

**ON THE REGULARIZED SIEGEL-WEIL FORMULA
(THE SECOND TERM IDENTITY)
AND NON-VANISHING OF THETA LIFTS FROM ORTHOGONAL GROUPS**

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ABSTRACT. We derive a (weak) second term identity for the regularized Siegel-Weil formula for the even orthogonal group, which is used to obtain a Rallis inner product formula in the “second term range”. As an application, we show the following non-vanishing result of global theta lifts from orthogonal groups. Let π be a cuspidal automorphic representation of an orthogonal group $O(V)$ with $\dim V = m$ even and $r + 1 \leq m \leq 2r$. Assume further that there is a place v such that $\pi_v \cong \pi_v \otimes \det$. Then the global theta lift of π to Sp_{2r} does not vanish up to twisting by automorphic determinant characters if the (incomplete) standard L -function $L^S(s, \pi)$ does not vanish at $s = 1 + \frac{2r-m}{2}$. Note that we impose no further condition on V or π . We also show analogous non-vanishing results when $m > 2r$ (the “first term range”) in terms of poles of $L^S(s, \pi)$ and consider the “lowest occurrence” conjecture of the theta lift from the orthogonal group.

1. Introduction

Let F be a number field with ring of adèles \mathbb{A} . Let $O(\mathcal{V}) = O_{m,m}$ be the split orthogonal group of rank m over F and let $\mathrm{Sp}(W) = \mathrm{Sp}_{2r}$ be the symplectic group of rank r . The group $O(\mathcal{V}) \times \mathrm{Sp}(W)$ is a dual reductive pair in the symplectic group $\mathrm{Sp}(\mathcal{V} \otimes W)$ and possesses a distinguished representation known as the Weil representation (modulo the choice of a non-trivial character of $F \backslash \mathbb{A}$). If one chooses a maximal isotropic subspace \mathcal{V}^+ of \mathcal{V} , then this Weil representation can be realized on the space $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ of Schwarz functions on $(\mathcal{V}^+ \otimes W)(\mathbb{A})$. Moreover, it has a natural “automorphic” realization

$$\theta : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) \longrightarrow \{\text{Functions on } O(\mathcal{V})(F) \backslash O(\mathcal{V})(\mathbb{A}) \times \mathrm{Sp}(W)(F) \backslash \mathrm{Sp}(W)(\mathbb{A})\}$$

which is an $O(\mathcal{V}) \times \mathrm{Sp}(W)$ -equivariant map.

In this context, the classical Siegel-Weil formula is an identity of automorphic forms on $O(\mathcal{V})$ which relates the theta integral

$$I(g, \varphi) = \int_{\mathrm{Sp}(W)(F) \backslash \mathrm{Sp}(W)(\mathbb{A})} \theta(\varphi)(gh) dh$$

to a special value of a Siegel Eisenstein series $E(g, s_0; \varphi)$ for each $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$. When both the theta integral and the sum defining the Eisenstein series are absolutely convergent, such an identity was proved by Weil ([We]). Subsequent pioneering works of Kudla-Rallis (with significant refinements by Ikeda [Ik] and Ichino [Ich1]) extend this identity to situations where these convergent conditions are not satisfied, culminating in a regularized Siegel-Weil formula ([Kd-R2]) where one requires a regularization of the divergent theta integral $I(-, \varphi)$. Strictly speaking, the work of Kudla-Rallis deals with the case where one considers a theta integral over $O(\mathcal{V})$, so that

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the Siegel-Weil formula is an identity of automorphic forms on $\mathrm{Sp}(W)$. In the particular context of this paper, the regularized Siegel-Weil formula was established by Mœglin ([Mo]) and later Jiang-Soudry ([JS]). As we shall explain below, the identity established in all these papers is the so-called “first term identity”.

To explain this terminology, let us briefly recall the setup of the regularized Siegel-Weil formula in [Mo] and [JS]. The regularized theta integral $I(g, s, \varphi)$ (whose definition we recall in §3) turns out to be equal to a non-Siegel Eisenstein series $\mathcal{E}^{(m,r)}(g, s; \varphi)$ on $\mathrm{O}(\mathcal{V})$. Suppose that

$$m > 2r,$$

so that we are in what we call “first term range”. Then the regularized theta integral has at most a simple pole at $s = \rho'_r = \frac{r+1}{2}$ and one may consider its Laurent expansion there:

$$\mathcal{E}^{(m,r)}(s, g; \varphi) = \frac{B_{-1}^{(m,r)}(\varphi)(g)}{s - \rho'_r} + B_0^{(m,r)}(\varphi)(g) + \cdots .$$

On the other hand, one has the Siegel Eisenstein series $E^{(m,m)}(g, s; \varphi)$ and one may consider its Laurent expansion at $s = \rho_{m,r} = \frac{2r-m+1}{2}$:

$$E^{(m,m)}(g, s; \Phi_\varphi) = \frac{A_{-1}^{(m,r)}(\varphi)(g)}{s - \rho_{m,r}} + A_0^{(m,r)}(\varphi) + A_1^{(m,r)}(\varphi)(g)(s - \rho_{m,r}) + \cdots .$$

The regularized Siegel-Weil formula proved in the papers [Mo] and [JS] is an identity between the first (or leading) terms in these Laurent expansions. Hence such a formula is referred to as a “first term identity”.

However, the main concern of this paper is what we call “second term range”:

$$r + 1 \leq m \leq 2r.$$

In this case, the regularized theta integral has a double pole at the point of interest $s = \rho'_r = \frac{r+1}{2}$, so that its Laurent expansion there looks like:

$$\mathcal{E}^{(m,r)}(s, g; \varphi) = \frac{B_{-2}^{(m,r)}(\varphi)(g)}{(s - \rho'_r)^2} + \frac{B_{-1}^{(m,r)}(\varphi)(g)}{s - \rho'_r} + B_0^{(m,r)}(\varphi)(g) + \cdots .$$

The Laurent expansion of the Siegel Eisenstein series at $s = \rho_{m,r} = \frac{2r-m+1}{2}$ is as before:

$$E^{(m,m)}(g, s; \Phi_\varphi) = \frac{A_{-1}^{(m,r)}(\varphi)(g)}{s - \rho_{m,r}} + A_0^{(m,r)}(\varphi) + A_1^{(m,r)}(\varphi)(g)(s - \rho_{m,r}) + \cdots .$$

In this case, we shall show that there is still a first term identity relating B_{-2} with A_{-1} . But the chief contribution of this paper is to show that there is also a relationship between the second terms of the two Laurent series above (for a certain class of φ). Hence we shall establish a (weak) second term identity relating B_{-1} with A_0 . More precisely, we will prove the following. (Again see §3 for the notations.)

Theorem 1.1 (Weak Second Term Identity). *Let $\dim \mathcal{V}^+ = m$ with $r + 1 \leq m \leq 2r$. Then for all $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ that are in the $\mathrm{O}(\mathcal{V}, \mathbb{A})$ -span of a particular spherical vector φ^0 , the*

following identity holds:

$$\begin{aligned} B_{-1}^{(m,r)}(\varphi) + |D|^{-m\rho_{m,r}} \prod_{i=0}^{2r-m} \frac{\xi(i)}{\xi(2r-2i)} B_0^{(m,m-r-1)}(\mathrm{Ik}(\varphi)) \\ \equiv A_0^{(m,r)}(\varphi) \pmod{\mathrm{Im} A_{-1}^{(m,r)}}, \end{aligned}$$

where $\mathrm{Im} A_{-1}^{(m,r)}$ is the image of the $A_{-1}^{(m,r)}$ map.

Here we call this ‘‘the weak second term identity’’ because of the assumption we have to impose on φ . It is expected that this assumption can be removed, but it seems that our method is unable to do this. We note that this weak second term identity was proven by A. Ichino and the first author in their recent preprint [GI] for the case $r = 2$ and $m = 4$. Strictly speaking, in [GI] they consider the similitude group $\mathrm{GO}(\mathcal{V})$, but exactly the same computation yields the same formula for the isometry case. We should also mention that the full second term identity was obtained in certain other low rank cases by Kudla-Rallis-Soudry [KRS] (on the symplectic group Sp_4 for the dual pair $\mathrm{O}_4 \times \mathrm{Sp}_4$) and V. Tan [T] (on the unitary group $\mathrm{U}(2, 2)$ for the dual pair $\mathrm{U}(3) \times \mathrm{U}(2, 2)$). Also the spherical second term identity on the symplectic group has been known to Kudla already since 1989, whose precise statement appeared in [K, Thm. 3.13]. This is essentially the symplectic analogue of our Theorem 4.8. However his proof has never appeared anywhere to our best knowledge.

The main application of the (regularized) Siegel-Weil formula is in the derivation of the Rallis inner product formula for the theta lift from O_m to Sp_{2r} . In the first term range, using the first term identity, we obtain in §6 such a Rallis inner product formula and use it to obtain a sufficient condition for the non-vanishing of this theta lift in terms of the location of poles of standard L-functions of $\mathrm{O}(m)$ (see below). In this way, we recover certain results of Mœglin ([Mo]) and Ginzburg-Jiang-Soudry ([GJS]) in a somewhat more direct manner.

As we explain in §6, it is necessary to establish a ‘‘second term identity’’ in the ‘‘second term range’’ in order to obtain a Rallis inner product formula for the theta lift from O_m to Sp_{2r} ; the first term identity in this range would give one nothing! This Rallis inner product formula relates the inner product of the theta lifts to the values of the standard L-function at certain points, rather than the residues of the standard L-function at these points. Thus, as an application of our second term identity, we prove the following non-vanishing result for the global theta lift.

Theorem 1.2. *Let π be a cuspidal automorphic representation of $\mathrm{O}(V, \mathbb{A})$ with $\dim V = m$ even and $r + 1 \leq m \leq 2r$. Assume that*

- (i) *there is a place v where $\pi_v \cong \pi_v \otimes \det$;*
- (ii) *the (incomplete) standard L-function $L^S(s, \pi)$ does not vanish at $s = 1 + \frac{2r-m}{2}$ (a pole is allowed).*

Then there exists an automorphic character μ on $\mu_2(F) \backslash \mu_2(\mathbb{A})$ such that the global theta lift $\theta_{2r}(\pi \otimes (\mu \circ \det))$ of $\pi \otimes (\mu \circ \det)$ to the symplectic group $\mathrm{Sp}_{2r}(\mathbb{A})$ of rank r does not vanish.

Another way of stating the above theorem is that if σ is a cuspidal automorphic representation of $\mathrm{SO}(V, \mathbb{A})$ which satisfies the analogous conditions in the theorem, then σ has a nonzero theta lift to Sp_{2r} .

Let us mention that the condition (i) in the theorem is merely technical, though we do not know how to suppress it. As a result, we were not able to obtain the analogous theorem when $\dim V$ is

odd. Indeed, if $\dim V = m$ is odd, then $\pi_v \not\cong \pi_v \otimes \det$ for all v , and hence the assumption (i) of the theorem is never satisfied. However, we would like to emphasize that our non-vanishing result applies to an orthogonal group $O(V)$ for any quadratic space V , and moreover we do not impose any further assumption on π such as genericity or temperedness.

In the first term range $m > 2r$, the analogous results on non-vanishing of theta lifts were first shown by Mœglin ([Mo]) and Ginzburg-Jiang-Soudry ([GJS]), using Mœglin’s idea of “generalized doubling method” ([Mo]). As we mentioned above, these results can also be shown more directly using the Rallis inner product formula:

Theorem 1.3. *Let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$ for $\dim V = m$ where $m \geq 2r + 1$.*

- (a) *Assume $m \geq 2r + 2$. Further assume that the (incomplete) standard L -function $L^S(s, \pi)$ has a pole at $s = \frac{m-2r}{2}$. Then there is a character μ on $\mu_2(F) \backslash \mu_2(\mathbb{A})$ such that $\theta_{2r}(\pi \otimes (\mu \circ \det)) \neq 0$.*
- (b) *Assume $m = 2r + 1$. Further assume that the (incomplete) standard L -function $L^S(s, \pi)$ does not vanish at $s = \frac{1}{2}$. Then there is a character μ on $\mu_2(F) \backslash \mu_2(\mathbb{A})$ such that $\theta_{2r}(\pi \otimes (\mu \circ \det)) \neq 0$.*

For this theorem, we do not need the assumption $\pi_v \cong \pi_v \otimes \det$ for some place v , and hence $\dim V$ can be odd. This is because of the crucial Proposition 7.2.

Now once these theorems have been proven, the natural question to ask is whether the converses are true. We are not able to answer this question. In [GJS], a conjecture related to this issue was made by using the notion of the “lowest occurrence” for the range $m \geq 2r + 1$. In the very last section, we consider this conjecture in some detail not only for the range $m \geq 2r + 1$, but for any range.

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2. Notations and Preliminaries

In this paper, F is a number field with ring of integers \mathcal{O} and ring of adeles \mathbb{A} . Fix a non-trivial additive character $\psi = \otimes_v \psi_v$ on $F \backslash \mathbb{A}$. For each finite v , let c_v be the conductor of the additive character ψ_v , so that ψ_v is trivial on $\mathcal{P}_v^{-c_v}$. We fix the Haar measure dx_v on F_v (for all v) to be self-dual with respect to ψ_v , and hence the volume of \mathcal{O}_v is $q_v^{-c_v/2}$.

Let

$$\xi(s) = |D|^{\frac{s}{2}} \prod_{v \leq \infty} \zeta_v(s)$$

be the complete normalized zeta function of F , where D is the discriminant of F . It satisfies the functional equation

$$\xi(1-s) = \xi(s).$$

Note that ξ has residues at $s = 0, 1$. In this paper, we write

$$\xi(0) := \operatorname{Res}_{s=0} \xi(s) \quad \text{and} \quad \xi(1) := \operatorname{Res}_{s=1} \xi(s).$$

Here $\operatorname{Res}_{s=s_0} \xi(s)$ of course means the residue of $\xi(s)$ at $s = s_0$. Similarly for any meromorphic function $F(s)$, $\operatorname{Res}_{s=s_0} F(s)$ refers to the residue of $F(s)$ at $s = s_0$. We also denote by $\operatorname{Val}_{s=s_0} F(s)$ the “value” of $F(s)$, *i.e.* the constant term of the Laurent series of $F(s)$ at $s = s_0$. Note that $\operatorname{Val}_{s=s_0} F(s)$ makes sense even though $F(s)$ is not holomorphic at $s = s_0$.

For an algebraic group G over F , we occasionally write

$$[G] := G(F) \backslash G(\mathbb{A})$$

for the sake of saving space. Also we denote by $\mathcal{A}(G)$ the space of automorphic forms on G . We do not impose the K -finiteness condition on the elements of $\mathcal{A}(G)$ so that the full group $G(\mathbb{A})$ acts on $\mathcal{A}(G)$.

Let $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ be a split quadratic space with \mathcal{V}^\pm maximal isotropic subspaces of dimension m , and let $W = W^+ \oplus W^-$ be the symplectic space with $\dim W^\pm = r$. Because the spaces \mathcal{V} and W are split, one can fix self-dual lattices of \mathcal{V} and W which are compatible with the decompositions $\mathcal{V}^+ \oplus \mathcal{V}^-$ and $W^+ \oplus W^-$, thereby endowing the spaces \mathcal{V}^\pm and W^\pm with \mathcal{O} -integral structures. Choosing bases for these lattices also gives us identifications $\mathcal{V} \cong F^{2m}$ and $W \cong F^{2r}$, which are well-defined up to the natural action of $\operatorname{GL}_{2m}(\mathcal{O})$ and $\operatorname{GL}_{2r}(\mathcal{O})$ respectively. Via these identifications and using our fixed Haar measure dx_v on F_v , we obtain additive Haar measures on the spaces \mathcal{V}^\pm and W^\pm .

Set

$$H = \operatorname{Sp}_{2r} = \operatorname{Sp}(W) \quad \text{and} \quad G = \operatorname{O}_{m,m} = \operatorname{O}(\mathcal{V}).$$

The \mathcal{O} -integral structures on \mathcal{V} and W also endow the groups G and H with \mathcal{O} -integral structures. This determines maximal compact subgroups $G(\hat{\mathcal{O}})$ and $H(\hat{\mathcal{O}})$ of $G(\mathbb{A}_f)$ and $H(\mathbb{A}_f)$ respectively. Picking maximal compact subgroups arbitrarily for the archimedean places, we thus obtain maximal compact subgroups K_G and K_H of $G(\mathbb{A})$ and $H(\mathbb{A})$ respectively.

Now let $\omega = \omega_\psi$ be the Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$, which can be realized on the space $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ of Schwartz functions on the adelic vector space $(\mathcal{V}^+ \otimes W)(\mathbb{A})$. Recall that one has the partial Fourier transform

$$\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) \rightarrow \mathcal{S}((\mathcal{V} \otimes W^+)(\mathbb{A}))$$

defined by

$$\hat{\varphi}(u \oplus v) = \int_{(\mathcal{V}^+ \otimes W^-)(\mathbb{A})} \varphi(x \oplus u) \cdot \psi(\langle x, v \rangle) dx$$

where $u \in (\mathcal{V}^+ \otimes W^+)(\mathbb{A})$ and $v \in (\mathcal{V}^- \otimes W^+)(\mathbb{A})$ and dx is the fixed additive Haar measure on $\mathcal{V}^+ \otimes W^-$.

Following [GI, Sec. 7], we define **the spherical Schwartz function**

$$\varphi^0 = \otimes_v \varphi_v^0 \in (\mathcal{V}^+ \otimes W)(\mathbb{A})$$

to be such that the partial Fourier transform $\hat{\varphi}_v^0$ is

- the characteristic function of $(\mathcal{V} \otimes W^+)(\mathcal{O}_v)$ if v is non-archimedean

- the Gaussian if v is archimedean, namely for $x \in (\mathcal{V} \otimes W^+)(F_v)$

$$\hat{\varphi}_v^0(x) = \begin{cases} \exp(-\pi \langle x, x^* \rangle) & \text{if } v \text{ is real} \\ \exp(-2\pi \langle \bar{x}, x^* \rangle) & \text{if } v \text{ is complex,} \end{cases}$$

where x^* is the image of x under the map $\mathcal{V} \otimes W^+ \rightarrow \mathcal{V} \otimes W^-$ given by $v \otimes w \mapsto v \otimes w^*$ with w^* defined as follows: if $\{e_1, \dots, e_r, f_1, \dots, f_r\}$ is the symplectic basis giving the integral \mathcal{O} -structure on $W = W^+ \oplus W^-$ such that $\langle e_i, f_i \rangle = 1$, then $e_i^* = f_i$.

It is not difficult to show that

$$\varphi_v^0 = \begin{cases} q_v^{-c_v r m / 2} \times \text{the characteristic function of} \\ (\mathcal{V}^+ \otimes W^-)(\mathcal{P}_v^{-c_v}) \oplus (\mathcal{V}^+ \otimes W^+)(\mathcal{O}_v) & \text{if } v \text{ is non-archimedean,} \\ \text{the Gaussian if } v \text{ is archimedean.} \end{cases}$$

Then because

$$\prod_v q_v^{c_v} = |D|,$$

we see that

$$\varphi^0(0) = |D|^{-r m / 2}.$$

Now we define

$$\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ = \text{the } \mathcal{O}(\mathcal{V})(\mathbb{A})\text{-span of the spherical } \varphi^0.$$

Also for each $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$, we define

$$\varphi_{K_H}(v \otimes w) = \int_{K_H} \varphi(v \otimes kw) dk \quad \text{for } v \otimes w \in (\mathcal{V}^+ \otimes W)(\mathbb{A}).$$

3. The Regularized Siegel-Weil formula

The first term identity of the regularized Siegel-Weil formula identifies the leading term of the Siegel Eisenstein series with the regularized theta integral ([Kd-R2, Ik, Ich1, Mo]). In this section, we will review this theory to the extent we need it. Essentially everything in this section is already known.

3.1. Eisenstein series associated with degenerated principal series. First, let us fix some of the notations and review the basics of the Eisenstein series on the split orthogonal group $O_{m,m}$ of rank m associated with a family of degenerate principal series.

For each integer r with $1 \leq r \leq m$, we let P_r be the standard maximal parabolic subgroup of $O_{m,m}$ stabilizing an isotropic subspace of rank r , so that the Levi factor is isomorphic to $GL_r \times O_{m-r, m-r}$. As usual, we denote the degenerate principal series by

$$I^{m,r}(s) = \text{Ind}_{P_r(\mathbb{A})}^{O_{m,m}(\mathbb{A})} ||^s \quad (\text{normalized induction}).$$

Here $||^s$ is the character $|\det|^s$ with \det is on GL_r and extended to P_r trivially on the rest. Recall that a section $f(-, s) \in I^{m,r}(s)$ is said to be standard (or flat) if its restriction to the maximal compact K of $O_{m,m}(\mathbb{A})$ is independent of s . For each standard section $f(-, s) \in I^{m,r}(s)$, we form the Eisenstein series $E^{(m,r)}(g, s; f)$ by

$$E^{(m,r)}(g, s; f) = \sum_{\gamma \in P_r(F) \backslash O_{m,m}(F)} f(\gamma g, s).$$

It is well-known that this sum converges absolutely for $\Re(s) \gg 0$ and admits a meromorphic continuation.

When $r = m$, we call $E^{(m,r)}(g, s; f)$ the Siegel Eisenstein series, and otherwise a non-Siegel Eisenstein series. Also when f is a spherical standard section with $f(1) = 1$, we call the corresponding Eisenstein series the spherical Eisenstein series.

3.2. The Siegel Eisenstein series. Next we will review the theory of the Siegel Eisenstein series of the split orthogonal group and its relations to the Weil representation.

