}_\alpha$ . In particular, if  $n$  is in  $N_\alpha \cong \mathbb{G}_a$ , then

$$n : (1, x_0, x_0^\#, 0) \mapsto (1, x_0 + nx_0^*, 0)$$

This clearly coincides with the action of  $U$ , via the map  $\mathbf{n}$ . The lemma is proved.  $\square$

Note that the kernel of  $\mathbf{n}$  is precisely  $U_0$ . It follows that the expression for  $\Theta(\varphi, f)_{N_2, \psi_\beta}(1)$  can be rewritten as

$$\Theta(\varphi, f)_{N_2, \psi_\beta}(1) = \int_{U(F) \backslash PGL_3(\mathbb{A})} f(h) \sum_{\delta \in N_\alpha(F)} \varphi_X(\delta h) dh,$$

where  $\delta = \mathbf{n}(\gamma)$ . Integrating over  $N_\alpha(F) \backslash N_\alpha(\mathbb{A})$ , it follows that

$$\Theta(\varphi, f)_{N_2, \psi_\beta}(1) = \int_{U(F) \backslash PGL_3(\mathbb{A})} f(h) \cdot \left[ \int_{N_\alpha(\mathbb{A})} \varphi_X(nh) dn \right] dh.$$

The inner integral is now a left  $U(\mathbb{A})$ -invariant function of  $h$ . The vanishing of  $\Theta(\varphi, f)_{N_2, \psi_\beta}$  now follows from the cuspidality of  $f$ . This proves Theorem 13(i).

## 7. Non-Vanishing

In this section, we prove Theorem 13(ii). For this, we take a generic character  $\psi_N$  of  $N(\mathbb{A})$ . Without loss of generality, we may assume that the restriction of  $\psi_N$  to  $N_2$  is represented by the element  $(1, 0, 0, 0) \in V_2$  and its restriction to  $N_\alpha$  is a non-trivial character  $\psi^{-1}$ . Now we need to show that  $\Theta(\varphi, f)_{N, \psi_N}$  is non-zero for some  $f$  and  $\varphi$ .

This integral is similar to that in the previous section and the same computation gives:

$$\Theta(\varphi, f)_{N, \psi_N}(1) = \int_{U(F) \backslash PGL_3(\mathbb{A})} f(h) \left[ \int_{N_\alpha(\mathbb{A})} \psi(n) \varphi_X(nh) dn \right] dh,$$

where  $X = (1, x_0, x_0^\#, 0)$ , for the fixed nilpotent rank 2 matrix  $x_0$  as in the previous section. This time, however, the inner integral is not left  $U(\mathbb{A})$ -invariant, but it transforms according to the generic character  $\psi_U^{-1}$  with:

$$\psi_U(u) = \psi(\mathbf{n}(u)).$$

In particular, if  $f_{U, \psi_U}$  is the Whittaker coefficient of  $f$  relative to  $\psi_U$ , then

$$\Theta(\varphi, f)_{N, \psi_N}(1) = \int_{U(\mathbb{A}) \backslash PGL_3(\mathbb{A})} f_{U, \psi_U}(h) \cdot \left[ \int_{N_\alpha(\mathbb{A})} \psi(n) \varphi_X(nh) dn \right] dh.$$

We want to show that  $\Theta(\varphi, f)_{N, \psi_N}(1) \neq 0$  for some choices of  $\varphi$  and  $f$ . More precisely, if  $f_{U, \psi_U}(1) \cdot \varphi_X(1) \neq 0$  (which clearly can be assumed), we shall show that the Whittaker coefficient does not vanish if we replace  $\varphi$  by

$$\Phi * \varphi(g) = \int_{\mathbf{N}_{\alpha+\beta}(\mathbb{A})} \Phi(y) \varphi(gy) dy,$$

for some Schwarz function  $\Phi$  on  $\mathbf{N}_{\alpha+\beta}(\mathbb{A})$ . Indeed, after commuting  $nh$  and  $y$ , we get

