Non-tempered Arthur Packets of G_2

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to Professor S. Rallis
with admiration and best wishes

§1. Introduction

In his famous conjecture, J. Arthur gave a description of the discrete spectrum $L^2_{disc}(G(F)\backslash G(\mathbb{A}))$ of a reductive linear algebraic group G defined over a number field F. According to this conjecture, the irreducible constituents of L^2_{disc} can be partitioned into equivalence classes called A-packets. In this survey article, we describe some recent construction and characterization of the (non-tempered) A-packets of the split exceptional group of type G_2. These results are contained in the papers [GGJ], [G] and [GG]. For the classical groups of small rank, the construction of these packets has been carried out by the work of various people. These include: the work of J. Rogawski [R] for $U(3)$, the work of I. Piatetski-Shapiro [PS] and D. Soudry [So] for $PGSp_4$, and the recent work of K. Konno and T. Konno [KK] for $U(2,2)$. As we shall see in the article, the case of G_2 offers a number of (pleasant) surprises which do not appear for classical groups.

We begin by giving a brief description of Arthur's conjecture for the discrete spectrum, assuming for simplicity that G is split and simply-connected, so that the dual group has no center. The references are of course [A1] and [A2].

(1.1) Decomposition of discrete spectrum. Loosely speaking, Arthur's conjecture is a classification of the constituents of L^2_{disc} into near equivalence classes (this is not entirely true, but for the group G_2, it is expected to be so). Here, we say that two representations π_1 and π_2 of $G(\mathbb{A})$ are nearly equivalent if for almost all places v, $\pi_{1,v}$ and $\pi_{2,v}$ are isomorphic: this is an equivalence relation.

More precisely, Arthur speculated that the discrete spectrum possesses a decomposition

$$L^2_{disc}(G(F)\backslash G(\mathbb{A})) = \widehat{\bigoplus}_{\psi} L^2_{\psi},$$

where the Hilbert space direct sum runs over equivalence classes of A-parameters ψ, i.e. admissible maps

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

where L_F denotes the conjectural Langlands group of F and $\hat{G}$ is the complex dual group of G. For any ψ, the space L^2_{ψ} is a direct sum of nearly equivalent representations and can be more precisely described as follows.

(1.2) Local A-packets. The global A-parameter ψ gives rise to a local A-parameter

$$\psi_\nu : L_{F_\nu} \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

for each place ν of F. Denote by S_{ψ_v} the group of components of $Cent_{\hat{G}}(Im(\psi_\nu))$. To each irreducible representation η_v of S_{ψ_v}, Arthur speculated that one can attach a unitarizable admissible (possibly reducible, possibly zero) representation π_{η_v} of $G(F_v)$. The set

$$A_{\psi_\nu} = \left\{ \pi_{\eta_\nu} \, : \, \eta_\nu \in \widehat{S}_{\psi_\nu} \right\}$$

is called a local A-packet. It is required that

- for almost all v, π_{η_v} is irreducible and unramified if η_v is the trivial character 1_v. In fact, for almost all v, π_{η_v} is the unramified representation whose Satake parameter is:

$$s_{\psi_v} = \psi_v \left(Fr_v \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right),$$

  where Fr_v is a Frobenius element at v and q_v is the number of elements of the residue field at v;

- the distribution

$$\sum_{\eta_v} \epsilon_{\eta_v} dim(\eta_v) Tr(\pi_{\eta_v}), \quad \text{for explicitly determined } \epsilon_v \in \{\pm 1\},$$

  is stable, and certain identities involving transfer of character distributions to endoscopic groups of $G(F_v)$ should hold.

These requirements may not characterize the set A_{ψ_v} but they come pretty close; for more details, see [A2].

(1.3) Global A-packets. With the local packets A_{ψ_v} at hand, we may define the global A-packet by:

$$A_\psi = \{\pi = \otimes_v \pi_{\eta_v} : \pi_{\eta_v} \in A_{\psi_v} \; \eta_v = 1_v \text{ for almost all } v\}.$$

It is a set of nearly equivalent representations of $G(\mathbb{A})$, indexed by the irreducible representations of the compact group $\mathcal{S}_{\psi,\mathbb{A}} := \prod_v S_{\psi,v}$. If $\eta = \bigotimes_v \eta_v$ is an irreducible character of $\mathcal{S}_{\psi,\mathbb{A}}$, then we may set

$$\pi_\eta = \bigotimes_v \pi_{\eta_v}.$$

This is possible because for almost all v, $\eta_v = \mathbf{1_v}$ and $\pi_{\mathbf{1_v}}$ is required to be unramified by the above.

(1.4) Multiplicity formula. The space L^2_ψ will be the sum of the elements of A_ψ with some multiplicities. More precisely, Arthur attached to ψ a quadratic character ϵ_ψ of S_ψ. Now if η is an irreducible character of $S_{\psi,\mathbb{A}}$, we set

$$m_\eta = \frac{1}{\# S_\psi} \cdot \left(\sum_{s \in S_\psi} \epsilon_\psi(s) \cdot \eta(s) \right).$$

Then Arthur conjectures that there is a $G(\mathbb{A})$-equivariant embedding

$$\iota_\psi : \bigoplus_\eta m_\eta \pi_\eta \hookrightarrow L^2_{disc}(G(F)\backslash G(\mathbb{A})).$$

The image of this embedding is the subspace L^2_ψ.

(1.5) Tempered and non-tempered parameters. An A-parameter ψ is called *tempered* if ψ is trivial when restricted to $SL_2(\mathbb{C})$. In this case, the representations in A_ψ are conjectured to be tempered. A non-tempered A-paramter tends to factor through a subgroup $\hat{H} \subset \hat{G}$ with H an endoscopic group of G. Of course, this is the case for some tempered parameters too. In any case, the representations corresponding to such a parameter are considered more degenerate. One might thus hope to construct the packets A_{ψ_ν} and the embedding $L^2_\psi \hookrightarrow L^2_{disc}$ by lifting from a group smaller than G.

(1.6) CAP representations. Some of the earlier constructions of A-packets for classical groups of low rank were couched in the language of **CAP** representations, rather than in the framework of Arthur's conjectures. Recall that a cuspidal representation π of G is said to be **CAP** *with respect to a parabolic subgroup* P of G if it is nearly equivalent to the irreducible constituents of an induced representation $Ind_P^G \tau$ with τ a cuspidal representation of the Levi factor of P. Arthur's conjecture implies that the CAP representations are precisely the cuspidal representations belonging to the non-tempered A-packets of G.

(1.7) A list of problems. Given an A-parameter ψ, let us formulate a list of problems which describe more precisely what we would like to achieve.

PROBLEM A: Give a natural construction of the local A-packets A_{ψ_v}.

How can we justify that our naturally constructed set A_{ψ_v} is the correct A-packet? Ideally, we would like to be able to verify that the set A_{ψ_v} that was constructed is stable. Unfortunately, we have nothing to offer on this. However, one can give strong global justification for the authenticity of A_{ψ_v} by addressing:

PROBLEM B: With the local packets defined, construct the subspace $L^2_\psi \subset L^2_{disc}$ as prescribed by Arthur's conjecture.

After resolving Problem B, one knows that if η is an irreducible character of $S_{\psi,\mathbb{A}}$, then the multiplicity $m_{disc}(\pi_\eta)$ of π_η in L^2_{disc} satisfies

$$m_{disc}(\pi_\eta) \geq m_\eta,$$

at least when π_η is irreducible. Though this is nice, it does not rule out the occurence of π_η when $m_\eta = 0$. Thus, we would like to resolve:

PROBLEM C: Find the true discrete multiplicity of π_η. For example, decide if $m_{disc}(\pi_\eta) = m_\eta$.

Note that it is not always the case that $m_{disc}(\pi_\eta) = m_\eta$. For example, for the group SL_n ($n \geq 3$), Blasius has shown in [B] that there are cuspidal representations which occur as members of global A-packets corresponding to inequivalent A-parameters. Another example of this phenomenon is the group SO_{2n}, where one has inequivalent A-parameters which are conjugate by the outer automorphism group.