Recall that $G(\mathbb{A})$ has an Iwasawa decomposition

$$G(\mathbb{A}) = P(\mathbb{A})K_G,$$

where P is the Siegel parabolic subgroup that fixes \mathcal{V}^+ . Then for each $g \in G(\mathbb{A})$, we write $g = nm(a)k$ and $|a(g)| = |\det(a)|_{\mathbb{A}}$ following the convention of Kudla and Rallis ([Kd-R2, p.9-10]). Also we denote

$$\rho_m = \frac{m-1}{2}$$

and

$$\rho_{m,r} = r - \rho_m = \frac{2r - m + 1}{2}$$

Now consider the degenerate principal series $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} |\cdot|^s$ (normalized induction). For each standard section Φ , we form the Siegel Eisenstein series

$$E^{(m,m)}(g, s; \Phi) = \sum_{\gamma \in P(F) \backslash G(F)} \Phi(\gamma g, s).$$

It is known that this sum is absolutely convergent for $\Re(s) > \rho_m = \frac{m-1}{2}$, and it has meromorphic continuation together with the functional equation given by

$$E^{(m,m)}(g, s; \Phi) = E^{(m,m)}(g, -s; M_m(s)\Phi),$$

where $M_m(s)$ is the intertwining operator defined by

$$M_m(s)\Phi(g, s) = \int_{N(\mathbb{A})} \Phi(wng, s) dn,$$

for $\Re(s)$ sufficiently large and by meromorphic continuation in general.

The following is due to Kudla and Rallis (Theorem 1.0.1 of [Kd-R1]).

Proposition 3.1. *Let Φ be a standard section. Then the Siegel Eisenstein series $E^{(m,m)}(g, s; \Phi)$ has at most simple poles, and for $\Re(s) > 0$ those occur only in the set*

$$\{\hat{0}, \dots, \rho_m - 1, \rho_m\} = \begin{cases} \{\frac{1}{2}, \frac{3}{2}, \dots, \rho_m\} & \text{if } m \text{ is even;} \\ \{1, \dots, \rho_m\} & \text{if } m \text{ is odd,} \end{cases}$$

where $\hat{0}$ means 0 is omitted.

We are interested in the residues and values of $E^{(m,m)}(g, s; \Phi)$ at $s = \rho_{m,r} = \frac{2r-m+1}{2}$ for various r . We write the Laurent expansion of $E^{(m,m)}(g, s; \Phi)$ at $\rho_{m,r} = \frac{2r-m+1}{2}$ as

$$\begin{aligned} E^{(m,m)}(g, s; \Phi) &= \sum_{d \geq -1}^{\infty} A_d^{(m,r)}(\Phi)(g)(s - \rho_{m,r})^d \\ &= \frac{A_{-1}^{(m,r)}(\Phi)(g)}{s - \rho_{m,r}} + A_0^{(m,r)}(\Phi) + A_1^{(m,r)}(\Phi)(g)(s - \rho_{m,r}) + \dots \end{aligned}$$

Note that each $A_d^{(m,r)}(\Phi)$ is an automorphic form on $G(\mathbb{A})$.

The Siegel Eisenstein series is related to the Weil representation in the following way. Let $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ be the space of Schwartz functions giving rise to the Weil representation, where $\dim W = 2r$ and $\dim \mathcal{V} = \dim(\mathcal{V}^+ \oplus \mathcal{V}^-) = 2m$. Then for each $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$, we have a map

$$\begin{aligned} \Phi^{(m,r)} : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) &\longrightarrow \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} ||^s \\ \varphi &\longmapsto \Phi_\varphi^{(m,r)} \end{aligned}$$

given by

$$\Phi_\varphi^{(m,r)}(g, s) = \omega(g)\varphi(0) \cdot |a(g)|^{s-\rho_{m,r}}.$$

Here we should emphasize that this map is *not* $G(\mathbb{A})$ -intertwining unless $s = \rho_{m,r}$. Also notice that $\Phi_\varphi^{(m,r)}$ is clearly a standard section, and so the poles of the Eisenstein series $E^{(m,m)}(g, s; \Phi_\varphi^{(m,r)})$ are at most simple. By abuse of notation, we sometimes write

$$A_d^{(m,r)}(\Phi_\varphi^{(m,r)}) = A_d^{(m,r)}(\varphi),$$

whenever there is no danger of confusion. Then we have the map

$$A_d^{(m,r)} : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) \longrightarrow \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} ||^{\rho_{m,r}} \longrightarrow \mathcal{A}(G).$$

Now $A_d^{(m,r)}$ is $G(\mathbb{A})$ -intertwining if $A_d^{(m,r)}$ is the leading term of the Laurent expansion i.e. $A_0^{(m,r)}$ if $m = 2r + 1$ and $A_{-1}^{(m,r)}$ otherwise. (Let us also note that $A_d^{(m,r)}$ is $H(\mathbb{A})$ -invariant for all d . But we do not use this fact in this paper, even though it plays a pivotal role in the original work on the regularized Siegel-Weil formula by Kudla-Rallis ([Kd-R2]).)

Now consider the spherical Schwartz function $\varphi^0 \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ defined in §2. One sees that

$$\Phi_{\varphi^0}^{(m,r)}(1, s) = \varphi^0(0) = |D|^{-rm/2}.$$

Hence at $s = \rho_{m,r}$ we have

$$E^{(m,m)}(g, s; \Phi_{\varphi^0}^{(m,r)}) = |D|^{-rm/2} E^{(m,m)}(g, s; \Phi^0), \quad (1)$$

where $\Phi^0(-, s)$ is the spherical section in $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} ||^s$ normalized as $\Phi^0(1, s) = 1$.

3.3. The regularized theta integral. Next we will review the theory of the regularized theta integral and how it is related to the non-Siegel Eisenstein series. In this section (and indeed for this entire paper), we assume that we are outside the convergent range of Weil, namely

$$m > r.$$

(See [Kd-R2] regarding the convergent range.) First for each $\varphi \in \mathcal{S}(\mathcal{V} \otimes W^+)$ let us define the theta integral

$$I^{(m,r)}(g, s; \varphi) = \int_{H(F) \backslash H(\mathbb{A})} \theta(g, h; \varphi) E(h, s) dh,$$

where

$$\theta(g, h; \varphi) = \sum_{v \in (\mathcal{V}^+ \otimes W)(F)} \omega(g, h)\varphi(v),$$

and $E(h, s)$ is the spherical Siegel Eisenstein series on $H(\mathbb{A}) = \text{Sp}_{2r}(\mathbb{A})$ defined by

$$E(h, s) = \sum_{\gamma \in Q_r(F) \backslash H(F)} \Psi(\gamma h, s),$$

where

$$\Psi(h, s) = |a(h)|^{s+\rho'_r}, \quad \rho'_r = \frac{r+1}{2},$$

and Q_r is the Siegel parabolic subgroup of $\mathrm{Sp}_{2r}(\mathbb{A})$. The theta integral might not converge for all φ . However, there is an element z (resp. z') called a regularizing element in the center of universal enveloping algebra of \mathfrak{g}_v (resp. \mathfrak{h}_v) for v real ([Kd-R2]) or in the spherical Hecke algebra of $G(F_v)$ (resp. $H(F_v)$) for v non-archimedean ([Ich1]) so that

$$\begin{aligned} \omega(z) &= \omega(z'), \\ \omega(g, h)\omega(z) &= \omega(z)\omega(g, h) \end{aligned}$$

(i.e. the action of z (and hence z') commutes with the action of $G(\mathbb{A}) \times H(\mathbb{A})$), and such that the function $\theta(g, -, \omega(z)\varphi)$ is rapidly decreasing on any Siegel domain of $\mathrm{Sp}_{2r}(\mathbb{A})$. Thus the theta integral $I^{(m,r)}(g, s; \omega(z)\varphi)$ converges for $\Re(s) \gg 0$ for all φ . We call it the regularized theta integral.

By unfolding the Eisenstein series inside the regularized theta integral, we have

$$\begin{aligned} & I^{(m,r)}(g, s; \omega(z)\varphi) \\ &= \int_{\mathrm{GL}_r(F) \backslash \mathrm{GL}_r(\mathbb{A})} \sum_{\gamma \in P_r(F) \backslash G(F)} \sum_{\alpha \in \mathrm{GL}_r(F)} \omega(\gamma g, m(\alpha a)) \omega(z) \hat{\varphi}_{K_H}(w_0) |a|^{s-\rho'_r} da \\ &= \sum_{\gamma \in P_r(F) \backslash G(F)} \int_{\mathrm{GL}_r(\mathbb{A})} \omega(\gamma g, m(a)) \omega(z) \hat{\varphi}_{K_H}(w_0) |a|^{s-\rho'_r} da \\ &= \sum_{\gamma \in P_r(F) \backslash G(F)} \int_{\mathrm{GL}_r(\mathbb{A})} \omega(\gamma g, 1) \omega(z) \hat{\varphi}_{K_H}({}^t a w_0) |a|^{s+m-\rho'_r} da. \end{aligned}$$

Here w_0 is the element in $\mathcal{S}(\mathcal{V} \otimes W^+)$ corresponding to the element of the form $v_1 \otimes e_1 + v_2 \otimes e_2 + \cdots + v_r \otimes e_r$, where v_i 's and e_i 's are the obvious basis elements of \mathcal{V} and W , and P_r is the parabolic subgroup which stabilizes the isotropic r -plane in \mathcal{V} spanned by v_1, \dots, v_r . (See [Kd-R2, p.48].)

Now define

$$f^{(m,r)}(g, s; \varphi) = \int_{\mathrm{GL}_r(\mathbb{A})} \omega(g, 1) \hat{\varphi}({}^t a w_0) |a|^{s+m-\rho'_r} da$$

and so

$$I^{(m,r)}(g, s; \omega(z)\varphi) = \sum_{\gamma \in P_r(F) \backslash G(F)} f^{(m,r)}(\gamma g, s; \omega(z)\varphi_{K_H}).$$

Here note that

$$f^{(m,r)}(-, s; \varphi) \in \mathrm{Ind}_{P_r(\mathbb{A})}^{G(\mathbb{A})} | \cdot |^s \quad (\text{normalized induction}),$$

and the map

$$\begin{aligned} \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) &\rightarrow \mathrm{Ind}_{P_r(\mathbb{A})}^{G(\mathbb{A})} | \cdot |^s \\ \varphi &\mapsto f^{(m,r)}(-, s; \varphi) \end{aligned}$$

is a $G(\mathbb{A})$ -intertwining operator for $\Re(s)$ sufficiently large. Indeed, the integral for $f^{(m,r)}(g, s; \varphi)$ converges for $\Re(s) > \frac{2r-m+1}{2}$. Also

$$f^{(m,r)}(g, s; \omega(z)\varphi) = P_z(s) f^{(m,r)}(g, s; \varphi),$$

where $P_z(s)$ is a holomorphic function in s depending on z . In fact $P_z(s)$ is a polynomial in s if we choose our regularizing element z from the center of universal enveloping algebra at an archimedean

place, and $P_z(s) \in \mathbb{C}[q_v^{-s}, q_v^s]$ if z is from the spherical Hecke algebra at a non-archimedean v . For example, if we choose z from the center of universal enveloping algebra at a real place, it is explicitly given by

$$P_z(s) = \prod_{i=0}^{r-1} \left(\left(s - \frac{r-1}{2} + i \right)^2 - (m-r)^2 \right).$$

(See [Kd-R2, p.51].) In any case, we have

$$I^{(m,r)}(g, s; \omega(z)\varphi) = P_z(s)\mathcal{E}^{(m,r)}(g, s; \varphi),$$

where

$$\mathcal{E}^{(m,r)}(g, s; \varphi) = \sum_{\gamma \in P_r(F) \backslash G(F)} f^{(m,r)}(\gamma g, s; \varphi_{K_H}),$$

for the region $\Re(s) > \max\{r - \rho'_r, \rho'_r\}$. (See the second paragraph of [Kd-R2, p.53].)

Also, z' as an operator on the space of automorphic forms on H is self-adjoint for the Peterson inner product with the property

$$z' * E(h, s) = P_z(s)E(h, s).$$

It is known that at $s = \rho'_r = \frac{r+1}{2}$ the non-Siegel Eisenstein series $\mathcal{E}^{(m,r)}(g, s; \varphi)$ has at most a double pole, and if $m \geq 2r + 1$, then it only has a simple pole. (See [Kd-R2, bottom of p.53].) Accordingly, we call

$$\begin{aligned} m \geq 2r + 1 \quad i.e. \quad r \leq \frac{m}{2} - 1 : \mathbf{1st \ term \ range} \\ r + 1 \leq m \leq 2r \quad i.e. \quad \frac{m}{2} \leq r \leq m - 1 : \mathbf{2nd \ term \ range}. \end{aligned}$$

Also we sometimes call

$$m = 2r + 1 : \mathbf{boundary \ case}.$$

The rationale for this terminology is that for the 1st term range, we will need only the first term identity, and for the 2nd term range we will need the 2nd term identity. Also, for the boundary case, the first term identity differs from the non-boundary case. This will be clearer in due course.

We write the Laurent expansion of $\mathcal{E}^{(m,r)}(g, s; \varphi)$ at $s = \rho'_r = \frac{r+1}{2}$ as

$$\begin{aligned} \mathcal{E}^{(m,r)}(g, s; \varphi) &= \sum_{d \geq -2}^{\infty} B_d^{(m,r)}(\varphi)(g)(s - \rho'_r)^d \\ &= \frac{B_{-2}^{(m,r)}(\varphi)(g)}{(s - \rho'_r)^2} + \frac{B_{-1}^{(m,r)}(\varphi)(g)}{(s - \rho'_r)} + B_0^{(m,r)}(\varphi)(g) + \dots \end{aligned}$$

Note that each $B_d^{(m,r)}$ is a map from $\mathcal{S}((V^+ \otimes W)(\mathbb{A}))$ to $\mathcal{A}(G)$. Unlike the case of the Siegel Eisenstein series, however, all the $B_d^{(m,r)}$'s are $G(\mathbb{A})$ -intertwining. This follows from the fact that $\omega(z)$ commutes with the action of $G(\mathbb{A})$ on $\mathcal{S}((V^+ \otimes W)(\mathbb{A}))$. To be more specific, note that

$$\begin{aligned} \mathcal{E}^{(m,r)}(g, s; \varphi) &= \frac{1}{P_z(s)} I^{(m,r)}(g, s; \omega(z)\varphi) \\ &= \frac{1}{P_z(s)} \int_{H(F) \backslash H(\mathbb{A})} \theta(g, h; \omega(z)\varphi) E(h, s) dh \\ &= \int_{H(F) \backslash H(\mathbb{A})} \theta(g, h; \omega(z)\varphi) \frac{E(h, s)}{P_z(s)} dh. \end{aligned}$$

If we write the Laurent series of $\frac{E(h,s)}{P_z(s)}$ at $s = \rho'_r$ as

$$\frac{E(h,s)}{P_z(s)} = \sum_{d=-2}^{\infty} C_d(h)(s - \rho'_r)^d = \frac{C_{-2}(h)}{(s - \rho'_r)^2} + \frac{C_{-1}(h)}{s - \rho'_r} + C_0(h) + \dots,$$

then one sees that

$$B_d^{(m,r)}(\varphi)(g) = \int_{H(F) \backslash H(\mathbb{A})} \theta(g, h; \omega(z)\varphi) C_d(h) dh.$$

Then the assertion follows from

$$\theta(gg', h; \omega(z)\varphi) = \theta(g, h; \omega(z)\omega(g', 1)\varphi)$$

for all $g' \in G(\mathbb{A})$.

Let us also note also that the first non-zero $B_d^{(m,r)}$, which turns out to be either $B_{-1}^{(m,r)}$ or $B_{-2}^{(m,r)}$ depending on the range, is $H(\mathbb{A})$ -invariant, but we do not use this fact, even though it is pivotal in the theory of the regularized Siegel-Weil formula.

Finally, let us assume $\varphi^0 \in \mathcal{S}((V^+ \otimes W)(\mathbb{A}))$ is our spherical Schwartz function as defined in §2. Then $f^{(m,r)}(-, s; \varphi^0) \in \text{Ind}_{P_r(\mathbb{A})}^{G(\mathbb{A})} | \cdot |^s$ is a spherical section if $\Re(s)$ is sufficiently large. Let $f^0 \in \text{Ind}_{P_r(\mathbb{A})}^{G(\mathbb{A})} | \cdot |^s$ be the spherical section with the property $f^0(1, s) = 1$. Then it was shown in [GI] that

Lemma 3.2.

$$\mathcal{E}^{(m,r)}(g, s; \varphi^0) = |D|^{-r(s+m-\frac{r+1}{2})/2} \prod_{i=1}^r \frac{\xi(s+m-\frac{r+1}{2}-(i-1))}{\xi(i)} E^{(m,r)}(g, s; f^0).$$

(Here recall our convention $\xi(1) = \text{Res}_{s=1} \xi(s)$.)

Proof. This follows from the fact that

$$f^{(m,r)}(1, s; \varphi^0) = |D|^{-r(s+m-\rho'_r)/2} \prod_{i=1}^r \frac{\xi(s+m-\frac{r+1}{2}-(i-1))}{\xi(i)}.$$

(See [GI, Lemma 7.4] for the detail computation.) \square

3.4. The regularized Siegel-Weil formula (first term identity). Finally, we can state the first term identity of the regularized Siegel-Weil formula in the first term range. To be precise, we have the following first term identity, which is essentially due to Moeglin ([Mo]) and completed by Jiang and Soudry ([J-S]), though their proof heavily depends on Kudla and Rallis ([Kd-R2]).

Proposition 3.3. *Assume $r < \frac{m-1}{2}$, i.e. $m > 2r + 1$. Then for all $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$, there exists a standard section $\Phi(-, s) \in \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} | \cdot |^s$ such that*

$$B_{-1}^{(m,r)}(\varphi)(g) = A_{-1}^{(m,m-r-1)}(\Phi)(g)$$

for all $g \in G(\mathbb{A})$. Moreover if φ is spherical, then Φ can be chosen to be a spherical section.

Proof. The first part is [J-S, Theorem 2.4]. The second part follows from the fact that $A_{-1}^{(m,m-r-1)}$ is a $G(\mathbb{A})$ -intertwining map from $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} | \cdot |^s$ to $\mathcal{A}(G)$. Namely, for any Φ corresponding to a

spherical φ in this formula, if we take Φ' to be such that $\Phi'(g) = \int_{K_G} \Phi(gk) dk$ then (assuming dk is chosen so that the measure of K_G is 1) we have

$$A_{-1}^{(m, m-r-1)}(\Phi')(g) = \int_{K_G} A_{-1}^{(m, m-r-1)}(\Phi)(gk) = \int_{K_G} B_{-1}^{(m, r)}(\varphi)(gk) = B_{-1}^{(m, r)}(\varphi)(g).$$

□

4. Spherical Second Term Identity

In this section and the next, we shall derive a certain form of the second term identity of the Siegel-Weil formula (the weak second term identity) in the second term range; it is an identity between the second terms of the Laurent series expansions of the Siegel Eisenstein series and the non-Siegel Eisenstein series resulting from the regularized theta integral on the orthogonal group. We first derive the spherical second term identity in this section and in the following section, we will extend it to the weak second term identity.

4.1. Spherical Eisenstein Series. Recall that at the beginning of the previous section, we defined the Eisenstein series $E^{(m, r)}(g, s; f)$ for $f(-, s) \in I^{m, r}(s)$. In this section, by studying this Eisenstein series when f is spherical, we will derive spherical Siegel-Weil formulas both for the first terms and the second terms. For this purpose, for each $s_0 \in \mathbb{C}$ we write the Laurent series expansion as

$$E^{(m, r)}(g, s; f) = \sum_{d \gg -\infty} (s - s_0)^d E_d^{(m, r)}(g, s_0; f).$$

Also for the spherical section f^0 with the property that $f^0(1, s) = 1$, we simply write

$$E^{(m, r)}(g, s) := E^{(m, r)}(g, s; f^0).$$

Now let $Q = P_1$, so that its Levi factor is $\mathbb{G}_m \times O_{m-1, m-1}$. We consider the constant term $E_Q^{(m, r)}$ of the above Eisenstein series $E^{(m, r)}(g, s; f^0)$ along the parabolic Q for the spherical section f^0 with the property that $f^0(1, s) = 1$. The constant term $E_Q^{(m, r)}$ is an automorphic form on $\mathbb{G}_m(\mathbb{A}) \times O_{m-1, m-1}(\mathbb{A})$ which can be computed as follows. (See also [GI, Appendix. B].)

Proposition 4.1. *The constant term $E_Q^{(m, r)}$ of the Eisenstein series $E^{(m, r)}$ for the spherical section f^0 , as an automorphic form on $\mathbb{G}_m(\mathbb{A}) \times O_{m-1, m-1}(\mathbb{A})$, can be expressed as follows.*

- If $1 < r < m$, then

$$\begin{aligned} E_Q^{(m, r)}((a, g), s) &= |a|^{s+m-\frac{r+1}{2}} E^{(m-1, r-1)}(g, s + \frac{1}{2}) \\ &\quad + |a|^r E^{(m-1, r)}(g, s) \frac{\xi(s+m-\frac{3r+1}{2})}{\xi(s+m-\frac{r+1}{2})} \\ &\quad + |a|^{-s+m-\frac{r+1}{2}} E^{(m-1, r-1)}(g, s - \frac{1}{2}) \frac{\xi(2s)}{\xi(2s+r-1)} \\ &\quad \cdot \frac{\xi(s+\frac{r-1}{2})}{\xi(s+m-\frac{r+1}{2})} \frac{\xi(s-m+\frac{3r+1}{2})}{\xi(s+\frac{r+1}{2})}. \end{aligned}$$

- If $r = m$, then

$$\begin{aligned} E_Q^{(m, m)}((a, g), s) &= |a|^{s+\frac{m-1}{2}} E^{(m-1, m-1)}(g, s + \frac{1}{2}) \\ &\quad + |a|^{-s+\frac{m-1}{2}} E^{(m-1, m-1)}(g, s - \frac{1}{2}) \frac{\xi(2s)}{\xi(2s+m-1)}. \end{aligned}$$

- If $r = 1$, then

$$\begin{aligned} E_Q^{(m,1)}((a, g), s) &= |a|^{s+m-1} \\ &+ |a| E^{(m-1,1)}(g, s) \frac{\xi(s+m-2)}{\xi(s+m-1)} \\ &+ |a|^{-s+m-1} \frac{\xi(s)}{\xi(s+m-1)} \cdot \frac{\xi(s-m+2)}{\xi(s+1)}. \end{aligned}$$

Here $\xi(s)$ is the complete normalized zeta function of the number field F with the functional equation $\xi(s) = \xi(1-s)$, and $(a, g) \in \mathbb{G}_m(\mathbb{A}) \times O_{m-1, m-1}(\mathbb{A})$.