$$\varphi_X(nhy) = \psi(\text{Tr}(h^{-1}n(x_0)h \cdot y)) \varphi_X(nh)$$

where  $n(x_0) = x_0 + nx_0^\#$ . Let  $\hat{\Phi}$  denote the Fourier transform of  $\Phi$ . It follows that

$$(\Phi * \varphi)_X(nh) = \hat{\Phi}(h^{-1}n(x_0)h) \cdot \varphi_X(nh)$$

and

$$\Theta(\Phi * \varphi, f)_{N, \psi_N}(1) = \int_{U(\mathbb{A}) \backslash PGL_3(\mathbb{A})} f_{U, \psi_U}(h) \left[ \int_{N_\alpha(\mathbb{A})} \psi(n) \hat{\Phi}(h^{-1}n(x_0)h) \varphi_X(nh) dn \right] dh.$$

For any finite place  $v$ , let  $A_v$  be the maximal order in  $F_v$ . Let  $S$  be the set of places, including the archimedean ones, such that for every place  $v \notin S$ , our functions  $f$ ,  $\varphi$  and the character  $\psi$  are unramified. We shall choose  $\Phi$  of the form

$$\Phi = \Phi_S \otimes (\otimes_{v \notin S} \Phi_v)$$

where  $\Phi_v$  is the charactersitic function of  $\mathbf{N}_{\alpha+\beta}(A_v)$  for all  $v \notin S$  and  $\Phi_S$  is to be chosen later.

Next, using the Iwasawa decomposition  $PGL_3 = UTK$ , we can write  $h = utk$  where  $t$  is represented by a  $3 \times 3$  matrix

$$t = \begin{pmatrix} a & & \\ & b & \\ & & 1 \end{pmatrix}.$$

A well-known fact is that the Whittaker coefficient  $f_{U\psi_U}(utk)$  could be non-zero only if for all  $v \notin S$

$$\text{ord}_v(a) \geq \text{ord}_v(b) \geq 0.$$

On the other hand,  $t^{-1}n(x_0)t$  is equal to

$$\begin{pmatrix} 0 & b/a & n/a \\ 0 & 0 & 1/b \\ 0 & 0 & 0 \end{pmatrix}.$$

If  $v \notin S$ , then  $\hat{\Phi}_v = \Phi_v$ . Since  $\Phi_v$  is supported on  $\mathbf{N}_{\alpha+\beta}(A_v)$  we also have the inequalities

$$\text{ord}_v(a) \leq \text{ord}_v(b) \leq 0 \quad \text{and} \quad \text{ord}_v(n) \geq \text{ord}_v(a).$$

In particular, our integrand is zero unless  $t_v$  is in  $T(A_v)$ , and  $n_v$  in  $N_\alpha(A_v)$ , for all  $v \notin S$ . It follows that our integral reduces to integration over groups with coefficients in  $\mathbb{A}_S$ .

Let  $\mathcal{O}$  be the set of rank 2 matrices in  $\mathbf{N}_{\alpha+\beta} \cong J$ . Let  $\omega$  be an element in  $\mathcal{O}(\mathbb{A}_S)$ , and write  $\omega = h^{-1}n(x_0)h$  for some  $h$  in  $PGL_3(\mathbb{A}_S)$  and  $n$  in  $N_\alpha(\mathbb{A}_S)$ . Then Lemma 17 implies that

$$g(\omega) = f_{U,\psi_U}(h)\psi(n)\varphi_X(nh)$$

is a well-defined continuous function on  $\mathcal{O}(\mathbb{A}_S)$ . It follows that

$$\Theta(\Phi * \varphi, f)_{N,\psi_N}(1) = \int_{\mathcal{O}(\mathbb{A}_S)} g(\omega)\hat{\Phi}_S(\omega) d\omega.$$

By our assumption,  $g(x_0) \neq 0$ . Because  $\mathcal{O}(\mathbb{A}_S)$  is a locally closed subset of  $\mathbf{N}_{\alpha+\beta}(\mathbb{A}_S)$ , we can find a Schwarz function  $\Phi_S$  such that its Fourier transform is non-negative, and supported in a sufficiently small neighborhood of  $x_0$  to assure the non-vanishing of the integral. The non-vanishing of the global theta lifting is now proved.