Finally, suppose that we have resolved the three problems above; is there still any doubt about the authenticity of our local packets A_{ψ_v}? Certainly. The reason is that the representations π_η are not required to be irreducible. It may happen that the true π_η is the sum of 2 irreducible representations, but the π_η we defined is irreducible; in other words, we have missed out one of the two constituents of the true π_η. In that case, we will not be able to detect the missing representation by resolving Problems A-C. This leads us to:

PROBLEM D: Decide if the subspace $L^2_{\psi_\tau}$ constructed in Problem B is a full near equivalence class. More precisely, if $\pi \subset L^2_{disc}$ is nearly equivalent to the representations in A_ψ (i.e. $\pi_v \cong \pi_{1_v}$ for almost all v), is $\pi \subset L^2_\psi$?

Again, we do not expect the statement in Problem D to hold all the time, in view of the examples of Blasius [B]. However, a positive solution to Problem D will show beyond any reasonable doubt that our local packets are the right ones. Indeed, if σ is a representation of $G(F_v)$ which should be in A_{ψ_v}, then σ should occur as the local component of a global representation $\pi \subset L^2_{disc}$ which is nearly equivalent to the representations in A_ψ. A positive solution to Problem D will show that we have already captured all such candidates σ.

In this paper, which is purely expository, we discuss our recent results on the above four problems for the non-tempered A-parameters of the split exceptional group G_2.

§2. The group G_2

We begin by introducing some basic notations and facts about the structure of G_2 and its representations. More importantly, we describe the 4 families of non-tempered A-parameters of G_2.

(2.1) The group. The group G_2 can be realized as the automorphism group of the (split) octonion algebra $\mathbb{O}$. Like the quaternion algebra, the octonion algebra carries a quadratic norm form N and a linear trace form Tr and these are preserved by G_2. Let V be the space of trace zero elements in $\mathbb{O}$, equipped with the quadratic form $q = -N$. Then G_2 acts as automorphisms of (V, q), giving an embedding $G_2 \hookrightarrow SO(V, q) \cong SO_7$.

(2.2) Maximal parabolics. The group G_2 has two conjugacy classes of maximal parabolic subgroups. One of them is the Heisenberg parabolic $P = M \cdot N$, with N a 5-dimensional Heisenberg group. Denote the other maximal parabolic by $Q = L \cdot U$. Its unipotent radical U is a 3-step nilpotent group. In both cases, the Levi subgroups are isomorphic to GL_2 and we fix these isomorphisms.

(2.3) The subgroups SU_3^K. Let K be an étale quadratic algebra, which determines a quasi-split special unitary group SU_3^K. There is a natural embedding $SU_3^K \hookrightarrow G_2$, which is unique up to conjugacy. To give one such embedding, we take a trace zero element x of $\mathbb{O}$ such that the subalgebra $F[x]$ generated by x is isomorphic to K. The stabilizer of x in G_2 is then isomorphic to SU_3^K.

(2.4) Representations. Now we introduce some notations for the representations of $G_2(F_v)$. Let τ be a tempered representation of $GL_2(F_v)$ and $s > 0$. In the following, we shall use standard notions for the representations of PGL_2. For example, St denotes the Steinberg representation and $\pi(\mu_1, \mu_2)$ denotes a principal series representation. The induced representations

$$I_P(\tau, s) = Ind_P^{G_2} \delta_P^{1/2} \cdot \tau \cdot |det|^s \quad \text{and} \quad I_Q(\tau, s) = Ind_Q^{G_2} \delta_Q^{1/2} \cdot \tau \cdot |det|^s$$

have unique irreducible quotient $J_P(\tau, s)$ and $J_Q(\tau, s)$.

(2.5) Fourier coefficients. Now we come to the global setting. Let $\mathcal{A}(G_2)$ be the space of automorphic forms on G_2 and let $\mathcal{A}_2 = \mathcal{A} \cap L^2$ be the space of square-integrable automorphic forms.

For $f \in \mathcal{A}(G_2)$, we will consider its generic Fourier coefficients along the unipotent radical N of P. More precisely, if χ is a unitary character of $N(\mathbb{A})$ trivial on $N(F)$, then the χ-th Fourier coefficient of f is the function on $G_2(\mathbb{A})$ given by:

$$f_\chi(g) = \int_{N(F)\backslash N(\mathbb{A})} \overline{\chi(n)} \cdot f(ng)\, dn.$$

We let l_χ be the linear map on $\mathcal{A}(G_2)$ given by $l_\chi(f) = f_\chi(1)$.

Now the Levi factor $M(F)$ acts naturally on the set of χ's which are trivial on $N(F)$. It is known that the generic $M(F)$-orbits are naturally parametrized by étale cubic F-algebras (cf. [GGJ]). If E is an étale cubic F-algebra, we let ψ_E be a character which is in the orbit indexed by E. If $\pi \subset \mathcal{A}(G_2)$ is irreducible, we say that π has non-zero *E-Fourier coefficient* if l_{ψ_E} is non-zero on π. It is known [G, Thm. 3.1] that if π is not the trivial representation, then π has a non-zero E-coefficient for some étale E.

We say that π is *E-distinguished* if the only non-zero (generic) Fourier coefficient of π along N is the one corresponding to E.

(2.6) The stabilizer M_{ψ_E}. The stabilizer in M of ψ_E is a finite group scheme S_E which is a twisted form of the finite constant group scheme S_3. More precisely, if B is any commutative F-algebra,

$$S_E(B) = Aut_B(E \otimes_F B).$$

Another way to define S_E is given in (3.2).

If π_v is a representation of $G_2(F_v)$, then $M_{\psi_E}(F_v)$ acts naturally on $Hom_{N(F_v)}(\pi_v, \mathbb{C}_{\psi_E})$. Similarly, if π is a representation of $G_2(\mathbb{A})$, then $M_{\psi_E}(\mathbb{A})$ acts naturally on $Hom_{N(\mathbb{A})}(\pi, \mathbb{C}_{\psi_E})$.

(2.7) The A-parameters of G_2. We conclude this section by describing the A-parameters of G_2. Recall that the complex dual group of G is $G_2(\mathbb{C})$, and there are 4 conjugacy classes of non-trivial homomorphisms $SL_2(\mathbb{C}) \longrightarrow G_2(\mathbb{C})$ corresponding to the 4 non-trivial unipotent conjugacy classes in $G_2(\mathbb{C})$:

$$\mathcal{O}_{long} < \mathcal{O}_{short} < \mathcal{O}_{subreg} < \mathcal{O}_{reg}.$$

There are thus 4 families of non-tempered A-parameters for G_2. We list them below:

- $\psi|_{SL_2}$ corresponds to the regular orbit. The corresponding local and global A-packets are singletons consisting of the trivial representation of $G(F_\nu)$ and $G(\mathbb{A})$ respectively.

- $\psi|_{SL_2}$ corresponds to the subregular orbit. The corresponding packets were studied in [GGJ] and [G]. The results are described in §3.
- $\psi|_{SL_2}$ gives the short root SL_2 in G_2. The corresponding packets were constructed in [GG], and the results are described in §5.
- $\psi|_{SL_2}$ gives the long root SL_2 in G_2. This is an ongoing project, which we hope to complete in the near future.

§3. Cubic Unipotent A-packets

In this section, we describe the construction of the A-packets A_ψ where $\psi|_{SL_2}$ is the subregular SL_2. The subregular map $SL_2(\mathbb{C}) \to G_2(\mathbb{C})$ is not injective; it factors through $SO_3(\mathbb{C})$. The centralizer of this $SO_3(\mathbb{C})$ in $G_2(\mathbb{C})$ is isomorphic to the symmetric group S_3 on 3 letters, so that $SO_3(\mathbb{C}) \times S_3$ is a subgroup of $G_2(\mathbb{C})$. With this information, we can now write down all subregular parameters ψ.

(3.1) Cubic unipotent parameters. Let E be an étale cubic F-algebra. Then E corresponds to a conjugacy class of maps

$$\rho_E : L_F \longrightarrow Gal(\overline{F}/F) \longrightarrow S_3.$$

Using ρ_E and the natural projection map from $SL_2(\mathbb{C})$ to $SO_3(\mathbb{C})$, we have an A-parameter

$$\psi_E : L_F \times SL_2(\mathbb{C}) \longrightarrow S_3 \times SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

These maps ψ_E are the cubic unipotent A-parameters.