Due to frequent use of those formulas, we simplify the notations by setting

$$\begin{aligned} F^{(m,r)}(s) &= \frac{\xi(s+m-\frac{3r+1}{2})}{\xi(s+m-\frac{r+1}{2})} \\ G^{(m,r)}(s) &= \frac{\xi(2s)}{\xi(2s+r-1)} \frac{\xi(s+\frac{r-1}{2})}{\xi(s+m-\frac{r+1}{2})} \frac{\xi(s-m+\frac{3r+1}{2})}{\xi(s+\frac{r+1}{2})} \\ H^{(m)}(s) &= \frac{\xi(2s)}{\xi(2s+m-1)}, \end{aligned}$$

for the factors containing the zeta functions in the first and second formulas above. Also we write the Laurent series expansion for $F^{(m,r)}(s)$ at $s = s_0$ as

$$F^{(m,r)}(s) = \sum_{d \gg -\infty} (s - s_0)^d F_d^{(m,r)}(s_0),$$

and similarly for $G^{(m,r)}(s)$ and $H^{(m)}(s)$. Also, throughout this section, we suppress g and a from the notation of the spherical Eisenstein series whenever there is no danger of confusion. Namely for $E^{(m,r)}(g, s)$ and $|a|$ we simply write $E^{(m,r)}(s)$ and $||$ respectively, and for the Laurent series of $E^{(m,r)}(g, s)$, we write

$$E^{(m,r)}(s) = \sum_{d \gg -\infty} (s - s_0)^d E_d^{(m,r)}(s_0).$$

4.2. A lemma. The following elementary lemma will be crucial for our computation and repeatedly used.

Lemma 4.2. *Let E_1, \dots, E_k and F_1, \dots, F_l be automorphic forms on $O_{m,m}(\mathbb{A})$. Also let r_1, \dots, r_k and s_1, \dots, s_l be real numbers such that all the r_i 's are distinct and all the s_j 's are distinct but some of the r_i 's might be the same as some of the s_j 's. Then if*

$$\sum_{i=1}^k ||^{r_i} E_i + \sum_{j=1}^l ||^{s_j} \ln || F_j = 0$$

as an automorphic form on $\mathbb{G}_m(\mathbb{A}) \times O_{m,m}(\mathbb{A})$, then all the E_i 's and F_j 's are zero.

Proof. Let us fix an embedding $\mathbb{R}^+ \subset \mathbb{A}^\times$ by using one of the archimedean places, and view the functions $||^{r_i}$ and $||^{s_j} \ln ||$ as functions on \mathbb{R}^+ . Then those functions are known to be linearly independent over \mathbb{C} . Thus the lemma follows. \square

4.3. Spherical first term identity for 1st term range $m \geq 2r + 1$. Using Proposition 4.1, we will compute the spherical first term identity between the spherical non-Siegel Eisenstein series $E^{(m,r)}(s)$ at $s = \frac{r+1}{2}$ and the spherical Siegel Eisenstein series $E^{(m,m)}(s)$ at $s = \frac{m-2r-1}{2} = -\rho_{m,r}$ in the 1st term range, i.e. when $m \geq 2r + 1$. It is basically a refinement of the spherical case of Proposition 3.3.

Lemma 4.3 (Spherical First Term Identity for 1st term range). *Assume $r < \frac{m-1}{2}$, i.e. $m > 2r + 1$. Then there exists a non-zero constant $c_{m,r}$ independent of g and f such that*

$$E_{-1}^{(m,r)}\left(\frac{r+1}{2}\right) = c_{m,r} E_{-1}^{(m,m)}\left(\frac{m-2r-1}{2}\right),$$

where $c_{m,r}$ can be explicitly computed as

$$c_{m,r} = \prod_{i=0}^{r-1} \frac{\xi(m-2r+i)}{\xi(m-i)} \cdot \prod_{i=r+1}^{m-(r+1)} \frac{\xi(2i)}{\xi(i)}.$$

Proof. The first part of the lemma is immediate from Proposition 3.3. So let us explicitly compute $c_{m,r}$. First we need to compute $c_{m,1}$. (Let us note that $m \geq 4$ since $m > 2r + 1$.) Note that the first term identity gives

$$E_{-1,Q}^{(m,1)}(1) = c_{m,1} E_{-1,Q}^{(m,m)}\left(\frac{m-3}{2}\right),$$

which together with Proposition 4.1 gives

$$\begin{aligned} & ||E_{-1}^{(m-1,1)}(1) \frac{\xi(m-1)}{\xi(m)} + ||^{m-2} \frac{\xi(1)}{\xi(m)} \frac{\xi(3-m)}{\xi(2)} \\ &= c_{m,1} \left(||^{m-2} E_{-1}^{(m-1,m-1)}\left(\frac{m-2}{2}\right) + ||E_{-1}^{(m-1,m-1)}\left(\frac{m-4}{2}\right) \frac{\xi(m-3)}{\xi(2m-4)} \right). \end{aligned}$$

(Here recall our convention that $\xi(1) = \operatorname{Res}_{s=1} \xi(s)$.)

Now Lemma 4.2 allows us to “extract” all the terms containing $||^{m-2}$. Namely we can equate the second term of the left hand side with the first term of the right hand side. Then we obtain

$$\frac{\xi(1)}{\xi(m)} \frac{\xi(3-m)}{\xi(2)} = c_{m,1} E_{-1}^{(m-1,m-1)}\left(\frac{m-2}{2}\right).$$

Notice that this implies that, as an automorphic form, $E_{-1}^{(m-1,m-1)}\left(\frac{m-2}{2}\right)$ is a constant function. Let

$$E_{-1}^{(m,m)}\left(\frac{m-1}{2}\right) = \lambda_m$$

so that

$$c_{m,1} = \frac{\xi(1)\xi(m-2)}{\xi(2)\xi(m)} \frac{1}{\lambda_{m-1}}.$$

(Here we used the functional equation $\xi(3-m) = \xi(m-2)$.) But λ_{m-1} can be computed as follows:

$$\lambda_m = E_{-1,Q}^{(m,m)}\left(\frac{m-1}{2}\right) = E_{-1}^{(m-1,m-1)}\left(\frac{m-2}{2}\right) \frac{\xi(m-1)}{\xi(2m-2)} = \lambda_{m-1} \frac{\xi(m-1)}{\xi(2m-2)}.$$

(To obtain this, we computed $E_{-1,Q}^{(m,m)}\left(\frac{m-1}{2}\right)$ by Proposition 4.1 and used $E_{-1}^{(m-1,m-1)}\left(\frac{m}{2}\right) = 0$ because $E^{(m-1,m-1)}(s)$ converges at $\frac{m}{2}$.) This gives

$$\lambda_m = \frac{\xi(m-1)\xi(m-2) \cdots \xi(2)}{\xi(2m-2)\xi(2m-4) \cdots \xi(4)} \cdot \lambda_2.$$

But it is easy to compute $\lambda_2 = \frac{\xi(1)}{\xi(2)}$. Hence we get the formula for $c_{m,1}$.

Now let us treat the case $r \geq 2$. First, since $E_{-1}^{(m,r)}(\frac{r+1}{2}) = c_{m,r} E_{-1}^{(m,m)}(\frac{m-2r-1}{2})$, we have $E_{-1,Q}^{(m,r)}(\frac{r+1}{2}) = c_{m,r} E_{-1,Q}^{(m,m)}(\frac{m-2r-1}{2})$. Then by Proposition 4.1,

$$\begin{aligned} & ||^m E_{-1}^{(m-1,r-1)}(\frac{r+2}{2}) + ||^r E_{-1}^{(m-1,r)}(\frac{r+1}{2}) \frac{\xi(m-r)}{\xi(m)} \\ & \quad + ||^{m-r-1} E_{-1}^{(m-1,r-1)}(\frac{r}{2}) \frac{\xi(r+1)}{\xi(2r)} \frac{\xi(r)}{\xi(m)} \frac{\xi(2r+1-m)}{\xi(r+1)} \\ & = c_{m,r} \left(||^{m-r-1} E_{-1}^{(m-1,m-1)}(\frac{m-2r}{2}) + ||^r E_{-1}^{(m-1,m-1)}(\frac{m-2r-2}{2}) \frac{\xi(m-2r-1)}{\xi(2m-2r-2)} \right). \end{aligned}$$

Note that $m-r-1 \neq r$ and $m-r-1 \neq m$. Then by lemma 4.2 we can extract the terms containing $||^{m-r-1}$, and obtain

$$E_{-1}^{(m-1,r-1)}(\frac{r}{2}) \frac{\xi(m-2r)}{\xi(m)} \frac{\xi(r)}{\xi(2r)} = c_{m,r} E_{-1}^{(m-1,m-1)}(\frac{m-2r}{2}).$$

(Here note that we used the functional equation $\xi(2r+1-m) = \xi(m-2r)$.) Also by definition of $c_{m,r}$ we have

$$E_{-1}^{(m-1,r-1)}(\frac{r}{2}) = c_{m-1,r-1} E_{-1}^{(m-1,m-1)}(\frac{m-2r}{2}).$$

Those two together give the recursive relation

$$c_{m,r} = c_{m-1,r-1} \cdot \frac{\xi(m-2r)}{\xi(m)} \frac{\xi(r)}{\xi(2r)},$$

which gives

$$c_{m,r} = \prod_{i=0}^{r-2} \frac{\xi(m-2r+i)}{\xi(m-i)} \cdot \prod_{i=2}^r \frac{\xi(i)}{\xi(2i)} \cdot c_{m-(r-1),1}.$$

Hence the formula for $c_{m,1}$ immediately gives the formula for $c_{m,r}$. \square

Next we will derive the first term identity on the boundary. First, we need the following lemma, which we will frequently use later.

Lemma 4.4.

- (i) $E_{-2}^{(m,r)}(s_0) = 0$ for $s_0 \geq \frac{r+2}{2}$.
- (ii) $E_{-2}^{(2r+1,r)}(\frac{r+1}{2}) = 0$.
- (iii) $E_{-1}^{(m,r)}(\frac{r+3}{2}) = 0$.

Proof.

- (i) We show this by induction on r . So first let us assume that $r = 1$. This can be shown by induction on m by using the third equation of Proposition 4.1. So assume the lemma holds for some r . Then by the first equation of Proposition 4.1 together with the induction hypothesis,

we get

$$\begin{aligned}
E_{-2,Q}^{(m,r+1)}(s_0) &= ||^{r+1}E_{-2}^{(m-1,r+1)}(s_0) \frac{\xi(s_0 + m - \frac{3r+4}{2})}{\zeta(s_0 + m - \frac{r+1}{2})} \\
E_{-2,Q}^{(m-1,r+1)}(s_0) &= ||^{r+1}E_{-2}^{(m-2,r+1)}(s_0) \frac{\xi(s_0 + m - 1 - \frac{3r+4}{2})}{\zeta(s_0 + m - 1 - \frac{r+1}{2})} \\
E_{-2,Q}^{(m-2,r+1)}(s_0) &= ||^{r+1}E_{-2}^{(m-3,r+1)}(s_0) \frac{\xi(s_0 + m - 2 - \frac{3r+4}{2})}{\zeta(s_0 + m - 2 - \frac{r+1}{2})} \\
&\vdots \\
&\vdots \\
&\vdots \\
E_{-2,Q}^{(r+2,r+1)}(s_0) &= ||^{r+1}E_{-2}^{(r+1,r+1)}(s_0) \frac{\xi(s_0 + r + 2 - \frac{3r+4}{2})}{\zeta(s_0 + r + 2 - \frac{r+1}{2})}.
\end{aligned}$$

Notice that in the last equation, $E_{-2}^{(r+1,r+1)}(s_0) = 0$ because the spherical Siegel Eisenstein series never has a double pole. (Also note that $\xi(s + r + 2 - \frac{3r+4}{2})$ is holomorphic at $s_0 \geq \frac{r+3}{2}$.) So $E_{-2,Q}^{(r+2,r+1)}(s_0) = 0$, i.e. $E_{-2}^{(r+2,r+1)}(s_0) = 0$. Hence $E_{-2,Q}^{(r+3,r+1)}(s_0) = 0$, i.e. $E_{-2}^{(r+3,r+1)}(s_0) = 0$ etc, and we get $E_{-2}^{(m,r+1)}(s_0) = 0$. Thus the induction is complete and (i) has been proven.

(ii) By proposition 4.1, we have

$$\begin{aligned}
E_{-2,Q}^{(2r+2,r)}\left(\frac{r+1}{2}\right) &= ||^{2r+2}E_{-2}^{(2r+1,r-1)}\left(\frac{r+2}{2}\right) + ||^rE_{-2}^{(2r+1,r)}\left(\frac{r+1}{2}\right)F^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\
&\quad + ||^rE_{-2}^{(2r+1,r-1)}\left(\frac{r}{2}\right)G^{(2r+2,r)}\left(\frac{r+1}{2}\right).
\end{aligned}$$

But $E_{-2,Q}^{(2r+2,r)}\left(\frac{r+1}{2}\right)$, $E_{-2}^{(2r+1,r-1)}\left(\frac{r+2}{2}\right)$ and $E_{-2}^{(2r+1,r-1)}\left(\frac{r}{2}\right)$ are all zero because, in general, the non-Siegel Eisenstein series $E^{(m,r)}(s)$ does not have a double pole when $m > 2r + 1$. Hence $E_{-2}^{(2r+1,r)}\left(\frac{r+1}{2}\right) = 0$.

(iii) This can be proven in a similar way as (i) by induction. But this time, we use the fact that the Siegel Eisenstein series $E^{(r+1,r+1)}(s)$ does not have a pole at $s = \frac{r+4}{2}$. The detail is left to the reader.

□

Then we have

Proposition 4.5 (Spherical First Term Identity on the boundary). *Assume $r = \frac{m-1}{2}$, i.e. $m = 2r + 1$. Then*

$$E_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) = c_r E_0^{(2r+1,2r+1)}(0),$$

where

$$c_r = c_{2r+2,r} \frac{\xi(1)}{2\xi(r+2)} = \frac{1}{2} \cdot \prod_{i=1}^{r+1} \frac{\xi(i)}{\xi(r+i)}.$$

(Recall $\xi(1) = \operatorname{Res}_{s=1} \xi(s)$.)

Proof. First assume $r > 1$. The first equation of Proposition 4.1 gives us

$$\begin{aligned} E_{-1,Q}^{(2r+2,r)}\left(\frac{r+1}{2}\right) &= ||^{2r+2}E_{-1}^{(2r+1,r-1)}\left(\frac{r+2}{2}\right) + ||^r E_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) F^{(2r+2,r)}\left(\frac{r+1}{2}\right) \\ &\quad + ||^{r+1} E_{-1}^{(2r+1,r-1)}\left(\frac{r}{2}\right) G^{(2r+2,r)}\left(\frac{r+1}{2}\right). \end{aligned} \quad (2)$$

(To compute this, we used Lemma 4.4 (ii).) Also Lemma 4.3 gives us

$$E_{-1,Q}^{(2r+2,r)}\left(\frac{r+1}{2}\right) = c_{2r+2,r} E_{-1,Q}^{(2r+2,2r+2)}\left(\frac{1}{2}\right),$$

which, combined with the second equation of Proposition 4.1, gives

$$\begin{aligned} E_{-1,Q}^{(2r+2,r)}\left(\frac{r+1}{2}\right) &= c_{2r+2,r} \left(||^{r+1} E_{-1}^{(2r+1,2r+1)}(1) + ||^r E_0^{(2r+1,2r+1)}(0) H_{-1}^{(2r+1,2r+1)} \right). \end{aligned} \quad (3)$$

Therefore by equating (2) and (3), and (again by Lemma 4.2) looking at the terms containing $||^r$, we obtain

$$E_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) \frac{\xi(r+2)}{\xi(2r+2)} = c_{2r+2,r} E_0^{(2r+1,2r+1)}(0) \frac{\xi(1)}{2\xi(2r+2)},$$

namely

$$E_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) = c_{2r+2,r} E_0^{(2r+1,2r+1)}(0) \frac{\xi(1)}{2\xi(r+2)},$$

where

$$c_r = c_{2r+2,r} \frac{\xi(1)}{2\xi(r+2)},$$

and Lemma 4.3 gives the explicit expression for c_r as in the proposition.

If $r = 1$, the same identity can be obtained by more direct computation, and is done in [GI]. \square

4.4. Identities in the second term range: Idea of the proof. Next we consider the 2nd term range $r+1 \leq m \leq 2r$. In this range, the non-Siegel Eisenstein series $E^{(m,r)}(g, s; f)$ can have a double pole, and indeed as we will show below, it does have a double pole for a spherical f . In this case, we shall establish not only a spherical first term identity but also a spherical second term identity. Both of these follow from the same idea which we have already exploited in the proof of the first term identities in the first term range (Lemma 4.3) and on the boundary (Proposition 4.5). We shall presently give a brief description of the idea before plunging into the details of the proof.

Roughly speaking, our proof of the identities for $(O_{m,m}, \mathrm{Sp}_{2r})$ is based on induction on the quantity

$$j = 2r - m.$$

Observe that one has

$$\begin{aligned} j < -1 &\iff \text{first term range;} \\ j = -1 &\iff \text{boundary;} \\ j > -1 &\iff \text{second term range.} \end{aligned}$$

We have shown the spherical first term identities when $j \leq -1$ and let us see how we can deduce the spherical first and second term identities for $j = 0$ from the case $j = -1$.

We start with the first term identity on the boundary (*i.e.* when $m = 2r + 1$ and $j = -1$), which is given in Proposition 4.5 and Lemma 4.4 (ii):

$$\begin{aligned} E_{-2}^{(2r+1,r)}\left(\frac{r+1}{2}\right) &= 0, \\ E_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) &= c_r E_0^{(2r+1,2r+1)}(0). \end{aligned}$$

These are identities of automorphic forms on the group $O_{2r+1,2r+1}(\mathbb{A})$. By considering the constant term along Q , we have

$$\begin{aligned} E_{-2,Q}^{(2r+1,r)}\left(\frac{r+1}{2}\right) &= 0, \\ E_{-1,Q}^{(2r+1,r)}\left(\frac{r+1}{2}\right) &= c_r E_{0,Q}^{(2r+1,2r+1)}(0). \end{aligned}$$

Now using Proposition 4.1, we can compute both sides of these two equations in terms of Eisenstein series on the lower rank group $O_{2r,2r}(\mathbb{A})$. This results in certain relations between Siegel and non-Siegel Eisenstein series on $O_{2r,2r}(\mathbb{A})$. The relation that follows from the first equation is simply the spherical first term identity for $m = 2r$ (*i.e.* $j = 0$), and the one that follows from the second equation gives the spherical second term identity for $m = 2r$ (*i.e.* $j = 0$).

Using $m = 2r$ as the base case, we will then show the spherical second term identity for the remaining cases $r + 1 \leq m < 2r$ by induction. For the first term identity, we further prove the case $m = 2r - 1$ (*i.e.* $j = 1$) by a similar method, and then show the remaining cases $r + 1 \leq m < 2r - 1$ by induction by using the $m = 2r - 1$ case as a base step.

While the above idea of deriving the spherical second term identity on $O_{m,m}$ from the first term identity of $O_{m+1,m+1}$ is quite simple and elegant, its execution requires some rather lengthy detailed computations. We apologize for not being able to package these computations more elegantly.

4.5. Spherical first term identities in 2nd term range $r < m \leq 2r$. Let us first consider the first term identity.

Proposition 4.6 (Spherical First Term Identity for 2nd term range). *Assume that we are in the 2nd term range $\frac{m-1}{2} \leq r \leq m - 1$, *i.e.* $r + 1 \leq m \leq 2r$. Then*

$$E_{-2}^{(m,r)}\left(\frac{r+1}{2}\right) = d_{m,r} E_{-1}^{(m,m)}\left(\frac{2r-m+1}{2}\right),$$

in which

$$\begin{aligned} d_{2r,r} &= c_{2r,r-1} \frac{\xi(1)}{\xi(r)} \frac{\xi(2r)}{\xi(r+1)} \\ d_{m-1,r} &= d_{m,r} \frac{\xi(m)}{\xi(m-r)}, \quad \text{for } m \leq 2r - 1 \end{aligned}$$

(Recall that $\xi(1) = \operatorname{Res}_{s=1} \xi(s)$.)

Proof. Let us note that the case $r = 1$ and $m = 2$ is essentially done in [GI, Appendix. B]. So we assume that $r > 1$.