**Remarks:** In the proof of [GJ, Thm. 3.3], the above argument was used directly over  $\mathbb{A}$  (rather than  $\mathbb{A}_S$ ). Translating to our setting, it was claimed there that “(the restriction of)  $\hat{\Phi}$  can be an arbitrary Schwarz function on  $\mathcal{O}(\mathbb{A})$ ”. This claim would be true if  $\mathcal{O}(\mathbb{A})$  is actually a closed subset of  $\mathbf{N}_{\alpha+\beta}(\mathbb{A})$ . It is not, and as a result, the restriction is never a Schwarz function. The situation here is analogous to restricting a Schwarz function on  $\mathbb{A}$  to the subset of ideles. This same error has been perpetuated in many papers.

## 8. Proof of the Main Theorem

We can now give the proof of the main theorem. For (i), we have shown that  $\Theta_{gen}(\pi)$  is non-zero and is contained in the space of cusp forms. Hence, it remains to show that  $\Theta_{gen}(\pi) \cong \sigma(\pi)$ . If  $\sigma = \otimes_v \sigma_v$  is any irreducible summand of  $\Theta_{gen}(\pi)$ , then  $\pi_v^\vee \otimes \sigma_v$  is a quotient of  $\Pi_v$  and thus  $\sigma_v = \sigma(\pi_v)$  by the local results of [GS] and Theorem 2. Thus any irreducible summand of  $\Theta_{gen}(\pi)$  is isomorphic to  $\sigma(\pi)$ . By the local uniqueness of Whittaker

functionals, any irreducible summand of  $\mathcal{A}_{\text{cusp,gen}}$  occurs with multiplicity one and thus we deduce that  $\Theta_{\text{gen}}(\pi)$  is irreducible and isomorphic to  $\sigma(\pi)$ . In particular, (i) is proved.

To prove (ii), we shall use the following result of Ginzburg-Jiang [GJ].

**Theorem 18.** *Let  $\sigma = \otimes_v \sigma_v$  be an irreducible submodule of  $\mathcal{A}_{\text{cusp,gen}}$ . Suppose that the partial (degree  $\gamma$ ) standard  $L$ -function of  $\sigma$  has a pole at  $s = 1$ . Then there exists a cuspidal representation  $\pi$  of  $PGL_3$  such that  $\sigma \subset \Theta_{\text{gen}}(\pi)$ .*

Now suppose that  $\sigma'$  is nearly equivalent to  $\sigma(\pi)$ . Then the partial standard  $L$ -function of  $\sigma'$  has a pole at  $s = 1$  and by the theorem of Ginzburg-Jiang, one concludes that  $\sigma' \subset \Theta_{\text{gen}}(\pi')$  for some cuspidal representation  $\pi'$  of  $PGL_3$ . It follows that  $\sigma(\pi_v)$  is isomorphic to  $\sigma(\pi'_v)$  for almost all  $v$ , and we deduce by [GS] that outside a finite set of places,

$$\pi'_v \cong \pi_v \text{ or } \pi_v^\vee.$$

Now we note:

**Lemma 19.** *If  $\pi$  and  $\pi'$  are two cuspidal representation of  $GL(n)$  such that  $\pi'_v \cong \pi_v$  or  $\pi_v^\vee$  for almost all  $v$ , then  $\pi' \cong \pi$  or  $\pi^\vee$ .*

*Proof.* Indeed, consider the induced representations  $\pi \boxplus \pi^\vee$  and  $\pi' \boxplus \pi'^\vee$  of  $GL(2n)$ . Any constituents of these are automorphic and nearly equivalent to each other. Hence, the lemma follows by the generalized strong multiplicity one theorem of Jacquet-Shalika [JS].  $\square$

With the lemma, we now know that  $\pi' \cong \pi$  or  $\pi^\vee$ , so that  $\sigma' = \Theta_{\text{gen}}(\pi') = \sigma(\pi')$ . Since  $\sigma(\pi) \cong \sigma(\pi^\vee)$ , we see that  $\sigma(\pi) = \sigma(\pi') = \sigma'$ , as desired. The main theorem is proved.