(3.2) The centralizer of ψ_E. Via ρ_E, the Galois group $Gal(\overline{F}/F)$ acts on S_3 by conjugation. This defines a finite group scheme S_E, which is a twisted form of the constant group scheme S_3. The group S_E was defined earlier in (2.6). It is not difficult to see that

$$S_{\psi_E} = S_E(F) \qquad \text{and} \qquad S_{\psi_{E,v}} = S_E(F_v).$$

Thus, $S_{\psi_E,\mathbb{A}} = S_E(\mathbb{A})$.

(3.3) The split case. As an example, consider the case when $E = E_0$ is the split algebra $F \times F \times F$. In this case, ρ_{E_0} is the trivial map, and so we have:

$$\begin{cases} \mathcal{S}_{\psi_{E_0}} = \mathcal{S}_{\psi_{E_0},v} = S_3 \\ \mathcal{S}_{\psi_{E_0},\mathbb{A}} = S_3(\mathbb{A}). \end{cases}$$

The map $\mathcal{S}_\psi \to \mathcal{S}_{\psi,\mathbb{A}}$ is simply the natural embedding $S_3(F) \hookrightarrow S_3(\mathbb{A})$. Thus, for each place v, the corresponding local A packet has 3 members indexed by the irreducible characters of S_3:

$$A_{\psi_{E_0}} = \{\pi_{\mathbf{1_v}}, \pi_{r_v}, \pi_{\epsilon_v}\}$$

where ϵ_v is the sign character of S_3 and r_v is the 2-dimensional one.

(3.4) Construction of local packets. We now explain the results of [GGJ], which give a construction of the cubic unipotent A-packets.

The étale cubic algebra E determines a simply-connected quasi-split group $Spin_8^E$ of type D_4, whose outer automorphism group is isomorphic to S_E. Let H_E be the disconnected linear algebraic group $Spin_8^E \rtimes S_E$. For each place v of F, the group $H_E(F_v)$ has a distinguished representation Π_{E_v} known as the minimal representation. For example, when $E = E_0$, $\Pi_{E_{0,v}}$ is a particular extension to $H_{E_0}(F_v)$ of the unramified representation of $Spin_8(F_v)$ whose Satake parameter is

$$\iota\begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix}$$

where $\iota : SL_2(\mathbb{C}) \longrightarrow PGSO_8(\mathbb{C})$ is the map associated to the subregular unipotent orbit of the dual group $PGSO_8(\mathbb{C})$.

Now the action of S_E on $Spin_8^E$ has pointwise stabilizer isomorphic to G_2. Thus H contains the subgroup $S_E \times G_2$, and one may restrict the representation Π_{E_v} to the subgroup $S_E(F_v) \times G_2(F_v)$ to get:

$$\Pi_{E_v} = \bigoplus_{\eta_v \in \widehat{S_E(F_v)}} \eta_v \otimes \pi_{\eta_v}.$$

In the papers [HMS] and [V], Huang-Magaard-Savin (for non-archimedean v) and Vogan (for archimedean v) showed the following theorem:

(3.5) Theorem *Each π_{η_v} is a non-zero irreducible unitarizable representation and the π_{η_v}'s are mutually distinct. Moreover, the representations π_{η_v} can be completely determined, and $\pi_{\mathbf{1_v}}$ is unramified with Satake parameter $s_{\psi_{E,v}}$.*

For example, when $E = E_0$ is the split algebra, then

$$\pi_{1_v} = J_Q(\pi(1,1),1), \qquad \pi_{r_v} = J_P(St, 1/2)$$

and π_{ϵ_v} is supercuspidal of depth zero if v is finite.

In view of these results, it is natural to take $A_{\psi_{E,v}}$ to be the set of representations π_{η_v} furnished by the theorem. This is our solution to Problem A.

(3.6) Construction of the subspace $L^2_{\psi_E}$. We come now to Problem B. Consider thus the global situation. The quadratic character ϵ_{ψ_E} is equal to the trivial character in this case. Thus, if our definition of the local packets is correct, we expect to construct a subspace of L^2_{disc} isomorphic to $\bigoplus_\eta \eta^{S_E(F)} \otimes \pi_\eta$.

If $\Pi_E = \otimes_v \Pi_{E_v}$, then as an abstract representation of $S_3(\mathbb{A}) \times G_2(\mathbb{A})$, we have:

$$\Pi_E = \bigoplus_\eta \eta \otimes \pi_\eta$$

as $\eta = \otimes_v \eta_v$ ranges over the irreducible representations of $S_E(\mathbb{A})$.

Using residues of Eisenstein series, one can construct a $Spin_8^E(\mathbb{A})$-equivariant embedding

$$\Theta : \Pi_E \hookrightarrow \mathcal{A}_2(Spin_8^E)$$

of Π_E into the space of square-integrable automorphic forms of $Spin_8^E$. We may now define a $G_2(\mathbb{A})$-equivariant map ι_E as follows:

$$\iota_E : \Pi_E = \oplus_\eta \eta \otimes \pi_\eta \xrightarrow{\Theta} \mathcal{A}_2(Spin_8^E) \xrightarrow{\text{restriction}} \{\text{functions on } G_2(F)\backslash G_2(\mathbb{A})\}.$$

Let us write $\eta = \eta^{S_E(F)} \oplus (\eta^{S_E(F)})^\perp$. Then it is easy to see that ι_E is the zero map when restricted to the subspace $\bigoplus_\eta (\eta^{S_E(F)})^\perp \otimes \pi_\eta$. There is thus no loss in restricting ι_E to the subspace $\bigoplus_\eta \eta^{S_E(F)} \otimes \pi_\eta$. Then the following is the main theorem proved in [GGJ]:

(3.7) Theorem *(i) The image of ι_E is contained in $\mathcal{A}_2(G_2)$.*

(ii) The restriction of ι_E to the subspace $\oplus_\eta \eta^{S_E(F)} \otimes \pi_\eta$ is injective.

Thus, the theorem provides a construction of the subspace $L^2_{\psi_E}$, which resolves Problem B and implies:

(3.8) Corollary *We have:*

$$m_{disc}(\pi_\eta) \geq m_\eta.$$

(3.9) Properties. We remark that the representations in $L^2_{\psi_E}$ are E-distinguished, and the cuspidal ones are CAP with respect to the Borel subgroup (resp. the maximal parabolic Q) if E is Galois (resp. non-Galois). Here, we say that E is non-Galois if it is a non-Galois cubic field extension of F, and is Galois otherwise. In the final section of the paper, we discuss whether these properties characterize the A-packets A_{ψ_E}.

(3.10) Some striking consequences. We briefly highlight two striking consequences of the theorem. Suppose for simplicity that $E = E_0$ is the split algebra, so that $S_{\psi_E} = S_{\psi_{E,v}} = S_3$. For S a non-empty finite set of places of F, let

$$\eta_S = (\otimes_{v\in S} r_v) \bigotimes (\otimes_{v\notin S} \mathbf{1_v}).$$

Then the representation

$$\pi_{\eta_S} = (\otimes_{v\in S} \pi_{r_v}) \bigotimes (\otimes_{v\notin S} \pi_{\mathbf{1_v}})$$

occurs in $L^2_{\psi_{E_0}}$ with multiplicity equal to the multiplicity of the trivial representation in $r \otimes r \otimes .... \otimes r$ ($\#S$ times). A quick computation gives:

$$m_{disc}(\pi_{\eta_S}) \geq \frac{1}{6} \cdot (2^{\#S} + (-1)^{\#S} 2),$$

On the other hand, the residual spectrum of G_2 has been determined by H. Kim [K] and S. Zampera [Z]. Their results show that L^2_{res} has the multiplicity one property, and further that $m_{res}(\pi_{\eta_S}) = 1$ if $\#S \neq 1$. Thus, we have:

(3.11) Theorem *(i) If $\#S \geq 3$, the representation π_{η_S} occurs in both the residual spectrum and the cuspidal spectrum.*

(ii) As $\#S \to \infty$, the cuspidal and discrete multiplicities of the π_{η_S}'s are unbounded.