The case $m = 2r$

As we mentioned above, we first prove the case $m = 2r$. Namely from the first formula of Proposition 4.1, we have

$$\begin{aligned} E_{-2,Q}^{(2r+1,r)}\left(\frac{r+1}{2}\right) &= ||^{2r+1}E_{-2}^{(2r,r-1)}\left(\frac{r+2}{2}\right) + ||^rE_{-2}^{(2r,r)}\left(\frac{r+1}{2}\right)F^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\ &\quad + ||^rE_{-1}^{(2r,r-1)}\left(\frac{r}{2}\right)G_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right). \end{aligned}$$

But as we mentioned at the beginning of the section,

$$E_{-2,Q}^{(2r+1,r)}\left(\frac{r+1}{2}\right) = 0$$

by Lemma 4.4 (ii). Then by extracting terms having $||^r$ (Lemma 4.2) we have

$$0 = E_{-2}^{(2r,r)}\left(\frac{r+1}{2}\right)F^{(2r+1,r)}\left(\frac{r+1}{2}\right) + E_{-1}^{(2r,r-1)}\left(\frac{r}{2}\right)G_{-1}^{(2r+1,r)},$$

from which, by computing $F^{(2r+1,r)}\left(\frac{r+1}{2}\right)$ and $G_{-1}^{(2r+1,r)}$ explicitly, we obtain

$$\begin{aligned} E_{-2}^{(2r,r)}\left(\frac{r+1}{2}\right)\frac{\xi(r+1)}{\xi(2r+1)} &= -E_{-1}^{(2r,r-1)}\left(\frac{r}{2}\right)\frac{\xi(r+1)}{\xi(2r)}\frac{\xi(r)}{\xi(2r+1)}\frac{\xi(0)}{\xi(r+1)} \\ &= -c_{2r,r-1}E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right)\frac{\xi(0)}{\xi(r)}\frac{\xi(2r)}{\xi(2r+1)} \quad (\text{by Lemma 4.3}), \end{aligned}$$

which gives

$$E_{-2}^{(2r,r)}\left(\frac{r+1}{2}\right) = -c_{2r,r-1}\frac{\xi(0)}{\xi(r)}\frac{\xi(2r)}{\xi(r+1)}E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right).$$

(Here recall our convention that $\xi(0) = \operatorname{Res}_{s=0} \xi(s)$.) We set

$$\begin{aligned} d_{2r,r} &= -c_{2r,r-1}\frac{\xi(0)}{\xi(r)}\frac{\xi(2r)}{\xi(r+1)} \\ &= c_{2r,r-1}\frac{\xi(1)}{\xi(r)}\frac{\xi(2r)}{\xi(r+1)} \quad \text{by } \xi(0) = -\xi(1). \end{aligned}$$

Hence the case $m = 2r$ is proven.

Base step of the induction: $m = 2r - 1$

Next we show the case $m \leq 2r - 1$. Recall

$$m = 2r - j \quad \text{where } 1 \leq j \leq r - 1.$$

Then we show this by induction on j . For the base step, consider

$$\begin{aligned} E_{-2,Q}^{(2r,r)}\left(\frac{r+1}{2}\right) &= ||^{2r}E_{-2}^{(2r-1,r-1)}\left(\frac{r+2}{2}\right) + ||^rE_{-2}^{(2r-1,r)}\left(\frac{r+1}{2}\right)F^{(2r,r)}\left(\frac{r+1}{2}\right) \\ &\quad + ||^{r-1}E_{-1}^{(2r-1,r-1)}\left(\frac{r}{2}\right)G_{-1}^{(2r,r)}\left(\frac{r+1}{2}\right). \end{aligned} \tag{4}$$

We know that

$$\begin{aligned} E_{-2,Q}^{(2r,r)}\left(\frac{r+1}{2}\right) &= d_{2r,r}E_{-1,Q}^{(2r,2r)}\left(\frac{1}{2}\right) \\ E_{-2}^{(2r-1,r-1)}\left(\frac{r+2}{2}\right) &= 0 \quad (\text{Lemma 4.4 (i)}) \\ E_{-1}^{(2r-1,r-1)}\left(\frac{r}{2}\right) &= c_{r-1}E_0^{(2r-1,2r-1)}(0). \end{aligned}$$

Hence (4) becomes

$$\begin{aligned} c_{2r,r-1} E_{-1,Q}^{(2r,2r)}\left(\frac{1}{2}\right) &= ||^r E_{-2}^{(2r-1,r)}\left(\frac{r+1}{2}\right) F^{(2r,r)}\left(\frac{r+1}{2}\right) \\ &\quad + ||^{r-1} c_{r-1} E_0^{(2r-1,2r-1)}(0) G_{-1}^{(2r,r)}\left(\frac{r+1}{2}\right). \end{aligned} \quad (5)$$

Now by the second equation of Proposition 4.1, we have

$$E_{-1,Q}^{(2r,2r)}\left(\frac{1}{2}\right) = ||^r E_{-1}^{(2r-1,2r-1)}(1) + ||^{r-1} E_0^{(2r-1,2r-1)}(0) H_{-1}^{(2r)}\left(\frac{1}{2}\right). \quad (6)$$

Then by (5) and (6), we have

$$\begin{aligned} d_{2r,r} &\left(||^r E_{-1}^{(2r-1,2r-1)}(1) + ||^{r-1} E_0^{(2r-1,2r-1)}(0) H_{-1}^{(2r-1)}\left(\frac{1}{2}\right) \right) \\ &= ||^r E_{-2}^{(2r-1,r)}\left(\frac{r+1}{2}\right) F^{(2r,r)}\left(\frac{r+1}{2}\right) + ||^{r-1} c_{r-1} E_0^{(2r-1,2r-1)}(0) G_{-1}^{(2r,r)}\left(\frac{r+1}{2}\right). \end{aligned} \quad (7)$$

By extracting the terms containing $||^r$ in (7) (Lemma 4.2), we obtain

$$d_{2r,r} E_{-1}^{(2r-1,2r-1)}(1) = E_{-2}^{(2r-1,r)}\left(\frac{r+1}{2}\right) F^{(2r,r)}\left(\frac{r+1}{2}\right),$$

i.e.

$$E_{-2}^{(2r-1,r)}\left(\frac{r+1}{2}\right) = d_{2r,r} F^{(2r,r)}\left(\frac{r+1}{2}\right) E_{-1}^{(2r-1,2r-1)}(1),$$

and

$$\begin{aligned} d_{2r-1,r} &= d_{2r,r} \frac{1}{F^{(2r,r)}\left(\frac{r+1}{2}\right)} \\ &= d_{2r,r} \frac{\xi(2r)}{\xi(r)}. \end{aligned}$$

This shows the base step.

The induction step

Now consider the induction step. First note that for $j \geq 1$,

$$\begin{aligned} E_{-2,Q}^{(2r-j,r)}\left(\frac{r+1}{2}\right) &= ||^{2r-j} E_{-2}^{(2r-j-1,r-1)}\left(\frac{r+2}{2}\right) + ||^r E_{-2}^{(2r-j-1,r)}\left(\frac{r+1}{2}\right) F^{(2r-j,r)}\left(\frac{r+1}{2}\right) \\ &\quad + ||^{r-j-1} E_{-2}^{(2r-j-1,r-1)}\left(\frac{r}{2}\right) G^{(2r-j,r)}\left(\frac{r+1}{2}\right) \\ &= ||^r E_{-2}^{(2r-j-1,r)}\left(\frac{r+1}{2}\right) F^{(2r-j,r)}\left(\frac{r+1}{2}\right) \\ &\quad + ||^{r-j-1} E_{-2}^{(2r-j-1,r-1)}\left(\frac{r}{2}\right) G^{(2r-j,r)}\left(\frac{r+1}{2}\right), \end{aligned} \quad (8)$$

where we used $E_{-2}^{(2r-j-1,r-1)}\left(\frac{r+2}{2}\right) = 0$ (Lemma 4.4 (i)). Also we have

$$\begin{aligned} E_{-1,Q}^{(2r-j,2r-j)}\left(\frac{j+1}{2}\right) &= ||^r E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{j+2}{2}\right) \\ &\quad + ||^{r-j-1} E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{j}{2}\right) H^{(2r-j)}\left(\frac{j+1}{2}\right). \end{aligned} \quad (9)$$

Now by the induction hypothesis

$$E_{-2,Q}^{(2r-j,r)}\left(\frac{r+1}{2}\right) = d_{2r-j,r} E_{-1,Q}^{(2r-j,2r-j)}\left(\frac{j+1}{2}\right),$$

together with (8) and (9) above, we obtain

$$\begin{aligned} & ||^r E_{-2}^{(2r-j-1,r)}\left(\frac{r+1}{2}\right) F^{(2r-j,r)}\left(\frac{r+1}{2}\right) + ||^{r-j} E_{-2}^{(2r-j,r-1)}\left(\frac{r}{2}\right) G^{(2r-j,r)}\left(\frac{r+1}{2}\right) \\ &= d_{2r-j,r} \left(||^r E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{j+2}{2}\right) + ||^{r-j-1} E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{j}{2}\right) H^{(2r-j)}\left(\frac{j+1}{2}\right) \right). \end{aligned}$$

By extracting the terms containing $||^r$ from both sides (Lemma 4.2), we obtain

$$E_{-2}^{(2r-j-1,r)}\left(\frac{r+1}{2}\right) F^{(2r-j,r)}\left(\frac{r+1}{2}\right) = d_{2r-j,r} E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{j+2}{2}\right),$$

i.e. , by computing $F^{(2r-j,r)}\left(\frac{r+1}{2}\right)$ explicitly,

$$E_{-2}^{(2r-j-1,r)}\left(\frac{r+1}{2}\right) = \frac{\xi(2r-j)}{\xi(r-j)} d_{2r-j,r} E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{j+2}{2}\right),$$

and

$$d_{2r-j-1,r} = \frac{\xi(2r-j)}{\xi(r-j)} d_{2r-j,r}, \quad \text{for } j \geq 1.$$

This completes the proof. \square

4.6. Spherical second term identities for 2nd term range $r < m \leq 2r$. Next we prove the spherical second term identity. As we mentioned at the beginning of the section, we prove the case $m = 2r$ as a base step and derive the rest by induction. Namely, first we prove

Proposition 4.7 (Spherical Second Term Identity for $m = 2r$). *The following identity holds for the spherical Eisenstein series:*

$$\begin{aligned} E_{-1}^{(2r,r)}\left(\frac{r+1}{2}\right) F_0^{(2r+1,r)}\left(\frac{r+1}{2}\right) + E_0^{(2r,r-1)}\left(\frac{r}{2}\right) G_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\ = 2c_r E_0^{(2r,2r)}\left(\frac{1}{2}\right) + \gamma_0 E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right). \end{aligned}$$

where γ_0 is some constant which depends only on r .

Proof. Recall as we mentioned at the beginning, we start with

$$E_{-1,Q}^{(2r+1,r)}\left(\frac{r+1}{2}\right) = c_r E_{0,Q}^{(2r+1,2r+1)}(0),$$

which is obtained by taking the constant term along Q of the spherical first term identity on the boundary (Proposition 4.5). Also by Proposition 4.1, we have

$$\begin{aligned} E_Q^{(2r+1,r)}(s) &= ||^{s+\frac{3r+1}{2}} E^{(2r,r-1)}\left(s+\frac{1}{2}\right) + ||^r E^{(2r,r)}(s) F^{(2r+1,r)}(s) \\ &\quad + ||^{-s+\frac{3r+1}{2}} E^{(2r,r-1)}\left(s-\frac{1}{2}\right) G^{(2r+1,r)}(s). \end{aligned}$$

Now we will compute the residue of both sides at $\frac{r+1}{2}$. Note that

$$\begin{aligned} E_{-1}^{(2r,r-1)}\left(\frac{r+1}{2} + \frac{1}{2}\right) &= 0 \quad (\text{Lemma 4.4 (iii)}), \\ E_{-2}^{(2r,r)}\left(\frac{r+1}{2}\right) &\neq 0, \\ E_{-1}^{(2r,r-1)}\left(\left(\frac{r+1}{2} - \frac{1}{2}\right)\right) &\neq 0, \\ G_{-1}^{(2r+1,r)}(s) &\neq 0. \end{aligned}$$

By taking all these into account, we obtain

$$\begin{aligned}
& E_{-1,Q}^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\
&= ||^r E_{-2}^{(2r,r)}\left(\frac{r+1}{2}\right) F_1^{(2r+1,r)}\left(\frac{r+1}{2}\right) + ||^r E_{-1}^{(2r,r)}\left(\frac{r+1}{2}\right) F_0^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\
&+ ||^r E_{-1}^{(2r,r-1)}\left(\frac{r}{2}\right) G_0^{(2r+1,r)}\left(\frac{r+1}{2}\right) + ||^r E_0^{(2r,r-1)}\left(\frac{r}{2}\right) G_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\
&- ||^r \ln || E_{-1}^{(2r,r-1)}\left(\frac{r}{2}\right) G_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right). \tag{10}
\end{aligned}$$

Next we will compute $E_{0,Q}^{(2r+1,2r+1)}(0)$ in terms of the Siegel Eisenstein series for the lower rank group. Namely by Proposition 4.1, we have

$$E_Q^{(2r+1,2r+1)}(s) = ||^{s+r} E^{(2r,2r)}\left(s + \frac{1}{2}\right) + ||^{-s+r} E^{(2r,2r)}\left(s - \frac{1}{2}\right) H^{(2r+1)}(s).$$

Then we have

$$\begin{aligned}
E_{0,Q}^{(2r+1,2r+1)}(0) &= ||^r E_0^{(2r,2r)}\left(\frac{1}{2}\right) + ||^r \ln || E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right) \\
&+ ||^r E_0^{(2r,2r)}\left(-\frac{1}{2}\right) H_0^{(2r+1)}(0) + ||^r E_1^{(2r,2r)}\left(-\frac{1}{2}\right) H_{-1}^{(2r+1)}(0) \\
&- ||^r \ln || E_0^{(2r,2r)}\left(-\frac{1}{2}\right) H_{-1}^{(2r+1)}(0).
\end{aligned}$$

Now recall the functional equation of the Siegel-Eisenstein series:

$$E^{(2r,2r)}(s) = \beta_{2r}(s) E^{(2r,2r)}(-s),$$

where, in general, $\beta_m(s)$ is given by

$$\beta_m(s) = \prod_{i=0}^{m-2} \frac{\xi(2s-i)}{\xi(2s+m-1-2i)}.$$

Then at $s = -\frac{1}{2}$, $\beta_{2r}(s)$ has a zero of order one and $E^{(2r,2r)}(s)$ has a simple pole. Accordingly, $E^{(2r,2r)}(-s)$ does not have a pole or zero at $s = \frac{1}{2}$. Hence

$$\begin{aligned}
E_0^{(2r,2r)}\left(-\frac{1}{2}\right) &= -\beta_{2r,1}\left(-\frac{1}{2}\right) E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right) \\
E_1^{(2r,2r)}\left(-\frac{1}{2}\right) &= \beta_{2r,1}\left(-\frac{1}{2}\right) E_0^{(2r,2r)}\left(\frac{1}{2}\right) - \beta_{2r,2}\left(-\frac{1}{2}\right) E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right),
\end{aligned}$$

where, as we have been doing for other meromorphic functions, we write the Laurent expansion of $\beta_{2r}(s)$ at s_0 by

$$\beta_{2r}(s) = \sum_{d \gg -\infty} (s - s_0)^d \beta_{2r,d}(s_0).$$

So we have

$$\begin{aligned}
E_{0,Q}^{(2r+1,2r+1)}(0) &= ||^r E_0^{(2r,2r)}\left(\frac{1}{2}\right) + ||^r \ln || E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right) - ||^r \beta_{2r,1}\left(-\frac{1}{2}\right) E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right) H_0^{(2r+1)}(0) \\
&+ ||^r \left(\beta_{2r,1}\left(-\frac{1}{2}\right) E_0^{(2r,2r)}\left(\frac{1}{2}\right) - \beta_{2r,2}\left(-\frac{1}{2}\right) E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right) \right) H_{-1}^{(2r+1)}(0) \\
&- ||^r \ln || E_0^{(2r,2r)}\left(-\frac{1}{2}\right) H_{-1}^{(2r+1)}(0). \tag{11}
\end{aligned}$$

Now by

$$E_{-1,Q}^{(2r+1,r)}\left(\frac{r+1}{2}\right) = c_r E_{0,Q}^{(2r+1,2r+1)}(0) \quad (\text{Proposition 4.5}),$$

together with (10) and (11), we obtain the following by extracting the terms having $||^r$ (Lemma 4.2):

$$\begin{aligned} & E_{-2}^{(2r,r)}\left(\frac{r+1}{2}\right)F_1^{(2+1,r)}\left(\frac{r+1}{2}\right) + E_{-1}^{(2r,r)}\left(\frac{r+1}{2}\right)F_0^{(2r+r,r)}\left(\frac{r+1}{2}\right) \\ & \quad + E_{-1}^{(2r,r-1)}\left(\frac{r}{2}\right)G_0^{(2r+1,r)}\left(\frac{r+1}{2}\right) + E_0^{(2r,r-1)}\left(\frac{r}{2}\right)G_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\ & = c_r \left[E_0^{(2r,2r)}\left(\frac{1}{2}\right) - \beta_{2r,1}\left(-\frac{1}{2}\right)E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right)H_0^{(2r+1)}(0) \right. \\ & \quad \left. + \left(\beta_{2r,1}\left(-\frac{1}{2}\right)E_0^{(2r,2r)}\left(\frac{1}{2}\right) - \beta_{2r,2}\left(-\frac{1}{2}\right)E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right) \right) H_{-1}^{(2r+1)}(0) \right] \\ & = c_r \left(1 + \beta_{2r,1}\left(-\frac{1}{2}\right)H_{-1}^{(2r+1)}(0) \right) E_0^{(2r,2r)}\left(\frac{1}{2}\right) \\ & \quad + c_r \left(-\beta_{2r,1}\left(-\frac{1}{2}\right) - \beta_{2r,2}\left(-\frac{1}{2}\right)H_{-1}^{(2r+1)}(0) \right) E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right), \end{aligned}$$

which is the first form of our spherical second term identity.

But the first term identities (Lemma 4.3 and Proposition 4.6) imply

$$\begin{aligned} E_{-1}^{(2r,r-1)}\left(\frac{r}{2}\right) & = c_{2r,r-1}E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right) \\ E_{-2}^{(2r,r)}\left(\frac{r+1}{2}\right) & = d_{2r,r}E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right). \end{aligned}$$

Thus our second term identity is now written as

$$\begin{aligned} & E_{-1}^{(2r,r)}\left(\frac{r+1}{2}\right)F_0^{(2r+1,r)}\left(\frac{r+1}{2}\right) + E_0^{(2r,r-1)}\left(\frac{r}{2}\right)G_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\ & = c_r \left(1 + \beta_{2r,1}\left(-\frac{1}{2}\right)H_{-1}^{(2r+1,2r+1)}(0) \right) E_0^{(2r)}\left(\frac{1}{2}\right) \\ & \quad + c_r \left[-\beta_{2r,1}\left(-\frac{1}{2}\right) - \beta_{2r,2}\left(-\frac{1}{2}\right)H_{-1}^{(2r+1)}(0) \right. \\ & \quad \left. - d_{2r,r}F_1^{(2r+1,r)}\left(\frac{r+1}{2}\right) - c_{2r,r-1}G_0^{(2r+1,r)}\left(\frac{r+1}{2}\right) \right] E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right), \end{aligned}$$

which we write as

$$\begin{aligned} & E_{-1}^{(2r,r)}\left(\frac{r+1}{2}\right)F_0^{(2r+1,r)}\left(\frac{r+1}{2}\right) + E_0^{(2r,r-1)}\left(\frac{r}{2}\right)G_{-1}^{(2r+1,r)}\left(\frac{r+1}{2}\right) \\ & = c_r \left(1 + \beta_{2r,1}\left(-\frac{1}{2}\right)H_{-1}^{(2r+1)}(0) \right) E_0^{(2r,2r)}\left(\frac{1}{2}\right) + \gamma_0 E_{-1}^{(2r,2r)}\left(\frac{1}{2}\right), \end{aligned}$$

where γ_0 is a constant independent of f and g .

Finally, we explicitly compute the coefficient of the first term of the right hand side as

$$1 + \beta_{2r,1}\left(-\frac{1}{2}\right)H_{-1}^{(2r+1)}(0) = 2.$$

To see this, first of all, we have

$$H_{-1}^{(2r+1)}(0) = \frac{\xi(0)}{2\xi(2r)}.$$

(Recall our convention $\xi(0) = \operatorname{Res}_{s=0} \xi(s)$.) Secondly, notice that

$$\beta_{2r}(s) = \frac{1}{\xi(2s+1)} \cdot \prod_{i=1}^{2r-2} \xi(2s-i) \prod_{i=0}^{r-2} \xi(2s+2r-1-2i) \prod_{i=r}^{2r-2} \xi(2s+2r-1-2i),$$

where the numerator of this fraction does not have a pole or zero at $s = -\frac{1}{2}$. So

$$\beta_{2r,1}\left(-\frac{1}{2}\right) = \frac{2}{\xi(0)} \cdot \prod_{i=1}^{2r-2} \xi(-1-i) \prod_{i=0}^{r-2} \xi(-1+2r-1-2i) \prod_{i=r}^{2r-2} \xi(-1+2r-1-2i).$$

Recall that our $\xi(s)$ is normalized so that $\xi(s) = \xi(1-s)$. Then we see most of the ξ 's get canceled out and we obtain

$$\beta_{2r,1}\left(\frac{1}{2}\right) = \frac{2}{\xi(0)} \xi(2r).$$

So

$$1 + \beta_{2r,1}\left(-\frac{1}{2}\right) H_{-1}^{(2r+1)}(0) = 1 + \frac{2}{\xi(0)} \xi(2r) \cdot \frac{\xi(0)}{2\xi(2r)} = 2.$$

Hence we obtain the spherical second term identity for the case $m = 2r$. \square

Now using this second term identity as our base step, we will show the following more general second term identity by induction. Recall

$$m = 2r - j \quad \text{for} \quad 0 \leq j \leq r - 1.$$

Then by induction on j , we prove

Theorem 4.8 (Spherical Second Term Identity). *For $j = 0, \dots, r - 1$, the following spherical second term identity holds:*

$$\begin{aligned} E_{-1}^{(2r-j,r)}\left(\frac{r+1}{2}\right) \prod_{i=0}^j F^{(2r-i+1,r)}\left(\frac{r+1}{2}\right) + E_0^{(2r-j,r-j-1)}\left(\frac{r-j}{2}\right) \prod_{i=0}^j G^{(2r-i+1,r-i)}\left(\frac{r-i+1}{2}\right) \\ = 2c_r E_0^{(2r-j,2r-j)}\left(\frac{1+j}{2}\right) + \gamma_j E_{-1}^{(2r-j,2r-j)}\left(\frac{1+j}{2}\right), \end{aligned}$$

where γ_j is some constant which depends only on j (and r). Note that explicitly

$$F^{(2r-i+1,r)}\left(\frac{r+1}{2}\right) = \frac{\xi(r-i+1)}{\xi(2r-i+1)}$$

and

$$G^{(2r-i+1,r-i)}\left(\frac{r-i+1}{2}\right) = \frac{\xi(r-i)\xi(i)}{\xi(2r-2i)\xi(2r-i+1)},$$

where for $i = 0$ and 1 we agree that by $G^{(2r-i+1,r-i)}\left(\frac{r-i+1}{2}\right)$ we mean $G_{-1}^{(2r-i+1,r-i)}\left(\frac{r-i+1}{2}\right)$.

