

A NOTE ON KOTTWITZ'S INVARIANT $e(G)$

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ABSTRACT. Kottwitz has defined an invariant $e(G)$ for any reductive group G . In this note, we give an interpretation of $e(G)$ in terms of the Killing forms of G and its quasi-split inner form.

1. THE INVARIANTS $d(G)$ AND $e(G)$

Let G be a reductive group over a field k with $\text{char}(k) \neq 2$, and let G_s be its split form. Hence, G defines an element a_G in $H^1(k, \text{Aut}_{k^s}(G_s))$. Let G_s^{ad} be the quotient of G_s by its center. Also, let Φ be the based root datum of G_s . Then we have the following exact sequence of Galois modules:

$$1 \rightarrow G_s^{ad} \rightarrow \text{Aut}_{k^s}(G_s) \rightarrow \text{Aut}(\Phi) \rightarrow 1$$

We also have a determinant map $\det : \text{Aut}(\Phi) \rightarrow \langle \pm 1 \rangle$. From the composite:

$$d : H^1(k, \text{Aut}_{k^s}(G_s)) \rightarrow H^1(k, \text{Aut}_{k^s}(\Phi)) \rightarrow H^1(k, \langle \pm 1 \rangle)$$

we get an elementary cohomological invariant of G , which is given by:

$$d(G) = d(a_G) \in H^1(k, \langle \pm 1 \rangle) = H^1(k, \mu_2)$$

since $\text{char}(k) \neq 2$. Here, the groups or pointed sets $H^*(k, -)$ refer to those of Galois cohomology, or equivalently those with respect to the étale topology on $\text{Spec}(k)$.

On the other hand, Kottwitz [K] has defined an invariant $e(G) \in H^2(k, \mu_2)$ as follows. Let G' be the quasi-split inner form of G . So G gives a well-defined $\text{Aut}_k(G'_{ad})$ -orbit in $H^1(k, G'_{ad})$. Here again, G'_{ad} is the quotient of G' by its center. From the exact sequence:

$$1 \rightarrow Z \rightarrow G'_{sc} \rightarrow G'_{ad} \rightarrow 1$$

of sheaves in the flat topology on $\text{Spec}(k)$, we get a map

$$\delta : H^1(k, G'_{ad}) \rightarrow H_f^1(k, G'_{ad}) \rightarrow H_f^2(k, Z)$$

where $H_f^*(k, -)$ refers to flat cohomology.

Let (T, B) be a pair consisting of a maximal torus and a Borel subgroup of G'_{sc} . Let ρ be the character of T given by half the sum of the positive roots (positive with respect to B), and let λ be the restriction of ρ to Z . Then Kottwitz showed that λ is independent of the choice of (T, B) , is defined

over k , and is stable under the action of $\text{Aut}_k(G'_{sc})$. Moreover, since ρ^2 is the identity on Z , we have $\lambda : Z \rightarrow \mu_2$. Hence the composite:

$$\lambda \circ \delta : H^1(k, G'_{ad}) \rightarrow H_f^2(k, Z) \rightarrow H_f^2(k, \mu_2) \cong H^2(k, \mu_2)$$

gives an invariant $e(G) = \lambda \circ \delta(c_G)$, where c_G is any element of $H^1(k, G'_{ad})$ in the orbit determined by G . As discussed in [K], the need to use flat cohomology arises because $\text{char}(k)$ may divide the order of Z . Over a local field, the invariant $e(G)$ plays a role in the study of the trace formula and orbital integrals of G .

Now, let $W(k)$ be the Witt ring of non-degenerate quadratic forms over k , and let I be the ideal of even rank forms [S]. Then $I/I^2 \cong H^1(k, \mu_2)$ and $I^2/I^3 \cong H^2(k, \mu_2)$. Hence the invariants $d(G)$ and $e(G)$ can be regarded as elements in I/I^2 and I^2/I^3 respectively. It is thus natural to ask whether they can be interpreted in terms of quadratic forms.

2. KILLING FORM

Now assume that G is semi-simple and that the Killing form B_G of G is non-degenerate on the Lie algebra \mathfrak{g} of G . This latter assumption holds, for example, when $\text{char}(k) = 0$, or $\text{char}(k)$ is greater than the Coxeter numbers of each of the simple factors of G . Let O denote the orthogonal group of B_{G_s} and SO its connected component (of index 2). We have a natural morphism $\text{Aut}_{k_s}(G_s) \rightarrow O$ such that its restriction to G_s^{ad} is simply the adjoint representation $Ad : G_s^{ad} \rightarrow SO$. We thus have the following commutative diagram of Galois modules with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_s^{ad} & \longrightarrow & \text{Aut}_{k_s} G_s & \longrightarrow & \text{Aut} \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & SO & \longrightarrow & O & \longrightarrow & \langle \pm 1 \rangle & \longrightarrow & 1 \end{array}$$