## 9. Generic unitarizable representations

In this section, we list the generic unitarizable representations of  $PGL_3(\mathbb{R})$  and  $G_2(\mathbb{R})$  which were quoted in Theorem 2.

**9.1. Generic unitary representations of  $PGL_3(\mathbb{R})$ .** It will be convenient to think of representations of  $PGL_3(\mathbb{R})$  as representations of  $GL_3(\mathbb{R})$  with trivial central character. In particular, every representation  $\pi$  is a submodule of principal series  $\text{Ind}_B(\chi_1, \chi_2, \chi_3)$  for some three characters of  $\mathbb{R}^\times$  such that  $\chi_1 \cdot \chi_2 \cdot \chi_3 = 1$ . Since  $\pi$  is generic, it is completely determined by the character  $\chi = \chi_1 \times \chi_2 \times \chi_3$ .

Note that every character  $\chi_i$  can be written as  $\chi_i(x) = \epsilon(x)^{\delta_i} |x|^{t_i}$  where  $\epsilon$  is the sign character,  $\delta_i$  is an integer considered modulo 2, and  $t_i$  is a complex number. As usual, write the maximal torus as a product  $T(\mathbb{R}) = MA$ , where  $M \cong \langle \pm 1 \rangle \times \langle \pm 1 \rangle$  is the maximal compact subgroup of  $T(\mathbb{R})$ . The restriction of  $\chi$  to  $A$  determines the infinitesimal character of  $\pi$ , which can be identified with

$$\lambda = (t_1, t_2, t_3) \in \Lambda \otimes \mathbb{C},$$

where  $\Lambda$  is a root lattice of  $SL_3(\mathbb{C})$ . On the other hand, let  $\xi$  be the restriction of  $\chi$  to  $M$ , the maximal compact subgroup of  $T(\mathbb{R})$ . Then  $\xi$  is given by a triple of integers

$$\xi = (\delta_1, \delta_2, \delta_3),$$

such that  $\delta_1 + \delta_2 + \delta_3 \equiv 0 \pmod{2}$ . Note that the characters of  $M$  correspond to elements of order 2 in the torus  $\hat{T}(\mathbb{C}) = \Lambda \otimes \mathbb{C}^\times$  of the dual group  $SL_3(\mathbb{C})$ .

The following is a list of all generic unitary representations. (Parts (2) and (3) list representations up to contragredient.)

(1) *Tempered principal series*  $\pi(\lambda, \xi)$ . The infinitesimal character  $\lambda = (it_1, it_2, it_3)$  is purely imaginary. Assume that  $t_1 \geq t_2 \geq t_3$ . Then  $\pi(\lambda, \xi)$  is the fully induced representation

$$Ind_B(\epsilon^{\delta_1} |\cdot|^{it_1}, \epsilon^{\delta_2} |\cdot|^{it_2}, \epsilon^{\delta_3} |\cdot|^{it_3}).$$

Note that  $\pi(\lambda, \xi)$  is spherical if and only if  $\xi$  is trivial. Furthermore,  $\pi(\lambda, \xi) \cong \pi(\lambda, \xi^w)$  for every  $w$  in  $W(\lambda)$  (the stabilizer of  $\lambda$  in the Weyl group). In particular, there are up to 4 generic representations with the same infinitesimal character  $\lambda$ .