Proof. We will show this by induction on j . Clearly $j = 0$ is the spherical second term identity for $m = 2r$, which we have just proved. Assume the identity holds for a given j . Then by looking at the constant terms, we have

$$\begin{aligned} E_{-1,Q}^{(2r-j,r)}\left(\frac{r+1}{2}\right) \prod_{i=0}^j F^{(2r-i+1,r)}\left(\frac{r+1}{2}\right) + E_{0,Q}^{(2r-j,r-j-1)}\left(\frac{r-j}{2}\right) \prod_{i=0}^j G^{(2r-i+1,r-i)}\left(\frac{r-i+1}{2}\right) \\ = 2c_r E_{0,Q}^{(2r-j,2r-j)}\left(\frac{1+j}{2}\right) + \gamma_j E_{-1,Q}^{(2r-j,2r-j)}\left(\frac{1+j}{2}\right), \end{aligned}$$

where each of the terms $E_{-1,Q}^{(2r-j,r)}(\frac{r+1}{2})$, $E_{0,Q}^{(2r-j,r-j-1)}(\frac{r-j}{2})$, $E_{0,Q}^{(2r-j,2r-j)}(\frac{1+j}{2})$, and $E_{-1,Q}^{(2r-j,2r-j)}(\frac{1+j}{2})$ can be computed by Proposition 4.1 as

$$\begin{aligned}
& E_{-1,Q}^{(2r-j,r)}\left(\frac{r+1}{2}\right) \\
&= ||^{2r-j} E_{-1}^{(2r-j-1,r-1)}\left(\frac{r+2}{2}\right) \\
&+ ||^r E_{-1}^{(2r-j-1,r)} F^{(2r-j,r)}\left(\frac{r+1}{2}\right) + ||^r E_{-2}^{(2r-j-1,r)}\left(\frac{r+1}{2}\right) F_1^{(2r-j,r)}\left(\frac{r+1}{2}\right) \\
&+ ||^{r-j-1} E_{-1}^{(2r-j-1,r-1)}\left(\frac{r}{2}\right) G^{(2r-j,r)}\left(\frac{r+1}{2}\right) + ||^{r-j-1} E_{-2}^{(2r-j-1,r-1)}\left(\frac{r}{2}\right) G_1^{(2r-j,r)}\left(\frac{r+1}{2}\right) \\
&+ ||^{r-j-1} \ln || E_{-2}^{(2r-j-1,r-1)}\left(\frac{r}{2}\right) G^{(2r-j,r)}\left(\frac{r+1}{2}\right),
\end{aligned}$$

$$\begin{aligned}
& E_{0,Q}^{(2r-j,r-j-1)}\left(\frac{r-j}{2}\right) \\
&= ||^{2r} E_0^{(2r-j-1,r-j-2)}\left(\frac{r-j+1}{2}\right) + ||^{r-j-1} E_0^{(2r-j-1,r-j-1)}\left(\frac{r-j}{2}\right) F^{(2r-j,r-j-1)}\left(\frac{r-j}{2}\right) \\
&+ ||^{r-j-1} E_{-1}^{(2r-j-1,r-j-1)}\left(\frac{r-j}{2}\right) F_1^{(2r-j,r-j-1)}\left(\frac{r-j}{2}\right) \\
&+ ||^r E_0^{(2r-j-1,r-j-2)}\left(\frac{r-j-2}{2}\right) G^{(2r-j,r-j-1)}\left(\frac{r-j}{2}\right) \\
&- ||^r \ln || E_{-1}^{(2r-j-1,r-j-2)}\left(\frac{r-j-2}{2}\right) G^{(2r-j,r-j-1)}\left(\frac{r-j}{2}\right) \\
&+ ||^r E_{-1}^{(2r-j-1,r-j-2)}\left(\frac{r-j-2}{2}\right) G_1^{(2r-j,r-j-1)}\left(\frac{r-j}{2}\right),
\end{aligned}$$

$$\begin{aligned}
& E_{0,Q}^{(2r-j,2r-j)}\left(\frac{1+j}{2}\right) \\
&= ||^r E_0^{(2r-j-1,2r-j-1)}\left(\frac{2+j}{2}\right) + ||^r \ln || E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{2+j}{2}\right) \\
&+ ||^{r-1} E_0^{(2r-j-1,2r-j-1)}\left(\frac{j}{2}\right) H^{(2r-j)}\left(\frac{j+1}{2}\right) + ||^{r-1} E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{j}{2}\right) H_1^{(2r-j)}\left(\frac{j+1}{2}\right) \\
&- ||^{r-1} \ln || E_{-1}^{(2r-j,2r-j)}\left(\frac{j}{2}\right) H^{(2r-j)}\left(\frac{j+1}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
& E_{-1,Q}^{(2r-j,2r-j)}\left(\frac{1+j}{2}\right) \\
&= ||^r E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{2+j}{2}\right) + ||^{r-1} E_{-1}^{(2r-j-1,2r-j-1)}\left(\frac{j}{2}\right) H^{(2r-j)}\left(\frac{1+j}{2}\right).
\end{aligned}$$

Then by extracting the terms containing $||^r$ (Lemma 4.2), we obtain the identity

$$\begin{aligned} & \left(E_{-1}^{(2r-j-1,r)} F^{(2r-j,r)} \left(\frac{r+1}{2} \right) + E_{-2}^{(2r-j-1,r)} \left(\frac{r+1}{2} \right) F_1^{(2r-j,r)} \left(\frac{r+1}{2} \right) \right) \prod_{i=0}^j F^{(2r-i+1,r)} \left(\frac{r+1}{2} \right) \\ & + \left(E_0^{(2r-j-1,r-j-2)} \left(\frac{r-j-2}{2} \right) G^{(2r-j,r-j-1)} \left(\frac{r-j}{2} \right) \right. \\ & \quad \left. + E_{-1}^{(2r-j-1,r-j-2)} \left(\frac{r-j-2}{2} \right) G_1^{(2r-j,r-j-1)} \left(\frac{r-j}{2} \right) \right) \prod_{i=0}^j G^{(2r-i+1,r-i)} \left(\frac{r-i+1}{2} \right) \\ & = 2c_r E_0^{(2r-j-1,2r-j-1)} \left(\frac{2+j}{2} \right) + \gamma_j E_{-1}^{(2r-j-1,2r-j-1)} \left(\frac{2+j}{2} \right). \end{aligned}$$

Now by Lemma 4.3 and Proposition 4.6, we have

$$\begin{aligned} E_{-1}^{(2r-j-1,r-j-2)} \left(\frac{r-j-2}{2} \right) &= c_{2r-j-1,r-j-2} E_{-1}^{(2r-j-1,2r-j-1)} \left(\frac{2+j}{2} \right) \\ E_{-2}^{(2r-j-1,r)} \left(\frac{r+1}{2} \right) &= d_{2r-j-1,r} E_{-1}^{(2r-j-1,2r-j-1)} \left(\frac{2+j}{2} \right). \end{aligned}$$

Using those two, one can see that the above identity simplifies to the desired second term identity for $m = 2r - (j + i)$, which completes the induction step and thus the proof of the theorem. \square

Remark 4.9. In the above proof, to derive the formula for $m = 2r - 1$ from the one for $m = 2r$, the computation for the induction is slightly different due to the locations of poles of Eisenstein series and the zeta function. However the essential idea is the same. The verification for this case is left to the reader.

5. THE WEAK SECOND TERM IDENTITY

In this section, we will show that the spherical second term identity on the group $G = \mathrm{O}(\mathcal{V})$ which we derived in the previous section can be naturally extended to those sections which are in the $G(\mathbb{A})$ -span of the spherical section.

5.1. The complementary space and Ikeda's map. For the pair $(\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-, W = W^+ \oplus W^-)$ of split symmetric and symplectic spaces with $\dim \mathcal{V} = 2m$ and $\dim W = 2r$, the complementary space of W with respect to \mathcal{V} is defined to be the symplectic space W_c with

$$\dim W_c + \dim W = \dim \mathcal{V} - 2,$$

i.e. $\dim W_c = 2m - 2r - 2$. (Of course we have been assuming $m > r$.) Now assume $\dim W_c > \dim W$, namely

$$m > 2r + 1.$$

Fix an embedding $W \subset W_c$ so that the polarization $W_c = W_c^+ \oplus W_c^-$ is of the form

$$W_c^+ = U^+ \oplus W^+ \quad \text{and} \quad W_c^- = U^- \oplus W^+.$$

Let us denote each elements in $\mathcal{V}^+ \otimes W$ and $\mathcal{V}^+ \otimes W_c$ by the matrices

$$\begin{pmatrix} v_1 \otimes w^+ \\ v_2 \otimes w^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \otimes u^+ \\ v_1 \otimes w^+ \\ x_2 \otimes u^- \\ v_2 \otimes w^- \end{pmatrix},$$

respectively, where $v_1, v_2, x_1, x_2 \in \mathcal{V}^+$, $w^+ \in W^+$, $w^- \in W^-$, $u^+ \in U^+$ and $u^- \in U^-$. Then there is a nice $G(\mathbb{A}) = \mathrm{O}(\mathcal{V}, \mathbb{A})$ intertwining map

$$\mathrm{Ik}^{(m,r)} : \mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A})) \rightarrow \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$$

defined by

$$\mathrm{Ik}^{(m,r)}(\varphi) \begin{pmatrix} v_1 \otimes w^+ \\ v_2 \otimes w^- \end{pmatrix} = \int_{(\mathcal{V}^+ \otimes U^-)(\mathbb{A})} \varphi_{KH} \begin{pmatrix} x_1 \otimes u^+ \\ v_1 \otimes w^+ \\ 0 \\ v_2 \otimes w^- \end{pmatrix} d(x_1 \otimes u^+).$$

It can be checked that if φ_c^0 is the spherical function of our choice for $\mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))$, then $\mathrm{Ik}^{(m,r)}(\varphi_c^0)$ is the spherical function φ^0 of our choice for $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ as long as the measure $d(x_1 \otimes u^+)$ is chosen to be the Tamagawa measure on $(\mathcal{V}^+ \otimes U^-)(\mathbb{A})$.

It is our understanding that this map is due to Ikeda and hence our notation. (See [Ik] for the analogous map for the Siegel-Weil formula for the symplectic group.)

5.2. The (weak) first term identity for 1st term range $m \geq 2r + 1$. Before taking care of the second term identity, let us take care of the first term identities for the 1st term range $m \geq 2r + 1$. For the case $m > 2r + 1$, the first term identity is, of course, Proposition 3.3. But for the sake of our applications, we need to refine it by incorporating Ikeda's map $\mathrm{Ik}^{(m,r)}$ defined above.

First recall in Section 2 we have defined

$$\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ = \text{the } \mathrm{O}(\mathcal{V})(\mathbb{A})\text{-span of } \varphi^0,$$

where $\varphi^0 \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ is the spherical function of our choice. Similarly we define $\mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))^\circ$. Then we have

Proposition 5.1. *Assume $r < \frac{m-1}{2}$, i.e. $m > 2r + 1$. Then for all $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))^\circ$, there exists a constant $a_{m,r}$ independent of φ such that*

$$A_{-1}^{(m,m-r-1)}(\varphi) = a_{m,r} B_{-1}^{(m,r)}(\mathrm{Ik}^{(m,r)}(\varphi)),$$

namely $A_{-1}^{(m,m-r-1)}$ and $B_{-1}^{(m,r)} \circ \mathrm{Ik}^{(m,r)}$ are proportional as maps from $\mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))^\circ$ to $\mathcal{A}(G)$. Moreover the constant $a_{m,r}$ is explicitly given by

$$a_{m,r} = |D|^{m\rho_{m,r}} \prod_{i=1}^{m-2r-1} \frac{\xi(2(i+r))}{\xi(i)}.$$

Proof. This essentially follows from the spherical first term identity (Lemma 4.3) and the property of the map $\mathrm{Ik}^{(m,r)}$. Namely, first we have

$$\begin{aligned} A_{-1}^{(m,m-r-1)}(\varphi_c^0) &= |D|^{-\frac{(m-r-1)m}{2}} E_{-1}^{(m,m)}\left(\frac{m-r-1}{2}\right) \quad \text{by (1)} \\ &= |D|^{-\frac{(m-r-1)m}{2}} c_{m,r}^{-1} E_{-1}^{(m,r)}\left(\frac{r+1}{2}\right) \quad \text{by Lemma 4.3.} \end{aligned}$$

Also by Lemma 3.2, we have

$$\begin{aligned} B_{-1}^{(m,r)}(\mathrm{Ik}^{(m,r)}(\varphi_c^0)) &= B_{-1}^{(m,r)}(\varphi^0) \\ &= |D|^{-\frac{rm}{2}} \prod_{i=1}^r \frac{\xi(s+m-\frac{r+1}{2}-(i-1))}{\xi(i)} E^{(m,r)}\left(\frac{r+1}{2}\right). \end{aligned}$$

So by combining those, we have

$$\begin{aligned} & A_{-1}^{(m,m-r-1)}(\varphi_c^0) \\ &= |D|^{-\frac{(m-r-1)m}{2}} c_{m,r}^{-1} \cdot |D|^{\frac{rm}{2}} \prod_{i=1}^r \frac{\xi(i)}{\xi(s+m-\frac{r+1}{2}-(i-1))} B_{-1}^{(m,r)}(\mathrm{Ik}^{(m,r)}(\varphi_c^0)). \end{aligned}$$

Then one sees that the constant in front of $B_{-1}^{(m,r)}(\mathrm{Ik}^{(m,r)}(\varphi_c^0))$ can be simplified to $a_{m,r}$ as in the proposition.

Now notice that if $m > 2r + 1$, both $A_{-1}^{(m,m-r-1)}$ and $B_{-1}^{(m,r)}$ define $G(\mathbb{A})$ -intertwining maps

$$A_{-1}^{(m,m-r-1)} : \mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A})) \xrightarrow{\Phi^{(m,m-r-1)}} \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} | \rho_{m,m-r-1} \rightarrow \mathcal{A}(G),$$

and

$$B_{-1}^{(m,r)} : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) \xrightarrow{f^{(m,r)}} \mathrm{Ind}_{P_r(\mathbb{A})}^{G(\mathbb{A})} | | \frac{r+1}{2} \rightarrow \mathcal{A}(G).$$

Since the map $\mathrm{Ik}^{(m,r)}$ is a $G(\mathbb{A})$ -intertwining map such that $\mathrm{Ik}^{(m,r)}(\varphi_c^0) = \varphi^0$, we see that the two maps $A_{-1}^{(m,m-r-1)}$ and $B_{-1}^{(m,r)} \circ \mathrm{Ik}^{(m,r)}$ from $\mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))$ to $\mathcal{A}(G)$ are $G(\mathbb{A})$ -intertwining and

$$A_{-1}^{(m,m-r-1)}(\varphi) = a_{m,r} B_{-1}^{(m,r)} \circ \mathrm{Ik}^{(m,r)}(\varphi)$$

for all $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))^\circ$. □

Remark 5.2. Since the first term identity is shown to hold only for those φ 's in the $G(\mathbb{A})$ -span of the spherical function, we call it ‘‘the weak first term identity’’. For the Siegel-Weil formula of the symplectic group, Ikeda [Ik] and Ichino [Ich1] have shown this form of the first term identity, *i.e.* the first term identity with Ikeda’s map, for all Schwartz functions for the symplectic group. There is no doubt that it can be shown that the above first term identity holds for all $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))$ simply by modifying their method. However we show the first term identity only in this weak form, since it is sufficient for our main purposes.

Next consider the boundary case $m = 2r + 1$. For this, $W_c = W$ and so there is no need to introduce Ikeda’s map. Indeed, we have

Proposition 5.3. *Let $\dim \mathcal{V}^+ = 2r + 1$ and $\dim W = 2r$, and $\varphi \in \mathcal{S}(\mathcal{V}^+ \otimes W)^\circ$. Then we have*

$$A_0^{(2r+1,r)}(\varphi) = 2B_{-1}^{(2r+1,r)}(\varphi).$$

Proof. The proof is essentially the same as Proposition 5.1. This time, notice that the first term of the Siegel Eisenstein series is $A_0^{(2r+1,r)}$ and we use Proposition 4.5 for the spherical first term identity. Also one sees that the constant of proportionality is simplified to 2. The detail is left to the reader. □

Remark 5.4. Just like the case $m > 2r + 1$, it is highly likely that one can derive this identity for all $\varphi \in \mathcal{S}(\mathcal{V}^+ \otimes W)$ by using the method of Ikeda and Ichino.

5.3. The (weak) first term identity for 2nd term range $r < m \leq 2r$. For the 2nd term range $r < m \leq 2r$, the non-Siegel Eisenstein series has a double pole, *i.e.* $B_{-2}^{(m,r)} \neq 0$. Then the analogous first term identity holds between $B_{-2}^{(m,r)}$ and $A_{-1}^{(m,r)}$. Namely we have

Proposition 5.5. *Let $r < \dim \mathcal{V}^+ = m \leq 2r$ and $\dim W = 2r$, and $\varphi \in \mathcal{S}(\mathcal{V}^+ \otimes W)^\circ$. Then we have*

$$A_{-1}^{(m,r)}(\varphi) = b_{m,r} B_{-2}^{(m,r)}(\varphi),$$

where the constant $b_{m,r}$ is given by

$$b_{m,r} = d_{m,r}^{-1} \prod_{i=1}^r \frac{\xi(i)}{\xi(m-i+1)}.$$

Proof. Again the proof is essentially the same as Proposition 5.1. This time, we use Proposition 4.6 for the spherical first term identity. \square

5.4. The (weak) second term identity for 2nd term range $r < m \leq 2r$. Now let us consider the second term identity for the 2nd term range $r < m \leq 2r$.

First notice that $A_{-1}^{(m,r)}$ defines the map

$$\begin{aligned} A_{-1}^{(m,r)} : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) &\xrightarrow{\Phi^{(m,r)}} \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} | \cdot |^{\rho_{m,r}} \rightarrow \mathcal{A}(G) \\ \varphi &\mapsto A_{-1}^{(m,r)}(\varphi). \end{aligned}$$

This map is $G(\mathbb{A})$ -intertwining. Let

$$\mathcal{R} = \text{Im}(A_{-1}^{(m,r)})$$

be the image of $A_{-1}^{(m,r)}$. Now the map

$$\begin{aligned} \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) &\longrightarrow \mathcal{A}(G) \\ \varphi &\mapsto A_0^{(m,r)}(\varphi) \end{aligned}$$

is not $G(\mathbb{A})$ -intertwining. However it is $G(\mathbb{A})$ -intertwining modulo \mathcal{R} . So we consider the composite

$$\begin{aligned} A_0^{(m,r)} : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) &\longrightarrow \mathcal{A}(G) \longrightarrow \mathcal{A}(G)/\mathcal{R} \\ \varphi &\mapsto A_0^{(m,r)}(\varphi) \pmod{\mathcal{R}}. \end{aligned}$$

This is a $G(\mathbb{A})$ -intertwining map. Note that by (1) we have

$$A_0^{(m,r)}(\varphi^0)(g) = |D|^{-rm/2} E_0^{(m,m)}(g, \rho_{m,r}; \Phi^0).$$

Next let us consider the non-Siegel Eisenstein series $\mathcal{E}^{(m,r)}(g, s; \varphi)$. For the non-Siegel Eisenstein series, the point of our interest is $s = \frac{r+1}{2}$. Then at this point, $\mathcal{E}^{(m,r)}(s, g; \varphi)$ has at most a double pole. Recall that we write the Laurent expansion of $\mathcal{E}^{(m,r)}(s, g; \varphi)$ as

$$\mathcal{E}^{(m,r)}(s, g; \varphi) = \sum_{d=-2}^{\infty} \left(s - \frac{r+1}{2}\right)^d B_d^{(m,r)}(\varphi)(g).$$

We are interested in $B_{-1}^{(m,r)}$. As we mentioned in §3, for the non-Siegel Eisenstein series not only the first terms but all the terms (and so in particular $B_{-1}^{(m,r)}$) define $G(\mathbb{A})$ -intertwining maps. Then we consider the $G(\mathbb{A})$ -intertwining map

$$\begin{aligned} B_{-1}^{(m,r)} : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) &\longrightarrow \mathcal{A}(G) \longrightarrow \mathcal{A}(G)/\mathcal{R} \\ \varphi &\mapsto B_{-1}^{(m,r)}(\varphi) \pmod{\mathcal{R}}. \end{aligned}$$

Also note that by Lemma 3.2 we have

$$B_{-1}^{(m,r)}(\varphi^0)(g) = |D|^{-rm/2} \prod_{i=1}^r \frac{\xi(m-i+1)}{\xi(i)} E_{-1}^{(m,r)}(g, \frac{r+1}{2}; f_0).$$