(3.12) Exact multiplicity formula. Theorem 3.7 and its Corollary 3.8 already provide some justification for our definition of local packets. However, when $m_\eta = 0$, the inequality in Corollary 3.8 does not rule out the occurence of π_η in the discrete spectrum. In [G], this inequality is strengthened to an equality, thus resolving Problem C for the cubic unipotent A-packets:

(3.13) Theorem *Fix the étale cubic algebra E, and let η be an irreducible character of $S_E(\mathbb{A})$. Then*

$$m_{disc}(\pi_\eta) = m_\eta.$$

The proof of this theorem is reminescent of the proof of the multiplicity-one theorem for cusp forms on GL_n. It is a consequence of the following two results:

- Let ψ_E be a character of $N(\mathbb{A})$ trivial on $N(F)$ which lies in the $M(F)$-orbit indexed by E, and let l_{ψ_E} be as defined in (2.5). Then the assignment $\iota \mapsto l_{\psi_E} \circ \iota$ defines an *injective* map

$$\mathrm{Hom}_{G_2(\mathbb{A})}(\pi_\eta, \mathcal{A}(G_2)) \longrightarrow \mathrm{Hom}_{N(\mathbb{A})}(\pi_\eta, \mathbb{C}_{\psi_E})^{M_{\psi_E}(F)}.$$

- For each place v, we have an isomorphism of $M_{\psi_E}(F_v) \cong S_E(F_v)$-modules:

$$Hom_{N(F_v)}(\pi_{\eta_v}, \mathbb{C}_{\psi_E}) \cong \chi_{K_{E_v}} \cdot \eta_v^\vee,$$

  where $\chi_{K_{E_v}}$ is the quadratic character associated to the discriminant (quadratic) algebra K_{E_v} of E_v. Note that $\chi_{K_E} = \prod_v \chi_{K_{E_v}}$ is trivial when restricted to $M_{\psi_E}(F) = S_E(F)$.

We shall postpone the discussion of Problem D to §6.

§4. Tempered Representations with Unbounded Multiplicities

The representations π_{η_S} of Theorem 3.11 are very degenerate: their local components are non-tempered and non-generic, and are the so-called unipotent representations. This may lead one to think that the phenomenon of unbounded cuspidal multiplicities only happens for very degenerate representations. However, as we explain in this section, it should already occur for representations in some tempered L-packets. We shall discuss how we intend to construct these tempered representations of arbitrarily high cuspidal multiplicities.

(4.1) Some tempered parameters. We begin by considering some tempered A-parameters, i.e. those for which ψ is trivial on $SL_2(\mathbb{C})$. Let us start with a cuspidal representation τ of PGL_2 such that

$$\tau_v = \begin{cases} \text{Steinberg representation for } v \in S_\tau; \\ \text{an unramified representation for } v \notin S_\tau \end{cases}$$

for some non-empty finite set S_τ of finite places of F. Conjecturally, τ corresponds to a map $\phi_\tau : L_F \longrightarrow SL_2(\mathbb{C})$. Because of our assumptions, the map ϕ_τ is surjective; in fact, for $v \in S_\tau$, the local parameter ϕ_{τ_v} is already surjective, since it corresponds to the Steinberg representation.

Now we construct an A-parameter for G_2 using ϕ_τ as follows:

$$\psi_\tau : L_F \longrightarrow SL_2(\mathbb{C}) \to SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

Then we have:

$$\begin{cases} \mathcal{S}_{\psi_\tau} = \mathcal{S}_{\psi_\tau,v} = S_3 \quad \text{for all } v \in S_\tau. \\ \mathcal{S}_{\psi_\tau,v} = \{1\} \quad \text{for all } v \notin S_\tau. \end{cases}$$

In particular, Arthur's conjecture predicts that the local packets have the following form:

$$A_{\psi_\tau,v} = \begin{cases} \{\pi'_{\mathbf{1_v}}, \pi'_{r_v}, \pi'_{\epsilon_v}\} \quad \text{if } v \in S_\tau; \\ \{\pi'_{\mathbf{1_v}}\} \quad \text{if } v \notin S_\tau. \end{cases}$$

Moreover, the representations in the local packets should be tempered.

In fact, the parameter ψ_τ is an example of Langlands parameter considered by Lusztig. Hence, in this case, the local packet $A_{\psi_\tau,v}$ has already been defined, and it does consist of 3 discrete series representations (see [GrS]) when $v \in S_\tau$. Indeed, for $v \in S_\tau$, $A_{\psi_\tau,v}$ is related to the cubic unipotent packet $A_{\psi_{E_0},v}$ by the Aubert-Iwahori-Matsumoto involution.

Finally, if we set

$$\pi_\tau = \left(\otimes_{v\in S_\tau} \pi'_{r_v}\right) \bigotimes \left(\otimes_{v\notin S_\tau} \pi'_{\mathbf{1_v}}\right),$$

then Arthur's conjecture implies that

$$m_{disc}(\pi_\tau) \geq \frac{1}{6} \cdot (2^{\#S_\tau} + (-1)^{\#S_\tau} 2).$$

In fact, since the representation π'_τ is tempered, it cannot occur in the residual spectrum, and so we should have:

$$m_{cusp}(\pi_\tau) \geq \frac{1}{6} \cdot (2^{\#S_\tau} + (-1)^{\#S_\tau} 2).$$

Now one can find cuspidal representations τ of PGL_2 of the above type and with S_τ as big as one wishes (using the trace formula for example). Hence, Arthur's conjecture predicts that one can find a family of tempered representations of $G_2(\mathbb{A})$ whose cuspidal multiplicities are unbounded.

(4.2) Potential Construction. Finally, we would like to explain how we expect to demonstrate the lower bound on $m_{cusp}(\pi_\tau)$ predicted by Arthur's conjecture.

The parameter

$$\psi_\tau : L_F \longrightarrow SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$$

actually factors as:

$$\psi_\tau : L_F \longrightarrow SO_3(\mathbb{C}) \hookrightarrow SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

Hence, instead of lifting the cuspidal representation τ of PGL_2 directly to G_2, one may first lift it to a cuspidal representation of PGL_3. This is precisely the Gelbart-Jacquet lift, and we denote this cuspidal representation of PGL_3 by $GJ(\tau)$. Note that

$$GJ(\tau)_v = \begin{cases} \text{the Steinberg representation } St_v \text{ if } v \in S_\tau; \\ \text{a specific unramified representation if } v \notin S_\tau. \end{cases}$$

Now it turns out that $PGL_3 \times G_2$ is a dual pair in the split (adjoint) exceptional group of type E_6. This suggests that we may use exceptional theta correspondence to lift $GJ(\tau)$ from PGL_3 to G_2: hopefully we will get the representation π_τ. For this to work out, one should first verify that under local theta correspondence, the Steinberg representation St_v of $PGL_3(F_v)$ lifts to the representation π'_r of $G_2(F_v)$. However, it was shown in [GS] that the theta lift of St_v is equal to $\pi'_{\mathbf{1}} \oplus \pi'_\epsilon$. So this doesn't work out as planned.

Thankfully, a homomorphism $L_F \longrightarrow SL_3(\mathbb{C})$ is not just a Langlands parameter for PGL_3; it is also a parameter for any inner form of PGL_3. Such an inner form is of the form $PD^\times$ where D is a degree 3 division algebra. Over a p-adic field F_v, there are two such division algebras: D_v and its opposite D_v^{opp}. Being opposite algebras, their groups of invertible elements define isomorphic algebraic groups. Thus, locally, PGL_3 has precisely one inner form $PD^\times$.

Now under the local Jacquet-Langlands correspondence, the Steinberg representation St_v corresponds to the trivial representation $\mathbf{1_v}$ of $PD^\times(F_v)$. Moreover, $PD^\times \times G_2$ is a dual pair in an inner form of E_6. It was shown in [S] that the local theta lift of $\mathbf{1_v}$ is indeed equal to π'_r.

Hence we are led to the following strategy for embedding π_τ into L^2_{cusp}. Choose a global division algebra D of degree 3 which is ramified precisely at the set S_τ. Then one lifts τ from PGL_2 to G_2 as follows:

$$\begin{array}{ccccccc} PGL_2 & \xrightarrow{\text{Gelbart-Jacquet}} & PGL_3 & \xrightarrow{\text{Jacquet-Langlands}} & PD^\times & \xrightarrow{\text{theta lift}} & G_2 \\ \tau & \longrightarrow & GJ(\tau) & \longrightarrow & JL_D(GJ(\tau)) & \longrightarrow & \Theta(JL_D(GJ(\tau))). \end{array}$$

As an abstract representation, $\Theta(JL_D(GJ(\tau)))$ is either zero or is isomorphic to π_τ.