From this, we get the following commutative diagram:

$$\begin{array}{ccc} H^1(k, \text{Aut}_{k_s} G_s) & \longrightarrow & H^1(k, \text{Aut} \Phi) \\ \text{Ad} \downarrow & & \downarrow \text{det} \\ H^1(k, O) & \xrightarrow{\text{det}} & H^1(k, \langle \pm 1 \rangle) \end{array}$$

This implies that $d(G) = \text{det} \circ \text{Ad}(a_G)$. One checks that $\text{Ad}(a_G) \in H^1(k, O)$ is simply the class corresponding to the Killing form B_G of G , and that:

$$\text{det} \circ \text{Ad}(a_G) = \text{det}(B_G) / \text{det}(B_{G_s})$$

Hence $d(G)$ corresponds to the element $(\mathfrak{g} \oplus \mathfrak{g}_s, B_G \oplus B_{G_s}) \in I$.

Remarks: The assumption that G is semi-simple above is for convenience. If G is reductive, say $G = Z \cdot G_0$, with Z central and G_0 the derived group

of G , then we have:

$$d(G)/d(Z) = \det(B_{G_0})/\det(B_{(G_0)_s}).$$

Now, for $e(G)$, we shall use the following commutative diagram with exact rows of sheaves in the flat topology:

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z & \longrightarrow & G'_{sc} & \longrightarrow & G'_{ad} \longrightarrow 1 \\ & & \downarrow \lambda' & & \downarrow & & \downarrow Ad \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin & \longrightarrow & SO \longrightarrow 1 \end{array}$$

Here, SO denotes the special orthogonal group for the Killing form $B_{G'_{ad}}$. This gives:

$$\begin{array}{ccc} H_f^1(k, G'_{ad}) & \xrightarrow{\delta} & H_f^2(k, Z) \\ \downarrow Ad & & \downarrow \lambda' \\ H_f^1(k, SO) & \xrightarrow{\delta} & H_f^2(k, \mu_2) \end{array}$$

We define:

$$e'(G) = \delta \circ Ad(c_G) \in H_f^2(k, \mu_2) \cong H^2(k, \mu_2)$$

where c_G is any class in $H^1(k, G'_{ad})$ representing G , or rather its image in $H_f^1(k, G'_{ad})$. Note that the definition of e' uses only the Spin exact sequence. Hence, since $char(k) \neq 2$ by assumption, e' can be defined purely in terms of Galois cohomology. Moreover, the above commutative diagram shows that $e'(G) = \lambda' \circ \delta(c_G)$.

Of course we have to show that $e'(G)$ is well-defined, i.e. independent of the choice of c_G . To see this, note that if b_G is another class in $H^1(k, G'_{ad})$ representing G , then $Ad(b_G)$ and $Ad(c_G)$ are two classes in $H^1(k, SO)$, which have the same image in $H^1(k, O)$, since they both represent the Killing form B_G . From the commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin & \longrightarrow & SO \longrightarrow 1 \\ & & \downarrow id & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Pin & \longrightarrow & O \longrightarrow 1 \end{array}$$

we have:

$$\begin{array}{ccc} H^1(k, SO) & \longrightarrow & H^2(k, \mu_2) \\ \downarrow & & \downarrow id \\ H^1(k, O) & \longrightarrow & H^2(k, \mu_2) \end{array}$$

Hence the images of $Ad(b_G)$ and $Ad(c_G)$ in $H^2(k, \mu_2)$ are the same, i.e. $e'(G)$ is well-defined. Now for any reductive group G with centre C , we set $e'(G) := e'(G/C)$.

The following properties of $e'(G)$ are easy to check:

Proposition 1. (i) $e'(G) = w(B_G)/w(B_{G'})$, where $w(Q)$ is the Hasse-Witt invariant of the quadratic form Q (see [S] for the definition of w , and [Se] for this equality)

(ii) $e'(G_1 \times G_2) = e'(G_1)e'(G_2)$, using multiplicative notation in $H^2(k, \mu_2)$.

(iii) $e'(G) = e'(G/Z)$, where Z is central in G .

(iv) $e'(G \otimes_k E) = \text{Res}(e'(G))$, where E is a field extension of k , and $\text{Res} : H^2(k, \mu_2) \rightarrow H^2(E, \mu_2)$ is the usual restriction map.

(v) $e'(\text{Res}_{E/k} G_E) = \text{Cor}(e'(G_E))$, where E is a finite extension of k and $\text{Cor} : H^2(E, \mu_2) \rightarrow H^2(k, \mu_2)$ is the usual corestriction map.

We want to show that $e(G) = e'(G)$. It suffices to show that $\lambda = \lambda'$. By above proposition, we only have to do this for G absolutely quasi-simple.