Finally, we need to take care of the ‘‘complementary term’’ $E_0^{(m,m-r-1)}(\frac{m-r}{2})$. If W has a symplectic basis $\{e_1, \dots, e_r, f_1, \dots, f_r\}$, let

$$W_0 = \text{span}\{e_1, \dots, e_{m-r-1}, f_1, \dots, f_{m-r-1}\},$$

which has the obvious symplectic structure. We also denote W_0^+ and W_0^- for the spans of $\{e_1, \dots, e_{m-r-1}\}$ and $\{f_1, \dots, f_{m-r-1}\}$, respectively. Then notice the complementary space of W_0 with respect to \mathcal{V} is actually W . Hence we have Ikeda’s map

$$\text{Ik}^{(m,m-r-1)} : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) \rightarrow \mathcal{S}((\mathcal{V}^+ \otimes W_0)(\mathbb{A})).$$

Whenever there is no danger of confusion, we simply write

$$\text{Ik} = \text{Ik}^{(m,m-r-1)}.$$

Now for each $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ as in §3 we can define the non-Siegel Eisenstein series

$$\mathcal{E}^{(m,m-r-1)}(g, s; \text{Ik}(\varphi)) = \sum_{\gamma \in P_{m-r-1}(F) \backslash G(F)} f^{(m,m-r-1)}(\gamma g, s; \text{Ik}(\varphi)_{K_{H'}}),$$

where $H' = \text{Sp}(2(m-r-1))$. This has at most a simple pole at $s = \frac{m-r}{2}$ and we write its Laurent expansion at $s = \frac{m-r}{2}$ as

$$\mathcal{E}^{(m,m-r-1)}(g, s; \text{Ik}(\varphi)) = \sum_{d=-1}^{\infty} B_d^{(m,m-r-1)}(\text{Ik}(\varphi))(g) (s - \frac{m-r}{2})^d$$

as usual. Then $B_0^{(m,m-r-1)} \circ \text{Ik}$ defines a $G(\mathbb{A})$ -intertwining map

$$B_0^{(m,m-r-1)} : \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})) \longrightarrow \mathcal{A}(G) \longrightarrow \mathcal{A}(G)/\mathcal{R}.$$

Also note that for the spherical Schwartz function $\varphi^0 \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$, by lemma 3.2 we have

$$B_0^{(m,m-r-1)}(\text{Ik}(\varphi^0)) = |D|^{-m(m-r-1)/2} \prod_{i=1}^{m-r-1} \frac{\xi(m-i+1)}{\xi(i)} E^{(m,m-r-1)}(\frac{m-r}{2}).$$

Then the spherical second term identity (Theorem 4.8) can be expressed in terms of $A_0^{(m,r)}$, $B_{-1}^{(m,r)}$, and $B_0^{(m,m-r-1)}$ as follows.

$$\begin{aligned} & \prod_{i=0}^{2r-m} F^{(2r-i+1,r)}(\frac{r+1}{2}) \cdot |D|^{rm/2} \prod_{i=1}^r \frac{\xi(i)}{\xi(m-i+1)} \cdot B_{-1}^{(m,r)}(\varphi^0) \\ & + \prod_{i=0}^{2r-m} G^{(2r-i+1,r-i)}(\frac{r-i+1}{2}) \cdot |D|^{m(m-r-1)/2} \prod_{i=1}^{m-r-1} \frac{\xi(i)}{\xi(m-i+1)} B_0^{(m,m-r-1)}(\text{Ik}(\varphi^0)) \\ & \equiv |D|^{rm/2} \prod_{i=1}^{r+1} \frac{\xi(i)}{\xi(r+i)} A_0^{(m,r)}(\varphi^0)(g) \pmod{\mathcal{R}}. \end{aligned}$$

Here, when one applies Theorem 4.8, it is convenient to keep in mind that in Theorem 4.8

$$m = 2r - j \quad \text{and} \quad \rho_{m,r} = \frac{1+j}{2}.$$

One sees that many of the ξ 's get canceled out, and we can rewrite the spherical second term identity as

$$\begin{aligned} B_{-1}^{(m,r)}(\varphi^0) + |D|^{-m\rho_{m,r}} \prod_{i=0}^{2r-m} \frac{\xi(i)}{\xi(2r-2i)} B_0^{(m,m-r-1)}(\text{Ik}(\varphi^0)) \\ \equiv A_0^{(m,r)}(\varphi^0) \pmod{\mathcal{R}}, \end{aligned}$$

Hence we have

Theorem 5.6 (Weak Second Term Identity). *For all $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ that are in the $O(\mathcal{V}, \mathbb{A})$ -span of φ^0 , the following identity holds:*

$$\begin{aligned} B_{-1}^{(m,r)}(\varphi) + |D|^{-m\rho_{m,r}} \prod_{i=0}^{2r-m} \frac{\xi(i)}{\xi(2r-2i)} B_0^{(m,m-r-1)}(\text{Ik}(\varphi)) \\ \equiv A_0^{(m,r)}(\varphi) \pmod{\mathcal{R}}, \end{aligned}$$

Proof. This is immediate from the above form of the spherical second term identity together with the fact that $B_{-1}^{(m,r)}$, $A_0^{(2r,r)}$ and $B_0^{(m,m-r-1)} \circ \text{Ik}$ are $O(\mathcal{V}, \mathbb{A})$ -intertwining maps from $\mathcal{S}((V^+ \otimes W)(\mathbb{A}))$ to $\mathcal{A}(G)/\mathcal{R}$. \square

Remark 5.7. As we mentioned in the introduction, the above weak second term identity has been shown by Ichino and the first author [GI, Proposition 7.5] for the case $r = 2$ and $m = 4$.

Remark 5.8. Just as our first term identity, we are able to derive the second term identity only for those φ 's in the span of the spherical function. This is why we call it “the weak second term identity”. However, unlike the (weak) first term identity (Proposition 5.1 and 5.5), there seems to be no known method that possibly allows one to extend the weak second term identity to full generality so that the above second identity holds for all $\varphi \in \mathcal{S}((V^+ \otimes W)(\mathbb{A}))$.

6. Inner product formulas

By using our Siegel-Weil formula (both the first term and the second term identities), we will derive the Rallis inner product formula for the theta lift from $O(V, \mathbb{A})$ to Sp_{2r} if $\dim V$ is even and to $\widetilde{\text{Sp}}_{2r}$ if it is odd. For the rest of the paper, we let

$$\text{Mp}_{2r} = \begin{cases} \text{Sp}_{2r} & \text{if } \dim V \text{ is even} \\ \widetilde{\text{Sp}}_{2r} & \text{if } \dim V \text{ is odd.} \end{cases}$$

The reader will be able to tell which one is meant from the context.

Let us recall our setting. Namely, W is the symplectic space of $\dim W = 2r$ with a fixed polarization $W^+ \oplus W^-$, and V is a (not necessarily split) quadratic space of $\dim V = m$ defined over F and $\mathcal{V} = V \oplus (-V)$, where \mathcal{V} is the split quadratic space with the underlining space $V \oplus V$ and the quadratic form defined by

$$\langle (v_1, v_2), (v'_1, v'_2) \rangle_{\mathcal{V}} = \langle v_1, v'_1 \rangle_V - \langle v_2, v'_2 \rangle_V.$$

Let us again note that V is not necessarily split but \mathcal{V} is always split. Indeed, a maximal isotropic subspace is

$$\mathcal{V}^+ = \Delta V \subset V \oplus (-V).$$

There is a natural embedding

$$i : O(V) \times O(-V) \hookrightarrow O(\mathcal{V}),$$

which should be called the embedding of the doubling method. Also we have a $O(V, \mathbb{A}) \times O(-V, \mathbb{A})$ -intertwining map

$$\sigma : \mathcal{S}((V \otimes W^+)(\mathbb{A})) \hat{\otimes} \mathcal{S}((V \otimes W^+)(\mathbb{A})) \rightarrow \mathcal{S}((\mathcal{V} \otimes W^+)(\mathbb{A})) \simeq \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A})),$$

where we view $\mathcal{S}((\mathcal{V} \otimes W^+)(\mathbb{A}))$ and $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ as representations of $O(V, \mathbb{A}) \times O(-V, \mathbb{A})$ via the embedding i .

Now let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$. For a cusp form $f \in \pi$ and Schwartz function $\phi \in \mathcal{S}((V \otimes W^+)(\mathbb{A}))$, we define the theta lift $\theta_{\psi, 2r}(f, \phi) \in \mathcal{A}(\text{Mp}_{2r})$ of f to Mp_{2r} (with respect to ϕ) by

$$\theta_{\psi, 2r}(f, \phi)(h) = \int_{O(V, F) \backslash O(V, \mathbb{A})} \sum_{v \in (V \otimes W^+)(F)} \omega_{\psi}(h, g) \phi(v) dg.$$

Then we define the theta lift $\theta_{\psi, 2r}(\pi)$ of π to Mp_{2r} by

$$\theta_{\psi, 2r}(\pi) = \{\theta_{\psi, 2r}(f, \phi) : f \in \pi, \phi \in \mathcal{S}((V \otimes W^+)(\mathbb{A}))\}.$$

We often omit ψ from the notations and simply write $\theta_{2r}(f, \phi)$ and $\theta_{2r}(\pi)$. It is well-known that if $\theta_{2r_0}(\pi) \neq 0$ for some r_0 , then $\theta_{2r}(\pi) \neq 0$ for all $r \geq r_0$ (tower property), and if r is the smallest integer with $\theta_{2r}(\pi) \neq 0$, then $\theta_{2r}(\pi)$ is in the space of cusp forms.

Now we have

Proposition 6.1. *Let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$ with $\dim V = m$. For $f_1, f_2 \in \pi$ and $\phi_1, \phi_2 \in \mathcal{S}((V \otimes W^+)(\mathbb{A}))$, we let $\theta_{2r}(f_i, \phi_i)$ be the theta lift of f_i with ϕ_i to $\text{Mp}_{2r}(\mathbb{A})$. Further assume $\theta_{2r-2}(\pi) = 0$ so that both of the $\theta_{2r}(f_i, \phi_i)$'s are cusp forms (possibly zero). Then the Peterson inner product $\langle \theta_{2r}(f_1, \phi_1), \theta_{2r}(f_2, \phi_2) \rangle$ of the lifts is given by*

$$\begin{aligned} & \langle \theta_{2r}(f_1, \phi_1), \theta_{2r}(f_2, \phi_2) \rangle \\ &= \int_{\text{Sp}_{2r}(F) \backslash \text{Sp}_{2r}(\mathbb{A})} \theta_{2r}(f_1, \phi_1)(h) \overline{\theta_{2r}(f_2, \phi_2)(h)} dh \\ &= \frac{1}{\kappa} \cdot \text{Res}_{s=\frac{r+1}{2}} \int_{(G \times G)(F) \backslash (G \times G)(\mathbb{A})} f_1(g_1) \overline{f_2(g_2)} \mathcal{E}^{(m, r)}(i(g_1, g_2), s, \sigma(\phi_1 \otimes \overline{\phi_2})) dg_1 dg_2 \\ &= \frac{1}{\kappa} \cdot \int_{(G \times G)(F) \backslash (G \times G)(\mathbb{A})} f_1(g_1) \overline{f_2(g_2)} B_{-1}^{(m, r)}(\sigma(\phi_1 \otimes \overline{\phi_2}))(i(g_1, g_2)) dg_1 dg_2, \end{aligned}$$

where $G = O(V)$ and κ is the residue of the auxiliary Eisenstein series $E(h, s)$ on $\text{Sp}_{2r}(\mathbb{A})$. Note that even if $\dim V$ is odd, the integral for the inner product is over $\text{Sp}_{2r}(F) \backslash \text{Sp}_{2r}(\mathbb{A})$.

Proof. First recall that, for an algebraic group G , we write $[G] = G(F) \backslash G(\mathbb{A})$. Then we have

$$\begin{aligned} & \frac{1}{\kappa} \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} \mathcal{E}^{(m, r)}(i(g_1, g_2), s; \sigma(\phi_1 \otimes \overline{\phi_2})) dg_1 dg_2 \\ &= \frac{1}{\kappa P_z(s)} \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} I^{(m, r)}(i(g_1, g_2), s; \omega(z) \sigma(\phi_1 \otimes \overline{\phi_2})) dg_1 dg_2 \\ &= \frac{1}{\kappa P_z(s)} \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} \int_{[\text{Sp}]} \theta(i(g_1, g_2), h; \omega(z) \sigma(\phi_1 \otimes \overline{\phi_2})) E(h, s) dh dg_1 dg_2 \\ &= \frac{1}{\kappa P_z(s)} \int_{[\text{Sp}]} \left(\int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} \theta(i(g_1, g_2), h; \omega(z) \sigma(\phi_1 \otimes \overline{\phi_2})) dg_1 dg_2 \right) E(h, s) dh \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\kappa P_z(s)} \int_{[\mathrm{Sp}]} \left(\int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} \theta(i(g_1, g_2), h; \sigma(\phi_1 \otimes \overline{\phi_2})) dg_1 dg_2 \right) z' * E(h, s) dh \\
&= \frac{1}{\kappa} \int_{[\mathrm{Sp}]} \left(\int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} \theta(i(g_1, g_2), h; \sigma(\phi_1 \otimes \overline{\phi_2})) dg_1 dg_2 \right) E(h, s) dh \\
&= \frac{1}{\kappa} \int_{[\mathrm{Sp}]} \theta_{2r}(f_1, \phi_1)(h) \overline{\theta_{2r}(f_2, \phi_2)(h)} E(h, s) dh.
\end{aligned}$$

Here we used the Poisson summation formula to show that for $\phi_1, \phi_2 \in \mathcal{S}((V \otimes W^+)(\mathbb{A}))$

$$\theta(i(g_1, g_2), h; \sigma(\phi_1 \otimes \overline{\phi_2})) = \theta(g_1, h; \phi_1) \theta(g_2, h; \overline{\phi_2}).$$

Also we used the adjointness of the operator z' . Now the auxiliary Eisenstein series $E(h, s)$ has a constant residue κ at $s = \frac{r+1}{2}$. So we have

$$\begin{aligned}
&\mathrm{Res}_{s=\frac{r+1}{2}} \frac{1}{\kappa} \int_{\mathrm{Sp}(F) \backslash \mathrm{Sp}(\mathbb{A})} \theta_{2r}(f_1, \phi_1)(h) \overline{\theta_{2r}(f_2, \phi_2)(h)} E(h, s) dh \\
&= \int_{\mathrm{Sp}(F) \backslash \mathrm{Sp}(\mathbb{A})} \theta_{2r}(f_1, \phi_1)(h) \overline{\theta_{2r}(f_2, \phi_2)(h)} dh.
\end{aligned}$$

Thus the proposition follows. \square

As a corollary, which we will use later, we have

Corollary 6.2. *Let $\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$. Then*

$$\begin{aligned}
&\sum_i \langle \theta_{2r}(f_1, \phi_{1,i}), \theta_{2r}(f_2, \phi_{2,i}) \rangle \\
&= \frac{1}{\kappa} \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} B_{-1}^{(m,r)} \left(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \right) (i(g_1, g_2)) dg_1 dg_2.
\end{aligned}$$

Proof. This immediately follows from the above proposition because the map $B_{-1}^{(m,r)}$ is linear on $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$. \square

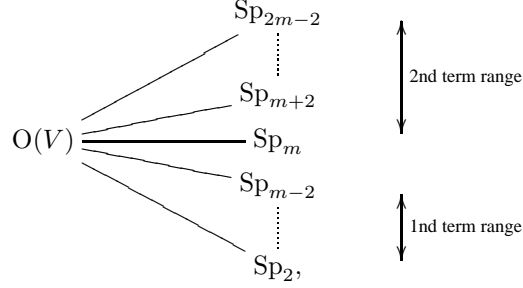
6.1. Inner product formula for the theta lift from $O(V)$ to Mp_{2r} . We derive the inner product formula for the theta lift from the orthogonal group $O(V)$ to Mp_{2r} , provided $\sigma(\phi_1 \otimes \phi_2)$ is in $\mathcal{S}((V^+ \otimes W)(\mathbb{A}))^\circ$. For this, we need to consider the following see-saw diagram:

$$\begin{array}{ccc}
O(V \oplus (-V)) & & \mathrm{Mp}_{2r} \times \mathrm{Mp}_{2r} \\
\uparrow i & \searrow & \uparrow \\
O(V) \times O(-V) & & \mathrm{Sp}_{2r}
\end{array}$$

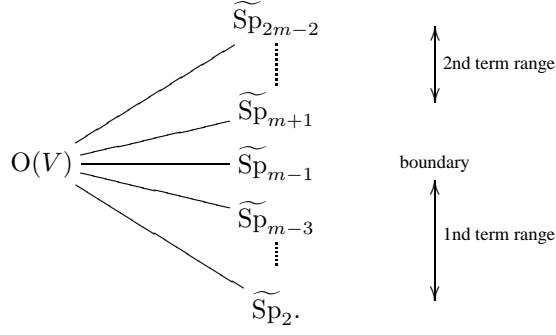
where Mp_{2r} is either Sp_{2r} or $\widetilde{\mathrm{Sp}}_{2r}$ depending on the parity of $\dim V$. Note that even if $\dim V$ is odd, at the lower right corner of the diagram we have Sp_{2r} .

Recall that the 1st term range is $m \geq 2r + 1$ and the 2nd term range is $r + 1 \leq m \leq 2r$. It will be helpful to keep the following diagram in mind regarding the theta lift from $O(V)$ to the symplectic

tower. Namely, when $\dim V = m$ is even, we consider



and when $\dim V = m$ is odd, we consider



First we have

Proposition 6.3. *Keeping all the assumptions and the notations of Proposition 6.1, let us further assume that $\dim V = m$ where $r + 1 \leq m \leq 2r$. Then if $\sum_i \sigma(\phi_{1,i} \otimes \phi_{2,i}) \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$, we have*

$$\begin{aligned} & \sum_i \langle \theta_{2r}(f_1, \phi_{1,i}), \theta_{2r}(f_2, \phi_{2,i}) \rangle \\ &= \frac{1}{\kappa} \int_{(G \times G)(F) \setminus (G \times G)(\mathbb{A})} f_1(g_1) \overline{f_2(g_2)} A_0^{(m,r)} \left(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \right) (i(g_1, g_2)) dg_1 dg_2. \end{aligned}$$

Proof. Corollary 6.2 together with our weak second term identity immediately implies

$$\begin{aligned} & \kappa \sum_i \langle \theta_{2r}(f_1, \phi_{1,i}), \theta_{2r}(f_2, \phi_{2,i}) \rangle \\ &= \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} B_{-1}^{(m,r)} \left(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \right) (i(g_1, g_2)) dg_1 dg_2 \\ &= \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} A_0^{(m,r)} \left(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \right) (i(g_1, g_2)) dg_1 dg_2 \\ &\quad - c \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} B_0^{(m,m-r-1)} (\text{Ik}(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}))) (i(g_1, g_2)) dg_1 dg_2 \\ &\quad + C \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} A_{-1}^{(m,r)} (\varphi') (i(g_1, g_2)) dg_1 dg_2, \end{aligned}$$

where c is the constant in the second term identity, C is some constant depending on $\sum_i \sigma(\phi_{1,i} \otimes \phi_{2,i})$ and r , and φ' is some Schwartz function in $\mathcal{S}((\mathcal{V}^+ \oplus W)(\mathbb{A}))$. We have only to show

$$\int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} B_0^{(m, m-r-1)} (\text{Ik}(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}))) (i(g_1, g_2)) dg_1 dg_2 = 0 \quad (12)$$

and

$$\int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} A_{-1}^{(m, r)} (\varphi') (i(g_1, g_2)) dg_1 dg_2 = 0. \quad (13)$$

Both of these are implied by our assumption $\theta_{2r-2}(\pi) = 0$. First let us define the natural isomorphism

$$\begin{aligned} \sigma' : \mathcal{S}((V \otimes W_0^+)(\mathbb{A})) \hat{\otimes} \mathcal{S}((V \otimes W_0^+)(\mathbb{A})) \\ \rightarrow \mathcal{S}((\mathcal{V} \otimes W_0^+)(\mathbb{A})) \simeq \mathcal{S}((\mathcal{V}^+ \otimes W_0)(\mathbb{A})). \end{aligned}$$

To show (12), let us write

$$\text{Ik}(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}})) = \sum_j \sigma'(\phi'_{1,j} \otimes \overline{\phi'_{2,j}}),$$

where $\phi'_{1,j} \otimes \overline{\phi'_{2,j}} \in \mathcal{S}((V \otimes W_0^+)(\mathbb{A})) \hat{\otimes} \mathcal{S}((V \otimes W_0^+)(\mathbb{A}))$.

Then by the analogous computation as the proof of Proposition 6.1, we can see

$$\begin{aligned} & \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} B_0^{(m, m-r-1)} (\text{Ik}(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}))) (i(g_1, g_2)) dg_1 dg_2 \\ &= \text{Val}_{s=\frac{m-r}{2}} \sum_j \int_{[\text{Sp}_{2(m-r-1)}]} \theta_{2(m-r-1)}(f_1, \phi'_{1,j})(h) \theta_{2(m-r-1)}(f_2, \overline{\phi'_{2,j}})(h) E'(h, s) dh, \end{aligned}$$

where $E'(h, s)$ is the auxiliary Eisenstein series on the group $\text{Sp}_{2(m-r-1)}(\mathbb{A})$. But

$$\theta_{2(m-r-1)}(f_1, \phi'_{1,j}) = \theta_{2(m-r-1)}(f_2, \overline{\phi'_{2,j}}) = 0$$

because by our assumption $\theta_{2r-2}(\pi) = 0$, and hence (12) follows.