How does the multiplicity $\frac{1}{6} \cdot (2^{\#S_\tau} + (-1)^{\#S_\tau} 2)$ arise in this case? The answer lies in the following lemma:

(4.3) Lemma *The number of global division algebras of degree 3 ramified precisely at a set S is equal to*

$$\frac{1}{3} \cdot (2^{\#S} + (-1)^{\#S} 2).$$

In particular, the number of inner forms of PGL_3 which are ramified at the set S is half of the above number.

Note that the various inner forms of the lemma are non-isomorphic as algebraic groups, but their groups of adelic points are abstractly isomorphic. Thus the reason for the high multiplicity here is the failure of Hasse principle for the inner forms of PGL_3!

In order for the above strategy to work, it remains to resolve the following problems:

PROBLEMS:

- Show the non-vanishing of the theta lift $\Theta(JL_D(GJ(\tau)))$;
- Show that the various $\Theta(JL_D(GJ(\tau)))$'s generate linearly independent copies of π_τ in L^2_{cusp}.

At the moment, these issues are unresolved.

§5. Short Root A-packets

In this section, we shall consider the short root A-packets, which have been constructed in [GG]. We first give a description of the short root A-parameters.

(5.1) Short root A-parameters. Observe that

$$SL_{2,l} \times_{\mu_2} SL_{2,s} \subset G_2$$

where $(SL_{2,l}, SL_{2,s})$ is a pair of commuting SL_2's corresponding to a pair of mutually orthogonal long and short roots. Indeed, the centralizer of one of these SL_2's is the other SL_2.

Now suppose that τ is a cuspidal representation of PGL_2. Conjecturally, τ corresponds to an irreducible representation

$$\rho_\tau : L_F \longrightarrow SL_2(\mathbb{C}).$$

We define an Arthur parameter by

$$\psi_\tau : L_F \times SL_2(\mathbb{C}) \xrightarrow{\rho_\tau \times id} SL_{2,l}(\mathbb{C}) \times SL_{2,s}(\mathbb{C}) \xrightarrow{\quad i \quad} G_2(\mathbb{C}).$$

Thus, the short root A-parameters are indexed by cuspidal representations of PGL_2. In particular, it makes sense to talk about them even though we don't know what L_F is.

The centralizer S_{ψ_τ} of ψ_τ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover, the local component groups $S_{\psi_{\tau,v}}$ are given by

$$S_{\psi_\tau,v} = \begin{cases} 1, & \text{if } \rho_{\tau,\mathrm{v}} \text{ is reducible} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \rho_{\tau,\mathrm{v}} \text{ is irreducible.} \end{cases}$$

Note the the condition $\rho_{\tau,v}$ is irreducible is equivalent to τ_v being a discrete series representation of $PGL_2(F_v)$.

(5.2) Local Arthur packets. Now Arthur's conjecture predicts that for each place v, the local A-packets $A_{\tau,v}$ has the form:

$$A_{\tau,v} = \begin{cases} \{\pi^+_{\tau_v}\}, & \text{if } \tau_{\rm v} \text{ is not discrete series} \\ \{\pi^+_{\tau_v}, \pi^-_{\tau_v}\}, & \text{if } \tau_{\rm v} \text{ is discrete series.} \end{cases}$$

Here, $\pi^+_{\tau_v}$ is indexed by the trivial character of $S_{\tau,v}$.

Of course, we know what π^+_v has to be for almost all v. Hence, our task is to define the remaining elements of $A_{\tau,v}$. In fact, it is not difficult to guess what π^+_v is in general. Indeed, for almost all v where τ_v is unramified, π^+_v is the spherical representation with Satake parameter $s_{\psi_{\tau,v}}$, and this representation can be alternatively expressed as $J_P(\tau_v, 1/2)$. Thus it is natural to guess that

$$\pi^+_v \cong J_P(\tau_v, 1/2) \qquad \text{for all } v.$$

As we explain later, this is essentially, but not completely, correct.

(5.3) Global A-packets. Let S_τ be the set of places v where τ_v is discrete series, so that the global A-packet has $2^{\#S_\tau}$ elements. To describe the multiplicity of $\pi_\eta \in A_\tau$ in $L^2_{\psi_\tau}$, we need to know the quadratic character ϵ_{ψ_τ} of S_{ψ_τ}. It turns out that ϵ_{ψ_τ} is the non-trivial character of $S_{\psi_\tau} \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\epsilon(\tau, 1/2) = -1$.

Now if $\pi = \otimes_v \pi^{\epsilon_v}_v \in A_\tau$, then the multiplicity associated to π by Arthur's conjecture is:

$$m(\pi) = \begin{cases} 1, & \text{if } \epsilon_\pi := \prod_v \epsilon_v = \epsilon(\tau, 1/2); \\ 0, & \text{if } \epsilon_\pi = -\epsilon(\tau, 1/2). \end{cases}$$

Thus, we expect to construct a subspace $L^2_{\psi_\tau}$ of L^2_{disc} which is isomorphic to

$$\bigoplus_{\pi \in A_\tau : \epsilon_\pi = \epsilon(\tau, 1/2)} \pi.$$

The structure of these A-packets should remind the reader of the Saito-Kurakawa packets for $PGSp_4$ [PS], and more importantly, of the results of Waldspurger on the discrete spectrum of the metaplectic group $\widetilde{SL}_2$. We review the relevant facts below, since they provide a clue for the construction of the short root A-packets.

(5.4) The Weil representations of $\widetilde{SL}_2(F_v)$. Fix a non-trivial unitary character ψ_v of F_v. Then associated to a quadratic character χ_v of $F_v^\times$ (possibly trivial) is a Weil representation ω_{χ_v} of the metaplectic group $\widetilde{SL}_2(F_v)$. The representation can be realized on the space $S(F_v)$ of Schwarz functions on F_v. As a representation of $\widetilde{SL}_2(F_v)$, ω_{χ_v} is reducible. In fact, it is the direct sum of two irreducible representations:

$$\omega_{\chi_v} = \omega^+_{\chi_v} \oplus \omega^-_{\chi_v},$$

where $\omega^+_{\chi_v}$ (resp. $\omega^-_{\chi_v}$) consists of the even (resp. odd) functions in $S(F_v)$. If v is a finite place, then ω^-_χ is supercuspidal and ω^+_χ is not.

(5.5) Waldspurger's lift and packets for $\widetilde{SL}_2(F_v)$. In [W1] and [W2], Waldspurger defined a surjective map Wd_{ψ_v} from the set of irreducible (genuine) representations of $\widetilde{SL}_2(F_v)$ which are not equal to $\omega^+_{\chi_v}$ for any χ_v to the set of infinite dimensional representations of $PGL_2(F_v)$. This leads to a partition of the set of such representations of $\widetilde{SL}_2(F_v)$, indexed by the infinite dimensional representations of $PGL_2(F_v)$. Namely, if τ_v is such a representation of $PGL_2(F_v)$, we set

$$\tilde{A}_{\tau_v} = \text{ inverse image of } \tau_v \text{ under } Wd_{\psi_v}.$$

It turns out that

$$\#\tilde{A}_{\tau_v} = \begin{cases} 2 \text{ if } \tau_v \text{ is discrete series;} \\ 1 \text{ if } \tau_v \text{ is not.} \end{cases}$$

In the first case, the set $\tilde{A}_{\tau_v}$ has a distinguished element $\sigma^+_{\tau_v}$, which is characterized by the fact that $\sigma^+_{\tau_v} \otimes \tau_v$ is a quotient of the Weil representation $\omega^{(3)}_{\psi_v}$ of $\widetilde{SL}_2(F_v) \times PGL_2(F_v)$. The other element of $\tilde{A}_{\tau_v}$ will be denoted by $\sigma^-_{\tau_v}$. In the second case, we shall let $\sigma(\tau_v)^+$ be the unique element in $\tilde{A}_{\tau_v}$ and set $\sigma(\tau_v)^- = 0$. Observe that these "packets" of $\tilde{SL}_2(F_v)$ has the same structure as the local A-packets associated to short root A-parameters. This similarity extends to the global situation as well, as we now explain.