Proposition 2. $\lambda = \lambda'$

Proof. This is just a question about coweight and coroot lattices. Indeed, the adjoint representation $ad : \mathfrak{g}' \rightarrow \mathfrak{so}$ induces a map from the coroot lattice Λ_r of \mathfrak{g}' to the coroot lattice Λ_r of \mathfrak{so} . It also maps the coweight lattice Λ_w of \mathfrak{g}' into a lattice Λ which contains Λ_r as a sublattice of index 2. Λ is simply the lattice corresponding to the group SO . Note that if $\dim(\mathfrak{g}')$ is odd, then Λ is simply the coweight lattice. Otherwise, it is a lattice of index 2 in the coweight lattice. Hence, we get an induced map:

$$Z(k^s) \cong \Lambda_w / \Lambda_r \longrightarrow \Lambda / \Lambda_r \cong \mu_2(k^s)$$

This induced map is simply λ' .

To compute this map of lattices, it suffices to assume that \mathfrak{g}' is defined over \mathbb{C} , so that we can regard the lattices above as sitting in \mathfrak{g}' and \mathfrak{so} . So let $\{H_\alpha, X_\alpha, Y_\alpha\}$ be a Chevalley basis of \mathfrak{g}' . Using the isomorphism $\mathfrak{so} \cong \wedge^2 \mathfrak{g}'$ described in [F-H] Pg. 303, we see that for any $t \in \mathfrak{t}$, the Lie algebra of the torus of G' , we have:

$$ad : t \mapsto \sum_{\alpha \in \Phi^+} \alpha(t) \frac{1}{2} X_\alpha \wedge Y_\alpha \in \mathfrak{t}(\mathfrak{so})$$

Let $t_\alpha = \frac{1}{2} X_\alpha \wedge Y_\alpha$. Then the t_α 's lie in Λ and can be completed to a basis of Λ . Also, the intersection of the \mathbb{Z} -span of the t_α 's with Λ_r is generated by $\{t_\alpha \pm t_\beta : \alpha, \beta \in \Phi^+\}$. Notice that if $t \in \Lambda_w$, then $\alpha(t) \in \mathbb{Z}$ for all α , and so $ad(t) \in \Lambda$. Similarly, if $t \in \Lambda_r$, then $\alpha(t) \in 2\mathbb{Z}$ for all α , and so $ad(t) \in \Lambda_r$.

We can check if an element of Λ_w lands in Λ_r as follows. There is an augmentation map:

$$\begin{aligned} \varepsilon : \quad \mathbb{Z}[t_\alpha] &\longrightarrow \frac{1}{2}\mathbb{Z} \\ c = \sum c_\alpha t_\alpha &\mapsto \frac{1}{2} \sum c_\alpha \end{aligned}$$

Then c lies in Λ_r if and only if $\varepsilon(c) \in \mathbb{Z}$, ie:

$$\varepsilon : \mathbb{Z}[t_\alpha] / \Lambda_r \cap \mathbb{Z}[t_\alpha] \cong \frac{1}{2}\mathbb{Z} / \mathbb{Z}$$

Hence $ad(t)$ lies in Λ_r if and only if $\frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha(t) \in \mathbb{Z}$, ie if and only if $\rho(t) \in \mathbb{Z}$. Hence $\varepsilon \circ \lambda' = \rho|_Z$, as required. \square

Corollary 3. $e(G) = e'(G)$.

Example: Suppose $k = \mathbb{R}$. Then we have the Cartan decomposition of \mathfrak{g} and \mathfrak{g}' :

$$\begin{aligned}\mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{p} \\ \mathfrak{g}' &= \mathfrak{k}' \oplus \mathfrak{p}'\end{aligned}$$

The Killing form is positive definite (respectively negative definite) on \mathfrak{p} and \mathfrak{p}' (respectively \mathfrak{k} and \mathfrak{k}'). Let $n = \dim(\mathfrak{k})$ and $m = \dim(\mathfrak{k}')$. Then we have:

$$\begin{aligned}w(B_G) &= (-1)^{\frac{1}{2}n(n-1)} \\ w(B_{G'}) &= (-1)^{\frac{1}{2}m(m-1)}\end{aligned}$$

Since G and G' are inner twists of each other, n and m must have the same parity. Hence,

$$w(B_G)/w(B_{G'}) = (-1)^{\frac{1}{2}(n-m)} = (-1)^{\frac{1}{2}(\dim G/K - \dim G'/K')}$$

where K (respectively K') is the maximal compact subgroup of G (respectively G').

Remarks: (1) This computation of $e(G)$ for $k = \mathbb{R}$ seems to be simpler than that given in [K].

(2) It would be nice to show using this definition that if k is p -adic, then $e'(G)$ is as computed in [K]. However I am unable to prove this.

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