To show (13), recall that the complementary space of W_0 is actually W . Hence by Proposition 5.1,

$$\begin{aligned} & \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} A_{-1}^{(m, r)} (\varphi') (i(g_1, g_2)) dg_1 dg_2 \\ &= a_{m, m-r-1} \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} B_{-1}^{(m, m-r-1)} (\text{Ik}(\varphi')) (i(g_1, g_2)) dg_1 dg_2. \end{aligned}$$

Again let us write $\text{Ik}(\varphi') = \sum_j \sigma'(\phi''_{1,j} \otimes \overline{\phi''_{2,j}})$, by Corollary 6.2 we have

$$\begin{aligned} & \kappa \sum_j \langle \theta_{2(m-r-1)}(f_1, \phi_{1,j}), \theta_{2(m-r-1)}(f_2, \phi_{2,j}) \rangle \\ &= \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} B_{-1}^{(m, m-r-1)} (\text{Ik}(\varphi')) (i(g_1, g_2)) dg_1 dg_2. \end{aligned}$$

But this is again zero because $\theta_{2(m-r-1)}(f_1, \phi''_{1,j}) = \theta_{2(m-r-1)}(f_2, \overline{\phi''_{2,j}}) = 0$ by our assumption that $\theta_{2r-2}(\pi) = 0$, and hence (13) follows. \square

Now this proposition implies the following inner product formula.

Theorem 6.4 (Inner Product Formula for 2nd term range). *Let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$ with $\dim V = m$ and $r + 1 \leq m \leq 2r$ such that $\theta_{2r-2}(\pi) = 0$. Let $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$ be such that $\Phi_\varphi^{(m,r)}$ is factorizable as $\otimes' \Phi_v$. Let us write $\varphi = \sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}})$. Then for $f_1, f_2 \in \pi$ we have*

$$\begin{aligned} & \sum_i \langle \theta_{2r}(f_1, \phi_{1,i}), \theta_{2r}(f_2, \phi_{2,i}) \rangle \\ &= \frac{1}{d_m^S(\rho_{m,r})} \text{Val}_{s=\rho_{m,r}} \left(L^S\left(s + \frac{1}{2}, \pi\right) \cdot \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) \right), \end{aligned}$$

where $d_m^S(s)$ is the normalizing factor for the doubling method, which is explicitly given by

$$d_m^S(s) = \begin{cases} \prod_{i=1}^{\frac{m}{2}} \zeta^S(2s + 2i - 1) & \text{if } m \text{ is even} \\ \prod_{i=1}^{\frac{m-1}{2}} \zeta^S(2s + 2i) & \text{if } m \text{ is odd.} \end{cases}$$

Proof. By Proposition 6.3 we have

$$\begin{aligned} & \sum_i \langle \theta_{2r}(f_1, \phi_{1,i}), \theta_{2r}(f_2, \phi_{2,i}) \rangle \\ &= \frac{1}{\kappa} \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} A_0^{(m,r)} \left(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \right) (i(g_1, g_2)) dg_1 dg_2 \\ &= \frac{1}{\kappa} \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} A_0^{(m,r)} (\Phi_\varphi^{(m,r)}) (i(g_1, g_2)) dg_1 dg_2, \end{aligned}$$

which is, by the doubling method, written as

$$\frac{1}{\kappa} \text{Val}_{s=\rho_{m,r}} \frac{1}{d_m^S(s)} \left(L^S\left(s + \frac{1}{2}, \pi\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) \right).$$

□

Remark 6.5. The assumption that $\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}})$ is in the $O(\mathcal{V}, \mathbb{A})$ -span of the spherical Schwartz function φ^0 is necessary because our second term identity works only those φ 's in $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$. Of course, if the second term identity can be extended to all the Schwartz functions in $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$, the inner product formula can also be completed in full generality.

Next we also consider the inner product formula for the 1st term range, *i.e.* $m \geq 2r + 1$. First of all, for the boundary case, *i.e.* $m = 2r + 1$, it is exactly the same as the inner product formula for the 2nd term range. Namely we have

Theorem 6.6 (Inner Product Formula on the boundary). *Let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$ with $\dim V = m = 2r + 1$ such that $\theta_{2r-2}(\pi) = 0$. Also let $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$ be such that $\Phi_\varphi^{(m,r)}$ is factorizable as $\otimes' \Phi_v$. Let us write $\varphi = \sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}})$. Then for $f_1, f_2 \in \pi$ we have*

$$\begin{aligned} & \sum_i \langle \theta_{2r}(f_1, \phi_{1,i}), \theta_{2r}(f_2, \phi_{2,i}) \rangle \\ &= \frac{2}{\kappa} \int_{(G \times G)(F) \backslash (G \times G)(\mathbb{A})} f_1(g_1) \overline{f_2(g_2)} A_0^{(2r+1,r)} \left(\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \right) (i(g_1, g_2)) dg_1 dg_2. \end{aligned}$$

Further if $\varphi = \sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}})$ is such that $\Phi_\varphi^{(m,r)} = \otimes' \Phi_v$, then the latter is equal to

$$\frac{2}{\kappa d_{2r+1}^S(0)} \text{Val}_{s=0} \left(L^S\left(s + \frac{1}{2}, \pi\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) \right).$$

Proof. This follows from a similar computation as the inner product formula for the 2nd term range by using Proposition 5.3. Actually this time is much easier and left to the reader. \square

Now for the 1st term range other than the boundary case, i.e. $m > 2r + 1$, by using the first term identity Proposition 5.1, we can derive the inner product formula as follows.

Theorem 6.7 (Inner Product Formula for 1st term range). *Let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$ with $\dim V = m > 2r + 1$ such that $\theta_{2r-2}(\pi) = 0$. Also let $\varphi' \in \mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))^\circ$ be such that $\Phi_{\varphi'}^{(m, m-r-1)}$ is factorizable as $\otimes' \Phi_v$. Let us write $\text{Ik}(\varphi') = \sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}})$. Then for $f_1, f_2 \in \pi$ we have*

$$\begin{aligned} & \sum_i \langle \theta_{2r}(f_1, \phi_{1,i}), \theta_{2r}(f_2, \phi_{2,i}) \rangle \\ &= \frac{a_{m,r}}{d_m^S(-\rho_{m,r})} \text{Res}_{s=-\rho_{m,r}} \left(L^S\left(s + \frac{1}{2}, \pi\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) \right). \end{aligned}$$

Proof. Left to the reader. \square

As the last thing in this section, let us mention the following. By looking at the equation (13) in the proof of Proposition 6.1, one notices that this immediately implies the following fact, which we will use later.

Lemma 6.8. *If π is a cuspidal representation of $O(V, \mathbb{A})$ with $\dim V = m$ such that $\theta_{2r-2}(\pi) = 0$ and $r + 1 \leq m \leq 2r$, i.e. the 2nd term range, then for $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$ we have*

$$\text{Res}_{s=\rho_{m,r}} \int_{[G \times G]} f_1(g_1) \overline{f_2(g_2)} E^{(m,m)}(i(g_1, g_2), s; \Phi_\varphi^{(m,r)}) dg_1 dg_2 = 0.$$

Hence if f_1, f_2 and $\Phi_\varphi^{(m,r)}$ are factorizable as $\otimes' f_{1,v}, \otimes' f_{2,v}$ and $\otimes' \Phi_v$, respectively, then by the doubling method, we have

$$\text{Res}_{s=\rho_{m,r}} \frac{1}{d_m^S(s)} L^S\left(s + \frac{1}{2}, \pi\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) = 0,$$

and so

$$L^S\left(s + \frac{1}{2}, \pi\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v)$$

is holomorphic at $s = \rho_{m,r}$. (Note that in this lemma, φ is not just in $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$ but in $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$.)

Proof. This is immediate from (13). The last statement follows from the fact that the Siegel Eisenstein series has at most a simple pole. Also note that the normalizing factor $d_m^S(s)$ is non-zero holomorphic at $s = \rho_{m,r}$. \square

7. Non-vanishing of theta lifts

In this section, as an application of our inner product formula, we show a certain non-vanishing result for the global theta lift from $O(V)$ to Mp_{2r} . In this section, $G = O_{m,m}$ and P is the Siegel parabolic of G . We often write $G_v = G(F_v)$ and $P_v = P(F_v)$.

7.1. The structure of degenerate principal series. First, we need to recall the structure of the local degenerate principal series $\mathrm{Ind}_{P_v}^{G_v} ||^s$ for any v . The detailed structure of this degenerate principal series is well-known. (See [B-J] for the non-archimedean case, [Lo] for the real case, and [L-H] for the complex case). For our purposes, we only need the following.

Proposition 7.1. *Let $s_0 = \rho_{m,r} = \frac{2r-m+1}{2}$, where $0 \leq s_0 \leq \frac{m-1}{2}$ i.e. $\frac{m-1}{2} \leq r \leq m-1$. Then the degenerate principal series $\mathrm{Ind}_{P_v}^{G_v} ||^{s_0}$ has two maximum subrepresentations σ_v^+ and σ_v^- generating $\mathrm{Ind}_{P_v}^{G_v} ||^{s_0}$, one of which, say σ_v^+ , is generated by a spherical vector. Moreover, for each $\epsilon = +$ or $-$,*

$$\tau_v^\epsilon := \sigma_v^\epsilon / (\sigma_v^+ \cap \sigma_v^-)$$

is irreducible. Also when $s_0 = 0$, $\sigma_v^+ \cap \sigma_v^- = 0$ and $\mathrm{Ind}_{P_v}^{G_v} ||^{s_0} = \sigma_v^+ \oplus \sigma_v^-$.

Proof. See [B-J, Proposition 3.3] for the non-archimedean case, [Lo, Theorem A.2.1] for the real case and [L-H, Theorem 1.3.2] for the complex case. \square

The important fact we use about the structure of the degenerate principal series, which easily follows from the above proposition, is the following. First of all, for $s_0 = \rho_{m,r}$,

$$\sigma_v^+ + \sigma_v^- = \mathrm{Ind}_{P_v}^{G_v} ||^{s_0}.$$

Here the sum is not necessarily the direct sum. Now for each $f \in \mathrm{Ind}_{P_v}^{G_v} ||^s$, let $\det \cdot f \in \mathrm{Ind}_{P_v}^{G_v} ||^s$ be the function defined by

$$(\det \cdot f)(g) = \det(g)f(g).$$

Then

$$\sigma_v^- = \{\det \cdot f : f \in \sigma_v^+\}.$$

Also notice that if W_v is the symplectic space with $\dim W_v = 2r$, then the image of the map

$$\mathcal{S}((\mathcal{V}_v^+ \otimes W_v)(F_v))^\circ \xrightarrow{\Phi^{(m,r)}} \mathrm{Ind}_{P_v}^{G_v} ||^{s_0}$$

is σ_v^+ , because σ_v^+ is generated by the spherical vector. In fact, not only the image of $\mathcal{S}((\mathcal{V}_v^+ \otimes W_v)(F_v))^\circ$ but of $\mathcal{S}((\mathcal{V}_v^+ \otimes W_v)(F_v))$ is σ_v^+ . This is because of the Howe duality for the trivial representation, which is known to be true for any residual characteristic. But we only need this weaker version for our purposes.

Next, we consider the global degenerate principal series $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} ||^s$. For this, let

$$\epsilon : \{\text{places } v \text{ of } F\} \rightarrow \{+, -\}$$

be a map such that $\epsilon(v) = +$ for almost all v . Then we say that

$$\epsilon \text{ is } \begin{cases} \text{coherent} & \text{if } \prod_v \epsilon(v) = +, \text{ i.e. } \epsilon(v) = - \text{ for an even number of } v\text{'s} \\ \text{incoherent} & \text{if } \prod_v \epsilon(v) = -, \text{ i.e. } \epsilon(v) = - \text{ for an odd number of } v\text{'s}. \end{cases}$$

Notice that by the above proposition, we know that for each ϵ the induced space $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} ||^{s_0}$ at $s_0 = \rho_{m,r}$ has a submodule and an irreducible quotient which are respectively isomorphic to

$$\sigma(\epsilon) := \bigotimes_v' \sigma_v^{\epsilon(v)} \quad \text{and} \quad \tau(\epsilon) := \bigotimes_v' \tau_v^{\epsilon(v)}.$$

Thus one has an equivariant projection

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \Big| \Big|^{s_0} \longrightarrow \bigoplus_{\epsilon} \tau(\epsilon).$$

We say that

$$\Phi \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \Big| \Big|^{s} \text{ is } \begin{cases} \text{coherent at } s = \rho_{m,r} \text{ if the image of } \Phi(-, \rho_{m,r}) \text{ lies in } \bigoplus_{\epsilon \text{ coherent}} \tau(\epsilon) \\ \text{incoherent at } s = \rho_{m,r} \text{ if the image of } \Phi(-, \rho_{m,r}) \text{ lies in } \bigoplus_{\epsilon \text{ incoherent}} \tau(\epsilon). \end{cases}$$

7.2. The residues of Siegel Eisenstein series. Recall that the Siegel Eisenstein series $E^{(m,m)}(g, s; \Phi)$ has a pole of order at most 1 at $s \in \{\hat{0}, \dots, \rho_m - 1, \rho_m\}$ where $\rho_m = \frac{m-1}{2}$, or equivalently it has a pole at $s = \rho_{m,r} = \frac{2r-m+1}{2} = \rho_{m,r}$ for $\frac{m}{2} \leq r \leq m-1$. Also recall that at each $s = \rho_{m,r}$ we write Laurent expansion of $E^{(m,m)}(g, s; \Phi)$ as

$$E^{(m,m)}(g, s; \Phi) = \sum_{d=-1}^{\infty} A_d^{(m,r)}(\Phi)(g)(s - \rho_{m,r})^d.$$

Then we have the following important proposition, which determines the image of the leading term $A_{-1}^{(m,r)}$.

Proposition 7.2. *Let $\Phi \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \Big| \Big|^{s}$. Then*

(a) $A_{-1}^{(m,r)}(\Phi) = 0$ if Φ is incoherent at $s = \rho_{m,r}$ where $\frac{m-1}{2} < r \leq m-1$. In particular,

$$\text{Image of } A_{-1}^{(m,r)} \cong \bigoplus_{\epsilon \text{ coherent}} \tau(\epsilon).$$

(b) $A_0^{(2r+1,r)}(\Phi) = 0$ if Φ is incoherent at $s = \rho_{2r+1,r} = 0$. In particular,

$$\text{Image of } A_0^{(2r+1,r)} \cong \bigoplus_{\epsilon \text{ coherent}} \tau(\epsilon).$$

The rest of this subsection is devoted to the proof of this proposition. The proof is by induction on m . For this proof, since we work with $O(m, m)$ with various m , let us write

$$G_m = O(m, m) \quad \text{and} \quad P_m = M_m \cdot N_m = \text{the Siegel parabolic subgroup of } G_m.$$

When $m \geq 2$, recall that $Q = Q_m$ is the parabolic subgroup of G_m whose Levi factor is $L = \mathrm{GL}_1 \times G_{m-1}$. We shall also set

$$I_m(s) := \mathrm{Ind}_{P_m(\mathbb{A})}^{G_m(\mathbb{A})} \Big| \Big|^{s} \quad (\text{normalized induction}).$$

We are interested in the behavior of the Siegel Eisenstein series $E^{(m,m)}(-, s; \Phi)$ associated to standard sections Φ_s of $I_m(s)$ for s in the set

$$\Sigma_m = \begin{cases} \{\frac{1}{2}, \frac{3}{2}, \dots, \rho_m\} & \text{if } m \text{ is even;} \\ \{0, 1, \dots, \rho_m\} & \text{if } m \text{ is odd,} \end{cases}$$

where $\rho_m = \frac{m-1}{2}$. A point in Σ_m is of the form

$$\rho_{m,r} = \frac{2r - m + 1}{2} \quad \text{for} \quad \frac{m}{2} \leq r \leq m - 1.$$

Also in this proof, the only Eisenstein series we shall work with is the Siegel Eisenstein series, and so we simply write

$$E^m(s; \Phi) := E^{(m,m)}(-, s; \Phi),$$

when there is no danger of confusion. The leading term of the Laurent expansion of $E^m(s; \Phi)$ at $s = \rho_{m,r}$ is denoted by

$$A_*^{(m,r)}(\Phi) = \begin{cases} A_{-1}^{(m,r)}(\Phi) & \text{if } \rho_{m,r} > 0; \\ A_0^{(m,r)}(\Phi) & \text{if } \rho_{m,r} = 0. \end{cases}$$

Consider the base step of the induction $m = 1$. In this case, $G_1 = \mathrm{O}(1, 1)$, $P_1 = \mathrm{SO}(1, 1)$ and $\Sigma_1 = \{0\}$, so that we are in situation (b) of the proposition with $r = 0$ and $s = 0$. In this case,

$$I_1(0) = \bigoplus_S \det_S$$

where S ranges over finite subsets of the set of places of F and

$$\det_S = \left(\bigotimes_{v \in S} \det_v \right) \otimes \left(\bigotimes_{v \notin S} 1_v \right).$$

The incoherent submodules of $I_1(0)$ are spanned by those \det_S 's with $\#S$ odd. The Eisenstein series attached to $\Phi \in I_1(0)$ is given at $s = 0$ by

$$E^{(1,1)}(g, 0; \Phi) = \Phi(g) + \Phi(\epsilon g),$$

where $\epsilon \in \mathrm{O}(1, 1)(F) \setminus \mathrm{SO}(1, 1)(F)$. It is thus clear that if Φ is incoherent, then $E(g, 0, \Phi) = 0$ because $\Phi(\epsilon g) = -\Phi(g)$. This proves the proposition when $m = 1$, which is the base step of the induction.

Now consider the case $m > 1$. We want to show that $A_*^{(m,r)}(\Phi)$ is zero when Φ is incoherent. For this, it suffices to show that

$$A_*^{(m,r)}(\Phi)_Q = 0,$$

where the LHS denotes the constant term of $A_*^{(m,r)}(\Phi)$ along Q . To compute $A_*^{(m,r)}(\Phi)$, we calculate the constant term of $E^m(s; \Phi)$ along Q and take the relevant term in its Laurent expansion at $s = \rho_{m,r}$. As an automorphic form of G_{m-1} , we have (cf. [GI, Appendix B]):

$$E_Q^m(s; \Phi) = E^{m-1}\left(s + \frac{1}{2}; \Phi|_{G_{m-1}}\right) + E^{m-1}\left(s - \frac{1}{2}; (M_{w_1}(s)\Phi)|_{G_{m-1}}\right),$$

where w_1 is the Weyl group element given by

$$w_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{2m-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Moreover, $M_{w_1}(s)$ is the standard intertwining operator defined by

$$M_{w_1}(s)(\Phi)(g) = \int_{(U \cap w_1 P w_1) \backslash U} \Phi_s(w_1 u g) du$$

when $\Re(s)$ is sufficiently large. Thus, when $s = \rho_{m,r}$, we will be interested in the Laurent expansion of

$$E^{m-1}(s; \Phi|_{G_{m-1}}) \quad \text{at } s = \rho_{m-1,r}$$

and

$$E^{m-1}(s; (M_{w_1}(s + 1/2)\Phi)|_{G_{m-1}}) \quad \text{at } s = \rho_{m-1,r-1}.$$

We first note the following lemma:

Lemma 7.3.

(i) Suppose that $\rho_{m-1,r} \in \Sigma_{m-1}$, i.e. $\rho_{m-1,r} \leq \rho_{m-1}$, then

$$\Phi \in I_m(\rho_{m,r}) \text{ incoherent} \implies \Phi|_{G_{m-1}} \in I_m(\rho_{m-1,r}) \text{ incoherent.}$$

(ii) The intertwining operator $M_{w_1}(s)$ is holomorphic at all $s \in \Sigma_m$, except when $s = 0$ or $1/2$, in which case it has a pole of order 1.

(iii) Denoting the leading term of the Laurent expansion of $M_{w_1}(s)$ at $s = \rho_{m,r} \in \Sigma_m$ by $M_{w_1}^{(m,r)}$, we have:

$$\Phi \in I_m(\rho_{m,r}) \text{ incoherent} \implies M_{w_1}^{(m,r)}\Phi|_{G_{m-1}} \in I_m(\rho_{m-1,r-1}) \text{ incoherent,}$$

when $\rho_{m-1,r-1} \in \Sigma_m$, i.e. when $\rho_{m-1,r-1} \geq 0$.

Proof.

(i) This is obvious.

(ii) Suppose that $s_0 \in \Sigma_m$. For each place v of F , the local representation $I_{m,v}(s_0)$ is generated as a $G_{m,v}$ -module by the spherical vector $f_{0,v}$ and its twist $f_{0,v} \cdot \det_v$. Hence, the global representation is generated as a $G_m(\mathbb{A})$ -module by

$$f_S = (\otimes_{v \in S} f_{0,v} \cdot \det_v) \otimes (\otimes_{v \notin S} f_{0,v})$$

as S ranges over all finite sets of places of F . Hence the analytic behavior of $M_{w_1}(s)$ at $s = s_0$ is determined by the analytic behavior of the collection of $M_{w_1}(s)(f_S)$ at $s = s_0$. But for $\Re(s) \gg 0$,

$$\begin{aligned} M_{w_1,v}(s)(f_{0,v} \cdot \det_v)(g) &= \int_{(U \cap w_1 P w_1) \backslash U} f_{0,v}(w_1 u g) \cdot \det_v(w_1 u g) du \\ &= -\det_v(g) \cdot M_{w_1,v}(s)(f_{0,v})(g). \end{aligned}$$

By meromorphic continuation, we thus have

$$M_{w_1,v}(s)(f_{0,v} \cdot \det_v) = -\det_v \cdot M_{w_1,v}(s)(f_{0,v}).$$

Globally, we deduce that

$$M_{w_1}(s)(f_S) = (-1)^{\#S} \cdot \det_S \cdot M_{w_1}(s)(f_0).$$

Hence, for any S , the analytic behavior of $M_{w_1}(s)(f_S)$ at $s = s_0$ is the same as that of $M_{w_1}(s)(f_0)$ at $s = s_0$. But by Proposition 4.1, we see that

$$M_{w_1}(s)(f_0)(1) = \frac{\zeta(2s)}{\zeta(2s + m - 1)},$$

which is holomorphic at all $s \in \Sigma_m$ except for $s = 0$ or $1/2$ where it has a pole of order 1.