(5.6) Cusp forms of $\widetilde{SL}_2(\mathbb{A})$. Let $\widetilde{SL}_2(\mathbb{A})$ be the two-fold cover of $SL_2(\mathbb{A})$, and fix a non-trivial unitary character $\psi = \prod_v \psi_v$ of $F\backslash\mathbb{A}$. Let $\widetilde{\mathcal{A}}_2$ denote the space of square-integrable genuine automorphic forms on $\widetilde{SL}_2(\mathbb{A})$. Then there is an orthogonal decomposition

$$\widetilde{\mathcal{A}}_2 = \widetilde{\mathcal{A}}_{00} \bigoplus \left(\bigoplus_{\chi} \widetilde{\mathcal{A}}_{\chi} \right).$$

Here, χ runs over all quadratic characters (possibly trivial) of $F^\times\backslash\mathbb{A}^\times$.

Let us describe the space $\widetilde{\mathcal{A}}_{\chi}$ more concretely. If $\omega_{\chi} = \otimes_v \omega_{\chi_v}$ is the global Weil representation attached to χ, then the formation of theta series gives a map $\theta_{\chi} : \omega_{\chi} \to \widetilde{\mathcal{A}}_2$, whose image is the space $\widetilde{\mathcal{A}}_{\chi}$. To describe the decomposition of $\widetilde{\mathcal{A}}_{\chi}$, for a finite set S of places of F, let us set

$$\omega_{\chi,S} = (\otimes_{v \in S} \omega^-_{\chi_v}) \otimes (\otimes_{v \notin S} \omega^+_{\chi_v})$$

so that

$$\omega_{\chi} = \bigoplus_S \omega_{\chi,S}.$$

Then we have

$$\widetilde{\mathcal{A}}_{\chi} \cong \bigoplus_{\#S \text{ even}} \omega_{\chi,S}.$$

Moreover, $\omega_{\chi,S}$ is cuspidal if and only if S is non-empty.

(5.7) Near equivalence classes. In the beautiful paper [W2], Waldspurger has described the near equivalence classes of representations in $\widetilde{\mathcal{A}}_{00}$. Earlier, in [W1], he has shown that $\widetilde{\mathcal{A}}_{00}$ satisfies multiplicity one. Let us describe his results.

Given a cuspidal automorphic representation $\tau = \otimes_v \tau_v$ of PGL_2, we define a set of irreducible unitary representations of $\widetilde{SL}_2(\mathbb{A})$ as follows. Recall that for each place v, we have a local "packet"

$$\tilde{A}_{\tau_v} = \{\sigma^+_{\tau_v}, \sigma^-_{\tau_v}\}$$

where $\sigma^-_{\tau_v} = 0$ if τ_v is not discrete series. Now set

$$\tilde{A}_\tau = \{\sigma = \otimes_v \sigma^{\epsilon_v}_{\tau_v} : \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v\}.$$

This is the global "packet" of $\widetilde{SL}_2(\mathbb{A})$ associated to τ.

For $\sigma = \otimes_v \sigma^{\epsilon_v}_{\tau_v} \in \tilde{A}_\tau$, let us set

$$\epsilon_\sigma = \prod_v \epsilon_v.$$

Then we have:

$$\widetilde{\mathcal{A}}_{00} = \bigoplus_{\text{cuspidal } \tau} \widetilde{\mathcal{A}}(\tau)$$

where each $\widetilde{\mathcal{A}}(\tau)$ is a near equivalence class of cuspidal representations and is given by:

$$\widetilde{\mathcal{A}}(\tau) = \bigoplus_{\sigma \in \tilde{A}_\tau : \epsilon_\sigma = \epsilon(\tau, 1/2)} \sigma.$$

Again, notice the formal similarity between $\widetilde{\mathcal{A}}(\tau)$ and $L^2_{\psi_\tau}$.

(5.8) Rallis-Schiffmann lifting. In view of the above results of Waldspurger on the discrete spectrum for $\widetilde{SL}_2(\mathbb{A})$, a reasonable strategy for the construction of the short root A-packets is construct a lifting of the "packets" of $\tilde{SL}_2$ to G_2. That such a lifting is possible was discovered by Rallis and Schiffmann.

In the paper [RS], Rallis and Schiffmann constructed a lifting of cuspidal automorphic forms from $\widetilde{SL}_2$ to G_2. This was achieved by exploiting the fact that $SL_2 \times G_2$ is a subgroup of $SL_2 \times SO(V,q) \cong SL_2 \times SO_7$, which is the classical dual pair in Sp_{14}. The lifting is then defined using the theta kernel furnished by the Weil representation $\omega^{(7)}_\psi$ of $\widetilde{Sp}_{14}$ (associated to ψ and the quadratic space (V,q)).

The surprising discovery of Rallis-Schiffmann is that, despite restricting from SO_7 to the smaller group G_2, one still obtains a correspondence of representations. More precisely, if σ is an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbb{A})$, let $V(\sigma)$ be the theta lift of σ; it is the (non-zero) subspace of $\mathcal{A}(G_2)$ generated by functions:

$$\theta(f\,\varphi)(g) = \int_{SL_2(F)\backslash \widetilde{SL}_2(\mathbb{A})} \theta(\varphi)(gh) \cdot \overline{f(h)}\, dh$$

where $f \in \sigma$, $\varphi \in \omega_\psi^{(7)}$ and $\theta : \omega_\psi^{(7)} \longrightarrow \mathcal{A}_2(\widetilde{Sp}_{14})$ is given by the formation of theta series. Now the main results of Rallis-Schiffmann are:

(5.9) Theorem *(i) $V(\sigma)$ is contained in the space of cusp forms if and only if the theta lift (associated to ψ) of σ to SO_3 (studied by Waldspurger) is zero.*

(ii) The cuspidal representations of G_2 which are not orthogonal to the lifts from $\widetilde{SL}_2$ are characterized as those having a non-zero period with respect to some quasi-split SU_3^K.

(iii) The local correspondence of unramified *representations is precisely determined. In particular, if $\sigma \subset \widetilde{A}(\tau)$, then for each irreducible constituent π of $V(\sigma)$, π_v has Satake parameter $s_{\psi_{\tau,v}}$ for almost all v.*

(iv) The cuspidal representations in $V(\sigma)$ are CAP with respect to P if $\sigma \subset \widetilde{A}_{00}$, and are CAP with respect to the Borel subgroup if $\sigma \subset \widetilde{A}_\chi$ for some χ. This gives the first construction of CAP representations of G_2.

The results of [RS], especially part (ii) of the theorem, certainly suggest that the short root parameters can be constructed by the Rallis-Schiffmann lifting. We should also mention that the Rallis-Schiffmann lifting was further investigated by Li-Schwermer [LS].

(5.10) The results of [GG]. In [GG], we completed the study initiated in [RS] by giving a complete analysis of the theta correspondence from $\widetilde{SL}_2$ to G_2, in the style of Waldspurger's analysis of the Shimura correspondence. In particular, we gave a precise determination of the representation $V(\sigma)$.

The first step in this is the complete determination of the *local* theta correspondence. More precisely, if v is a p-adic place of F and σ_v an irreducible representation of $\widetilde{SL_2}(F_v)$, the maximal σ_v-isotypic quotient of $\omega_{\psi_v}^{(7)}$ can be expressed as $\sigma_v \otimes \theta(\sigma_v)$, where $\theta(\sigma_v)$ is a smooth representation of $G_2(F_v)$. Let $\Theta(\sigma_v)$ be the maximal semisimple quotient of $\theta(\sigma_v)$. When v is archimedean, there is an analogous (but slightly different) definition of $\Theta(\sigma_v)$. The main local result of [GG] is:

(5.11) Theorem $\Theta(\sigma_v)$ *can be completely determined for any σ_v (to the extent that classification of representations of $G_2(F_v)$ is known). In particular, $\Theta(\sigma_v)$ is irreducible except when $\sigma_v = \omega_{\psi_v}^{\pm}$ (the even and odd Weil representations of $\tilde{S}L_2(F_v)$ associated to ψ_v). In these two exceptional cases, we have:*

$$\begin{cases} \Theta(\omega_{\psi_v}^{+}) = \pi_{1_v} \oplus \pi_{r_v} \\ \Theta(\omega_{\psi_v}^{-}) = \pi_{r_v} \oplus \pi_{\epsilon_v}. \end{cases}$$

Here, $\{\pi_{1_v}, \pi_{r_v}, \pi_{\epsilon_v}\}$ is the cubic unipotent packet attached to the split cubic algbera E_0.