(2) *Generalized principal series*  $\pi(k, t)$ . Here  $k = 1, 2, 3, \dots$  is a positive integer, and  $t$  in  $\mathbb{R}$ . The infinitesimal character is

$$\lambda = (k/2 + it, -k/2 + it, -2it).$$

The representation  $\pi(k, t)$  is a submodule of

$$Ind_B(\epsilon^{k+1} |\cdot|^{k/2+it}, |\cdot|^{-k/2+it}, \epsilon^{k+1} |\cdot|^{-2it}).$$

As a generalized principal series,  $\pi(k, t)$  is obtained by inducing  $\pi(k) \otimes |det|^{it}$  from a maximal parabolic subgroup, where  $\pi(k)$  a discrete series of  $GL(2)$  with minimal  $K$ -type  $k + 1$ . For a given pair  $(k, t)$ ,  $\pi(k, t)$  is the unique generic unitary representation with infinitesimal character  $\lambda$ .

(3) *Complementary series*  $\pi_\delta(s, t)$ . Here  $s$  is in  $(0, 1/2)$ ,  $t$  is in  $\mathbb{R}$ , and  $\delta = 0$  or  $1$ . The infinitesimal character is

$$\lambda = (s + it, -s + it, -2it).$$

The representation  $\pi_\delta(s, t)$  is the fully induced representation

$$Ind_B(\epsilon^\delta |\cdot|^{s+it}, \epsilon^\delta |\cdot|^{-s+it}, |\cdot|^{-2it}).$$

This representation is spherical if and only if  $\delta = 0$ . For a given pair  $(s, t)$ , there are two generic unitary representations with infinitesimal character  $\lambda$ .

**9.2. Generic unitary representations of  $G_2(\mathbb{R})$ .** We restrict ourselves only to representations with the infinitesimal character  $\lambda' = \gamma(\lambda)$  where  $\lambda$  is infinitesimal character of a unitary generic representation of  $PGL_3(\mathbb{R})$ .

(1) *Tempered principal series*  $\sigma(\lambda, \xi)$ . Here  $\xi$  can be identified with an element of order two in  $\hat{T}(\mathbb{C})$ . Again,  $\sigma(\lambda, \xi)$  is spherical if and only if  $\xi$  is trivial, and there are up to 4 generic representations with the same infinitesimal character  $\lambda$ .

(2) *Generalized principal series*  $\sigma(k, t)$ . Here  $k = 1, 2, 3, \dots$  is a positive integer, and  $t$  in  $\mathbb{R}$ . The infinitesimal character is

$$\lambda' = (k/2 + it, -k/2 + it, -2it).$$

The representation  $\sigma(k, t)$  is given in parts b), c) and d) of Theorem 10.9 in [V]. It is a unitary generalized principal series, obtained by inducing  $\pi(k) \otimes |\det|^{it}$  from a short-root maximal parabolic subgroup (i.e. the Heisenberg parabolic). Here  $\pi(k)$  a discrete series of  $GL(2)$  with minimal  $K$ -type  $k + 1$ . The induced representation is irreducible unless  $t = 0$  and  $k$  is even, in which case  $\sigma(k, 0)$  is taken to be the unique generic summand. For a given pair  $(k, t)$ ,  $\sigma(k, t)$  is the unique generic unitary representation with infinitesimal character  $\lambda$ .

(3) *Complementary series*  $\sigma_\delta(s, t)$ . Here  $s$  is in  $(0, 1/2)$ ,  $t$  is in  $\mathbb{R}$ , and  $\delta$  is 0 or 1. The infinitesimal character is

$$\lambda' = (s + it, -s + it, -2it).$$

Then  $\sigma_0(s, t)$  is the spherical representation in part b) of Theorem 12.1 in [V] (the short root case), and  $\sigma_1(s, t)$  is the non-spherical representation in Theorem 13.2 part b) in [V]. If  $t \neq 0$ , these are the unique generic unitary representations with infinitesimal character  $\lambda$ . If  $t = 0$ , there is one additional non-spherical unitary representation with the same infinitesimal character; see part d) of Theorem 13.2 in [V].

## 10. ACKNOWLEDGMENTS

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544, U.S.A.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84112, U.S.A.

*E-mail address:* `wtgan@math.princeton.edu`

*E-mail address:* `savin@math.utah.edu`