We should note that for a real place v, the lifting of most discrete series representations was already determined by Li-Schwermer [LS].

(5.12) Global results. We now turn to the global situation. For any cuspidal σ on $\widetilde{SL}_2$, we have defined the theta lift $V(\sigma)$. In fact, one can define the theta lift of any square-integrable automorphic representation of $\widetilde{SL}_2$ by a regularization of the theta integral. Thus, one can speak of the regularized theta lift $V(\sigma)$ of any $\sigma \subset \widetilde{\mathcal{A}}_2$.

One can show that $V(\sigma)$ is contained in the space of square-integrable automorphic forms. Thus $V(\sigma)$ is semisimple and is a non-zero summand of $\Theta(\sigma) := \otimes_v \Theta(\sigma_v)$. Theorem 5.11 immediately shows that $V(\sigma) \cong \Theta(\sigma)$, whenever $\sigma_v \neq \omega^{\pm}_{\psi_v}$ for any place v, since $\Theta(\sigma)$ is irreducible in that case. However, when $\Theta(\sigma)$ is reducible, there are more than one possibilities for $V(\sigma)$. The determination of $V(\sigma)$ in this case is easily the trickiest part of [GG]. The result is:

(5.13) Theorem *(i) If $\sigma \subset \mathcal{A}_{00}$, then $V(\sigma) \cong \Theta(\sigma)$.*

(ii) Suppose that K is an étale quadratic F-algebra with associated quadratic character χ_K. Then (the L^2-completion of) the regularized theta lift of $\widetilde{\mathcal{A}}_{\chi_K}$ is equal to the space $L^2_{\psi_{F\times K}}$.

Thus, part (ii) of the theorem gives us an alternative construction of the subspaces of L^2_{disc} associated to the cubic unipotent A-packets corresponding to those cubic algebras E which are not fields.

(5.14) Definition of local Arthur packets. Now we can use Theorem 5.11 to give a natural candidate for the packet $A_{\tau,v}$. Recall that τ_v determines a set $\tilde{A}_{\tau_v}$ of representations of $\widetilde{SL}_2(F_v)$, which has 2 or 1 elements $\sigma^{\pm}_{\tau_v}$, depending on whether τ_v is discrete series or not. We set

$$\pi^{\pm}_{\tau_v} = \Theta(\sigma^{\pm}_{\tau_v}).$$

This defines the packet $A_{\tau,v}$ and is our solution to Problm A.

(5.15) A special case. We would like to highlight the case when τ_v is the Steinberg representation St. In this case,

$$\sigma^{+}_{v} = \omega^{-}_{\psi_v}.$$

Thus, according to our definition,

$$\pi^{+}_{\tau_v} = \Theta(\omega^{-}_{\psi_v}) = \pi_{r_v} + \pi_{\epsilon_v} = J_P(St, 1/2) + \pi_{\epsilon}.$$

In particular, $\pi^{+}_{\tau_v}$ is reducible. For the case of split p-adic groups, this is the first instance we know in which the representation in a packet can be reducible, and this is quite surprising at first sight. As we mentioned before, the initial guess would be to take $\pi^{+}_{\tau_v}$ simply as $J_P(St, 1/2)$. For future reference, we set

$$\pi^{++}_{\tau_v} = \begin{cases} J_P(St, 1/2), & \text{if } \tau_v = St; \\ \pi^{+}_{\tau_v}, & \text{otherwise.} \end{cases}$$

Thus $\pi^{++}_{\tau_v}$ is a certain distinguished constituent of $\pi^{+}_{\tau_v}$.

(5.16) The subspace $L^2_{\psi_\tau}$. Why is the above definition of local packets reasonable? For one thing, when τ_v is unramified, $\Theta(\sigma^{+}_{\tau_v})$ is indeed irreducible and unramified with the required Satake parameter $s_{\psi_\tau,v}$. Secondly, Theorem 5.13(i) says that for a given τ, the global theta correspondence constructs a subspace of

L^2_{disc} isomorphic to

$$\bigoplus_{\sigma \in \tilde{A}_\tau : \epsilon_\sigma = \epsilon(\tau,1/2)} \Theta(\sigma).$$

With our definition of the local packets, this gives the desired subspace $L^2_{\psi_\tau}$, thus resolving Problem B. This provides compelling global justification for taking $\pi^+_{\tau_v}$ to be reducible when τ_v is Steinberg.

(5.17) SL_3-distinguished cusp forms. Let us highlight a corollary of Theorem 5.13. It pertains to the question of whether there are cuspidal representations of G_2 with non-zero SL_3-period. Such cuspidal representations should be very scarce. Examples are the cuspidal representations belonging to the cubic unipotent A-packet associatyed to the split algebra E_0.

It wasn't known previously if other SL_3-distinguished cuspidal representations exist. Now, as a consequence of our global theorem, we know that they do. Indeed, when $\sigma \subset \mathcal{A}_{00}$ is such that its theta lift to SO_3 is non-zero, we know from [RS] that $V(\sigma)$ is not totally contained in $\mathcal{A}_{cusp}(G_2)$. However, this does not exclude the possibility that $V(\sigma) \cap \mathcal{A}_{cusp}(G_2)$ is non-zero. In [RS, Pg. 823], Rallis-Schiffmann remarked that they do not know if such a possibility can actually happen. Theorem 5.13 implies that it does. Indeed, this occurs when σ_v is the odd Weil representation $\omega^-_{\psi_v}$ for some finite place v, because $V(\sigma) \cong \Theta(\sigma)$ and $\Theta(\sigma_v)$ contains the supercuspidal representation π_{ϵ_v} for such a place v. For such a σ, any irreducible constituent of $V(\sigma) \cap \mathcal{A}_{cusp}(G_2)$ will have a non-zero SL_3-period.

§6. Rankin-Selberg Integral and Near Equivalence Classes

In this section, we consider Problem D for the cubic unipotent and short root A-packets that we defined in the earlier sections. Recall that what we want is the following analog of the strong multiplicity one theorem:

- If π is an irreducible constituent of L^2_{disc} and π is nearly equivalent to the representations in an A-packet A_ψ, then π is contained in L^2_ψ. In other words, L^2_ψ is a full near equivalence class.

Let's consider the case of the short root A-packets. Hence, suppose that we have an irreducible constituent π of L^2_{disc} which is nearly equivalent to the representations in A_{ψ_τ}, for some cuspidal representation τ of PGL_2. Assume moreover that $\pi \subset \mathcal{A}_{cusp}(G_2)$. In view of Theorem 5.9, if we can show that π has a SU_3^K-period for some quadratic K, we will at least have shown that π is not orthogonal to $L^2_{\psi_\tau}$, and thus π is *isomorphic* to some irreducible constituent of $L^2_{\psi_\tau}$.

Because the cubic unipotent packets associated to $E = F \times K$ can also be constructed by the Rallis-Schiffmann lifting, the same consideration applies to those cuspidal π which are nearly equivalent to representations in $L^2_{\psi_{F\times K}}$.

(6.1) A Rankin-Selberg integral. The discussion above leads us to consider a certain Rankin-Selberg integral. More precisely, let B_K be a Borel subgroup of SU_3^K and consider the family of principal series representations

$$I_K(s) = Ind_{B_K(\mathbb{A})}^{SU_3^K(\mathbb{A})} \delta^s_{B_K}$$

For a standard section $f_s \in I_K(s)$, one has the Eisenstein series $E(g,s,g)$ which is defined for $Re(s) >> 0$ by

$$E(f,s,g) = \sum_{\gamma \in B_K(F)\backslash SU_3^K(F)} f_s(\gamma g)$$

and has a meromorphic cotinuation to $\mathbb{C}$. At $s = 1$, $E(f,s,g)$ has a pole of order at most 2 (resp. 1) if K is split (resp. non-split). Moreover, this highest possible order of pole is attained for some f, and the leading terms of the Laurent expansion at $s = 1$ of $E(f,s,g)$ (as f varies) span the space of constant functions in L^2_{disc}.

Thus we are led to consider the following integral

$$J_K(f,\varphi,s) = \int_{SU_3^K(F)\backslash SU_3^K(\mathbb{A})} \varphi(g) \cdot E(f,s,g)\, dg$$

where $\varphi \in \pi$. This defines a meromorphic function in s. In [GG, §15], the analytic properties of this Rankin-Selberg integral were studied in some detail. The main result is:

(6.2) Proposition *(i) Suppose that π is nearly equivalent to the representations in $L^2_{\psi_{F\times K}}$. Then*

$$J_K(f,\varphi,s) = \zeta_K^S(2s-1) \cdot L^S(\chi_K, 4s-2) \cdot d_S(f,\varphi,s)$$

where S is a large finite set of places of F containing the archimedean ones, and $d_S(f,\varphi,s)$ is a meromorphic function (the bad factor). We can choose f and φ so that $d_S(f,\varphi,s)$ is non-zero at $s = 1$.

(ii) Suppose that π is nearly equivalent to the representations in $L^2_{\psi_\tau}$ and π has non-zero $(F \times K)$-Fourier coefficient with K a quadratic field. Then

$$J_K(f,\varphi,s) = \frac{\zeta_K^S(2s-1)}{L^S(\chi_K, 4s-1)} \cdot L(\tau, 4s-3/2) \cdot d_S(f,\varphi,s)$$

where S is a large finite set of places of F containing the archimedean ones, and $d_S(f,\varphi,s)$ is a meromorphic function (the bad factor). We can choose f and φ so that $d_S(f,\varphi,s)$ is non-zero at $s = 1$.

As a consequence of the proposition, we have the following theorem, which resolves Problem D for some cubic unipotent A-packets.

(6.3) Theorem *Suppose that the étale cubic algebra $E = F \times K$ is not a field. Then $L^2_{\psi_E}$ is a full near equivalence class.*

Proof. After the exact multiplicity formula of Thm. 3.13, it remains to show that if $\pi \subset L^2_{disc}$ is nearly equivalent to the representations in $L^2_{\psi_E}$, then π is isomorphic to an irreducible constituent of $L^2_{\psi_E}$.

Suppose first that π has non-zero projection $P_{res}(\pi)$ to the residual spectrum. Now the residual spectrum has been determined completely by Kim [K] and Zampera [Z]. From their results, one sees that $P_{res}(\pi)$ is contained in $P_{res}(L^2_{\psi_E})$. Hence, we may assume that $\pi \subset \mathcal{A}_{cusp}(G_2)$. Then Prop. 6.2(i) shows that π has non-zero SU_3^K-period. Thus by Thm. 5.9(ii), π is not orthogonal to the theta lift of some cuspidal σ of $\widetilde{SL}_2$.

The local result Thm. 5.11 implies that such a σ must be contained in $\widetilde{A}_{\chi_K}$. Thus, π is not orthogonal to $L^2_{\psi_E}$, as desired. ∎

To apply Prop. 6.2 to the short root A-packet A_{ψ_τ}, we note:

(6.4) Proposition *If $\pi \subset \mathcal{A}_{cusp}(G_2)$ is nearly equivalent to the representations in A_{ψ_τ}, then π has non-zero $(F \times K)$-coefficient for some quadratic field K.*

The proof of this proposition is decidedly roundabout; we refer the reader to [GG, §16] for the details. In any case, we obtain:

(6.5) Theorem *Suppose that $\pi \subset \mathcal{A}_{cusp}(G_2)$ is nearly equivalent to the representations in $L^2_{\psi_\tau}$. Then*

(i) π is not orthogonal to $L^2_{\psi_\tau}$. In particular, π is isomorphic to a representation in $L^2_{\psi_\tau}$.

(ii) the projection of $L^2_{\psi_\tau}$ onto $L^2_{cusp}(G_2)$ is a full near equivalence class in $\mathcal{A}_{cusp}(G_2)$.

To conclude this section, we note the following corollary, which settles Problem C for most representations in the short root packet A_τ:

(6.6) Corollary *Suppose that π_0 is an irreducible constituent of $\pi \in A_\tau$. Then*

$$m_{disc}(\pi_0) = m(\pi)$$

except possibly in the following case: $\pi_0 \cong \pi_\tau^{++}$ and $L(\tau, 1/2) \neq 0$. *Here $\pi_\tau^{++} = \otimes_v \pi_{\tau_v}^{++}$ and $\pi_{\tau_v}^{++}$ was defined in (5.15). In this exceptional case, $m(\pi) = 1$ but we only know that $m(\pi_0) = 1$ or 2.*

PROOF. Away from the exceptional case, this follows from Theorem 6.5(ii) because such a π_0 does not occur in the residual spectrum, by the results of [K] and [Z]. So

$$m_{disc}(\pi_0) = m_{cusp}(\pi_0) = \text{multiplicity of } \pi_0 \text{ in } L^2_{\psi_\tau} = m(\pi).$$

In the exceptional case, $\pi_0 = \pi_\tau^{++}$ does occur in the residual spectrum, and the ambiguity in $m_{disc}(\pi_0)$ arises because we don't know that it cannot also occur in the cuspidal spectrum. By Theorem 6.5(ii), this ambiguity will be resolved if one shows that the copy of π_τ^{++} in $L^2_{\psi_\tau}$ (constructed by theta lifting) is *orthogonal* to the space of cusp forms. Unfortunately, we only know that it is not contained in the space of cusp forms. ∎

§7. Some Conjectures

In this final section, we highlight some unanswered questions regarding the cubic unipotent A-packets. This is an ongoing project of the authors and D.-H. Jiang.

(7.1) Another Rankin-Selberg integral. For the cubic unipotent A-packets associated to $E = F \times K$, Problems A-D have all been resolved. The resolution of Problem D, in particular, relies on the alternative construction of the packets by Rallis-Schiffmann lifting. When E is a field, however, this alternative construction is not available. Thus, it is desirable to have a more uniform treatment of Problem D for the cubic unipotent A-packets.

To this end, we are led to consider another Rankin-Selberg integral. Suppose that E is Galois. Then the minimal representation Π_E of $Spin_8^E(\mathbb{A})$ is the unique irreducible quotient of a certain degenerate principal series

$$I_{P_E}(\chi_E, s) = Ind_{P_E}^{Spin_8^E} \chi_E \cdot \delta_{P_E}^s,$$

for $s = 4/5$ and χ_E is a finite order character associated to E. Let $E_{\chi_E}(f, s, g)$ be the Eisenstein series attached to a standard section f_s. Then the leading terms of the Laurent expansion of $E_{\chi_E}(f, s, g)$ (as f varies) at $s = 4/5$ define the embedding of Π_E into $\mathcal{A}_2(Spin_8^E)$ used in §3.

Thus, if $\pi \subset \mathcal{A}_{cusp}(G_2)$ is irreducible, it is natural to consider the following integral, first investigated by Jiang:

$$R_E(\varphi, f, \chi_E, , s) = \int_{G_2(F)\backslash G_2(\mathbb{A})} \varphi(g) \cdot E_{\chi_E}(f, \frac{s+2}{5}, g)\, dg$$

for $\varphi \in \pi$, and more generally $R_E(\varphi, f, \chi, s)$ for any character χ and $f_s \in I_{P_E}(\chi, s)$. This defines a meromorphic function of s. On unwinding, we see that $R_E(\varphi, f, \chi, s)$ is zero unless π has non-zero E-Fourier coefficient.

Here is what we would like to show:

(7.2) Conjecture: Assume that π has non-zero E-Fourier coefficient, with E Galois. Then we have the identity:

$$R_E(\varphi, f, \chi, s) = d_\chi(\varphi, f, s) \cdot L^S(s, \pi, \chi),$$

where $L^S(s, \pi, \chi)$ is the partial (degree 7) standard L-function of π twisted by χ, and d_χ is the factor corresponding to ramified and archimedean places.

This conjecture should imply a positive solution to Problem D for the cubic unipotent A-packets associated to Galois E. Indeed, we expect it to give the following stronger result:

(7.3) Conjecture: Let $\pi \subset \mathcal{A}_{cusp}(G_2)$ be irreducible. Fix a Galois cubic algebra E. Then the following statements are equivalent.

(a) $\pi \subset L^2_{\psi_E}$.

(b) π is E-distinguished.

(c) the partial L-function $L^S(\pi, s, \chi_E)$ has a pole (of order 2 if E is split, and of order 1 otherwise) at $s = 2$.

Moreover, π is CAP with respect to the Borel subgroup if and only if π satisfies the above conditions for some Galois E.

Remarks: If E is non-Galois, we still expect the equivalence of (a) and (b) above.

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