(iii) For each place v of F , $M_{w_1}^{(m,r)}(f_{0,v})|_{G_{m-1}}$ is clearly, up to scaling, the spherical vector in $I_{m-1}(\rho_{m-1,r-1})$. Moreover, as we showed in (ii), $M_{w_1}^{(m,r)}(f_{0,v} \cdot \det_v)|_{G_{m-1}}$ is the twist by \det_v of the spherical vector in $I_{m-1}(\rho_{m-1,r-1})$ (again up to scaling). Since $\sigma_{m,r,v}^+$ is generated by $f_{0,v}$ and $\sigma_{m,r,v}^-$ by $f_{0,v} \cdot \det_v$ as $G_{m,v}$ -modules, we see that $M_{w_1}^{(m,r)}$ carries $\sigma_{m,r,v}^+$ (resp. $\sigma_{m,r,v}^-$) to $\sigma_{m-1,r-1,v}^+$ (resp. $\sigma_{m-1,r-1,v}^-$), and thus carries $\sigma_{m,r,v}^+ \cap \sigma_{m,r,v}^-$ to $\sigma_{m-1,r-1,v}^+ \cap \sigma_{m-1,r-1,v}^-$.

$\sigma_{m-1, r-1, v}^-$. This implies that, globally, one has a commutative diagram

$$\begin{array}{ccc} I_m(\rho_{m,r}) & \xrightarrow{M_{w_1}^{(m,r)}} & I_{m-1}(\rho_{m-1, r-1}) \\ \downarrow & & \downarrow \\ \bigoplus_{\epsilon \text{ coherent}} \tau_m(\epsilon) & \longrightarrow & \bigoplus_{\epsilon \text{ coherent}} \tau_{m-1}(\epsilon), \end{array}$$

where to be more precise the upper vertical arrow is given by $f_v \mapsto M_{w_1}^{(m,r)}(f_v)|_{G_{m-1}}$. This shows the desired result. \square

Going back to the proof of the proposition for $m > 1$, we first consider situation (a) so that $\rho_{m,r} > 0$. In this case, we have

$$A_{-1}^{(m,r)}(\Phi)_Q = A_{-1}^{(m-1,r)}(\Phi|_{G_{m-1}}) + A_*^{(m-1,r-1)}((M_{w_1}^{(m,r)}\Phi)|_{G_{m-1}}).$$

By the lemma and induction hypothesis, we know that the RHS of the above equation is zero. This establishes the induction step when $\rho_{m,r} > 0$.

It remains to treat the case when $\rho_{m,r} = 0$, so that we are in situation (b) of the proposition. In this case, $m = 2r + 1$ is odd and we have

$$A_0^{(m,r)}(\Phi)_Q = A_0^{(m-1,r)}(\Phi|_{G_{m-1}}) + [E^{m-1}(s - \frac{1}{2}; (M_{w_1}(s)\Phi)|_{G_{m-1}})]_{s=0}.$$

Examining the second term on the RHS above, we deduce by the functional equation for Eisenstein series that

$$E^{m-1}(s - \frac{1}{2}; (M_{w_1}(s)\Phi)|_{G_{m-1}}) = E^{m-1}(\frac{1}{2} - s; M_{m-1}(s - \frac{1}{2})((M_{w_1}(s)\Phi)|_{G_{m-1}})),$$

where

$$M_{m-1}(s) : I_{P_{m-1}}(s) \longrightarrow I_{P_{m-1}}(-s)$$

is the standard intertwining operator defined for $\Re(s) \gg 0$ by

$$M_{m-1}(s)\Phi(g) = \int_{N_{m-1}} \Phi(w_2 n g) dn,$$

with

$$w_2 = \begin{pmatrix} & & & 1 \\ & \mathbf{0} & & 1 \\ & & \ddots & \\ & 1 & & \mathbf{0} \\ 1 & & & \end{pmatrix}.$$

Now it is not difficult to see that

$$M_{m-1}(s - \frac{1}{2})((M_{w_1}(s)\Phi)|_{G_{m-1}}) = (M_m(s)\Phi)|_{G_{m-1}}.$$

Hence,

$$E^{m-1}(s - \frac{1}{2}; (M_{w_1}(s)\Phi)|_{G_{m-1}}) = E^{m-1}(\frac{1}{2} - s; (M_m(s)\Phi)|_{G_{m-1}}).$$

We are interested in

$$E^{m-1}(\frac{1}{2} + s; \Phi|_{G_{m-1}}) + E^{m-1}(\frac{1}{2} - s; (M_m(s)\Phi)|_{G_{m-1}}) \quad (14)$$

at $s = 0$.

Lemma 7.4.

- (i) *The intertwining operator $M_m(s) : I_{P_m}(s) \longrightarrow I_{P_m}(-s)$ is holomorphic at $s = 0$. It acts as $+1$, i.e. the identity, on the coherent submodules of $I_{P_m}(0)$ and as -1 on the incoherent submodules.*
- (ii) *If $M'_m(0)$ denotes the derivative of $M_m(s)$ at $s = 0$, then $M'_m(0)$ commutes with $M_m(0)$. In particular, $M'_m(0)$ preserves the incoherent submodule of $I_{P_m}(0)$.*

Proof.

- (i) We have the functional equation

$$M_m(-s) \circ M_m(s) = 1,$$

which implies that $M_m(s)$ is holomorphic at $s = 0$ and is nonzero as an operator. Moreover, $M_m(0)^2 = 1$, so that $M_m(0)$ acts as ± 1 on any irreducible submodule of $I_{P_m}(0)$.

Now it is not difficult to see that if f_0 is the spherical vector in $I_m(0)$, then $M_m(0)f_0 = f_0$. One way of seeing this is to observe that the functional equation for Eisenstein series

$$E^m(-s; M_m(s)f_0) = E^m(s; f_0)$$

implies that

$$A_0^{(m,r)}(M_m(0)f_0) = A_0^{(m,r)}(f_0).$$

As we observed in the proof of the spherical first term identity in §4 (see Proposition 4.5), $A_0^{(m,r)}(f_0) \neq 0$, so that we must have $M_m(0)f_0 = f_0$.

On the other hand, for a place v of F

$$\begin{aligned} M_{m,v}(s)(f_{0,v} \cdot \det_v)(g) &= \int_{N_m} f_{0,v}(w_2ng) \cdot \det_v(w_2ng) \, dn \\ &= -M_{m,v}(s)(f_{0,v})(g) \cdot \det_v(g) \\ &= -f_{0,v}(g) \cdot \det_v(g), \end{aligned}$$

since $\det(w_2) = -1$ as m is odd. Thus, globally, we have

$$M_m(0)(f_0 \cdot \det_S) = (-1)^{\#S} \cdot f_0 \cdot \det_S$$

for any finite subset of places of F . This shows that $M_m(0)$ acts as $+1$ on the coherent submodule of $I_m(0)$ which is generated by the $f_0 \cdot \det_S$'s for $\#S$ even and it acts as -1 on the incoherent submodule which is generated by the $f_0 \cdot \det_S$'s for $\#S$ odd.

- (ii) Differentiating

$$M_m(-s) \circ M_m(s) = 1$$

with respect to s and setting $s = 0$ gives

$$M_m(0) \circ M'_m(0) - M'_m(0) \circ M_m(0) = 0,$$

as desired. □

From the lemma, observe that both terms in (14) could have a pole of order 1 at $s = 0$ but their residues there cancel (as they should). More relevantly, when Φ is incoherent, the constant term in the Laurent expansion at $s = 0$ of (14) is:

$$A_0^{(m-1,r)}(\Phi|_{G_{m-1}}) - A_0^{(m-1,r)}(\Phi|_{G_{m-1}}) + A_{-1}^{(m-1,r)}((M'_m(0)\Phi)|_{G_{m-1}}).$$

The first two terms cancel whereas the third term vanishes by induction hypothesis since $(M'_m(0)\Phi)|_{G_{m-1}}$ is incoherent. Thus we conclude that $A_0^{(m,r)}(\Phi)_Q = 0$ when $\rho_{m,r} = 0$ as well.

We have thus shown that the leading term map $A_*^{m,r}$ vanishes on all incoherent Φ 's. Hence

$$\text{Image of } A_*^{m,r} \subset \bigoplus_{\epsilon \text{ coherent}} \tau(\epsilon).$$

On the other hand, we know that $A_*^{m,r}$ is nonzero on the spherical section, and thus by twisting by automorphic determinant characters, we see that equality must hold above. This completes the proof of Proposition 7.2

7.3. Non-vanishing results. In what follows, we will show our non-vanishing results for the theta lift. First let us prove a couple of lemmas which will be quite crucial for our proof.

Lemma 7.5. *Let π_v be an irreducible admissible representation of $O(V, F_v)$. (Here v can be either archimedean or non-archimedean.) Also let $\Phi_v(-, s) \in \text{Ind}_{P_v}^{G_v} ||^s$ be a $(K$ -finite) standard section, and $\langle \pi_v(g_v)f_1, f_2 \rangle$ a $(K$ -finite) matrix coefficient of π_v .*

(a) *Assume $\pi_v \cong \pi_v \otimes \det$. Then there exist a $(K$ -finite) matrix coefficient $\langle \pi_v(g_v)f'_1, f'_2 \rangle$ of π_v so that*

$$Z_v(s, f_1, f_2, \Phi_v, \pi_v) = Z_v(s, f'_1, f'_2, \det \cdot \Phi_v, \pi_v).$$

(b) *Assume $\pi_v \not\cong \pi_v \otimes \det$. Then*

$$Z_v(s, f_1, f_2, \Phi_v, \pi_v) = Z_v(s, f_1, f_2, \det \cdot \Phi_v, \pi_v \otimes \det).$$

Proof. Note that

$$\begin{aligned} Z_v(s, f_1, f_2, \Phi_v, \pi_v) &= \int_{O(V, F_v)} \Phi_v(i(g_v, 1)) \langle \pi_v(g_v)f_1, f_2 \rangle dg_v \\ &= \int_{O(V, F_v)} \det(i(g_v, 1)) \Phi_v(i(g_v, 1)) \langle \det(g_v)\pi_v(g_v)f_1, f_2 \rangle dg_v. \end{aligned}$$

So this proves the lemma if $\pi_v \not\cong \pi_v \otimes \det$. Now if $\pi_v \cong \pi_v \otimes \det$, then $\langle \det(g_v)\pi_v(g_v)f_1, f_2 \rangle$ is a matrix coefficient of π_v and so written as $\langle \det(g_v)\pi_v(g_v)f_1, f_2 \rangle = \langle \pi_v(g_v)f'_1, f'_2 \rangle$ for some f'_1 and f'_2 . Hence the lemma follows. \square

Lemma 7.6. *Let $\Phi(-, s) = \otimes' \Phi_v(-, s) \in \text{Ind}_P^G ||^s$ be a standard section such that for every place v , $\Phi_v(-, \rho_{m,r}) \in \sigma_v^+$. Then there is a Schwartz function $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$ of the form $\varphi = \sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}})$ such that $\Phi = \Phi_\varphi^{(m,r)}$.*

Proof. Since $\Phi_v(-, \rho_{m,r}) \in \sigma_v^+$, for each v we have $\Phi_v(-, \rho_{m,r}) = \Phi_{\varphi_v}^{(2r+j,r)}(-, \rho_{m,r})$ for some φ_v which is in the $O(\mathcal{V}, F_v)$ -span of the spherical Schwartz function $\varphi_{0,v}$. Clearly, we can choose $\varphi_v = \varphi_{0,v}$ for almost all v . Thus if we let $\varphi = \otimes' \varphi_v$, we have $\Phi = \Phi_\varphi^{(m,r)}$, which is of the form $\varphi = \sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}})$. \square

Now we are ready to prove our non-vanishing theorems. For this purpose, we need to introduce the following notation. Let μ_2 be the group of order two and so $\mu_2(\mathbb{A}) = \prod_v \{\pm 1\}$. For a finite set of places T , we define the character $\text{sign}(T)$ on $\mu_2(\mathbb{A})$ by

$$\text{sign}(T) := \otimes' \text{sign}(T)_v$$

where

$$\text{sign}(T)_v(-1) = \begin{cases} 1 & \text{if } v \notin T \\ -1 & \text{if } v \in T. \end{cases}$$

Notice that $\text{sign}(T)$ is an automorphic character on $\mu_2(F)\backslash\mu_2(\mathbb{A})$ if and only if the cardinality $|T|$ of T is even. And any automorphic character μ on $\mu_2(F)\backslash\mu_2(\mathbb{A})$ is of this form. From now on, by a character μ we always mean an automorphic character on $\mu_2(F)\backslash\mu_2(\mathbb{A})$. For each character μ and an automorphic representation π of $O(V, \mathbb{A})$, we write

$$\pi \otimes \mu := \pi \otimes (\mu \circ \det),$$

which is again automorphic.

We first prove the following non-vanishing result for the 1st term range. We should mention that the same theorem has been proven by Ginzburg-Jiang-Soudry in their recent paper ([GJS, Thm. 1.1]) by following Mœglin ([Mo]) and using her method of “generalized doubling method”. However, we will show that the same theorem quite simply follows from our inner product formula together with Proposition 7.2.

Theorem 7.7 (Non-vanishing theorem for 1st term range). *Let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$ for $\dim V = m$ where $m \geq 2r + 1$.*

- (a) *Assume $m > 2r + 1$. Further assume that the (incomplete) standard L -function $L^S(s, \pi)$ has a pole at $s = \frac{m-2r}{2}$. Then there is a character μ on $\mu_2(F)\backslash\mu_2(\mathbb{A})$ such that $\theta_{2r}(\pi \otimes \mu) \neq 0$.*
- (b) *Assume $m = 2r + 1$. Further assume that the (incomplete) standard L -function $L^S(s, \pi)$ does not vanish at $s = \frac{1}{2}$. Then there is a character μ on $\mu_2(F)\backslash\mu_2(\mathbb{A})$ such that $\theta_{2r}(\pi \otimes \mu) \neq 0$.*

Proof. First let us assume $m > 2r + 1$ i.e. situation (a). Suppose that there is a character μ such that $\theta_{2r-2}(\pi \otimes \mu) \neq 0$. Then by the tower property of theta lifting, the theorem follows. Hence we assume that $\theta_{2r-2}(\pi \otimes \mu) = 0$ for all μ , and hence $\theta_{2r}(\pi \otimes \mu)$ is in the space of cusp forms (possibly zero) for all μ .

Now it is known that if v is non-archimedean, one can always find a standard section Φ_v and vectors $f_{1,v}$ and $f_{2,v}$ so that

$$Z_v(-\rho_{m,r}, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) = 1,$$

and if v is archimedean, one can always find a K -finite section Φ_v and vectors $f_{1,v}$ and $f_{2,v}$ so that

$$Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v)$$

either has a pole or is non-zero holomorphic at $s = -\rho_{m,r}$. (See [Kd-R1, Theorem 2.0.2 and 2.0.3].) Moreover by Lemma 7.5, if $\pi_v \cong \pi_v \otimes \det$, then we can find a suitable $f'_{i,v}$ so that

$$Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) = Z_v(s, f'_{1,v}, f'_{2,v}, \det \cdot \Phi_v, \pi_v).$$

Since if $\Phi_v(-, -\rho_{m,r}) \in \sigma_v^-$ then $\det \cdot \Phi_v(-, -\rho_{m,r}) \in \sigma_v^+$, we may assume that at those v 's where $\pi_v \cong \pi_v \otimes \det$ the above choice of Φ_v is such that $\Phi_v(-, -\rho_{m,r}) \in \sigma_v^+$. Therefore one can find a global standard factorizable section $\Phi = \otimes' \Phi_v$, which can be chosen to be either coherent or incoherent at $s = -\rho_{m,r}$, and vectors $f_1 = \otimes' f_{1,v}$ and $f_2 = \otimes' f_{2,v}$ so that the product

$$\prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v)$$

either has a pole or is non-zero holomorphic at $s = -\rho_{m,r}$ for sufficiently large S containing all the archimedean v 's, and all the non-archimedean v 's where π_v is ramified, and moreover $\Phi_v(-, -\rho_{m,r}) \in \sigma_v^+$ if $\pi_v \cong \pi_v \otimes \det$.

Now by the doubling method, for this choice of Φ , f_1 and f_2 , we have

$$\begin{aligned} & \int_{(G \times G)(F) \backslash (G \times G)(\mathbb{A})} f_1(g_1) \overline{f_2(g_2)} E^{(m,m)}(i(g_1, g_2), s; \Phi) dg_1 dg_2 \\ &= \frac{1}{d_m^S(s)} L^S\left(s + \frac{1}{2}, \pi\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v). \end{aligned}$$

Now at $s = \rho_{m,m-r-1} = -\rho_{m,r}$, the left hand side has at most a simple pole. Hence by our assumption that $L^S(s, \pi)$ has a pole at $s = \frac{m-2r}{2}$, we know that the product of local zeta integrals on the right hand side cannot have a pole and so must be non-zero holomorphic. Then we have

$$\begin{aligned} & \int_{(G \times G)(F) \backslash (G \times G)(\mathbb{A})} f_1(g_1) \overline{f_2(g_2)} A_{-1}^{(m,m-r-1)}(\Phi)(i(g_1, g_2)) dg_1 dg_2 \\ &= \frac{1}{d_m^S(-\rho_{m,r})} \operatorname{Res}_{s=-\rho_{m,r}} \left(L^S\left(s + \frac{1}{2}, \pi\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) \right), \end{aligned}$$

where the right hand side and hence the left hand side are non-zero, and in particular

$$A_{-1}^{(m,m-r-1)}(\Phi) \neq 0.$$

Therefore by Proposition 7.2, we know that Φ is a coherent section. Hence if we let

$$T = \{v \in S : \Phi_v(-, -\rho_{m,r}) \notin \sigma^+\},$$

then the size $|T|$ of T is even by the coherence of Φ , and so if we let $\mu = \operatorname{sign}(T)$, then μ is automorphic. Since by our choice of Φ , we know that $\pi_v \not\cong \pi_v \otimes \det$ for all $v \in T$, and so by Lemma 7.5 we have

$$\prod_{v \in T} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) = \prod_{v \in T} Z_v(s, f_{1,v}, f_{2,v}, \det \cdot \Phi_v, \pi_v \otimes \det),$$

which, together with $\mu = \operatorname{sign}(T)$, gives

$$\prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) = \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \mu_v \cdot \Phi_v, \pi_v \otimes \mu_v).$$

Therefore since $L^S(s + \frac{1}{2}, \pi) = L^S(s + \frac{1}{2}, \pi \otimes \mu)$, we see that

$$L^S\left(s + \frac{1}{2}, \pi \otimes \mu\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \mu_v \cdot \Phi_v, \pi_v \otimes \mu_v)$$

has a simple pole at $s = -\rho_{m,r}$.

Now by the very definition of μ , we have

$$\mu_v \cdot \Phi_v \in \sigma_v^+ \quad \text{for all } v,$$

and so there is a $\phi' \in \mathcal{S}((\mathcal{V}^+ \otimes W_c)(\mathbb{A}))^\circ$ such that $\Phi_{\phi'}^{(m,m-r-1)} = \mu \cdot \Phi$ by Lemma 7.6. Hence by invoking Theorem 6.7 (Inner Product Formula for 1st term range), we obtain

$$\begin{aligned} & \sum_i \langle \theta_{2r}(f'_1, \phi_{1,i}), \theta_{2r}(f'_2, \phi_{2,i}) \rangle \\ &= \frac{a_{m,r}}{d_m^S(-\rho_{m,r})} \operatorname{Res}_{s=-\rho_{m,r}} \left(L^S\left(s + \frac{1}{2}, \pi \otimes \mu\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \mu_v \cdot \Phi_v, \pi_v \otimes \mu_v) \right). \end{aligned}$$

for some $\sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$ and for some $f'_i \in \pi \otimes \mu$. Since the right hand side is non-zero, we see that

$$\langle \theta_{2r}(f'_1, \phi_{1,i}), \theta_{2r}(f'_2, \phi_{2,i}) \rangle \neq 0$$

for some i . Thus part (a) of the theorem has been proven.

The part (b) of the theorem can be similarly shown by using the inner product formula on the boundary, and the detail of the proof is left to the reader. \square

Next we prove our non-vanishing result for the 2nd term range. Unfortunately, however, for this range we need to assume that $\dim V = m$ is even. This is because we need to impose on the cuspidal representation π the assumption that $\pi_v \cong \pi_v \otimes \det$ for at least one place v , and if $\dim V$ is odd, then $\pi_v \not\cong \pi_v \otimes \det$ for all v . Also the reason we can get away with this assumption for the 1st term range is Proposition 7.2. For the 2nd term range, we have not been able to prove the analogous proposition and we do not even know if such a statement even exists at all. In any case, the following is our main non-vanishing theorem for the 2nd term range.

Theorem 7.8 (Non-vanishing theorem for 2nd term range). *Let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$ for $\dim V = m$ where m is even with $r + 1 \leq m \leq 2r$. Assume that*

- (i) *there is a place v such that $\pi_v \cong \pi_v \otimes \det$;*
- (ii) *$L^S(1 + \frac{2r-m}{2}, \pi) \neq 0$. (A pole is allowed.)*

Then there is a character μ on $\mu_2(F) \backslash \mu_2(\mathbb{A})$ such that

$$\theta_{2r}(\pi \otimes \mu) \neq 0.$$

Proof. One can prove this theorem analogously to Theorem 7.7 by using the inner product formula of the 2nd term range. However, this time the set T as defined in the proof of Theorem 7.7 cannot be shown to have an even cardinality due to the absence of the analogue of Proposition 7.2. To get around this, we use a place v where $\pi_v \cong \pi_v \otimes \det$. The detailed proof is as follows.

First if $\theta_{2r-2}(\pi \otimes \mu) \neq 0$ for some character μ , then the theorem holds by the tower property of theta lifting. Also if the (incomplete) L -function has a pole at $s = \rho_{m,r}$, then by Theorem 7.7 we already know that $\theta_{2r-2\rho_{m,r}}(\pi \otimes \mu) \neq 0$, and so again by the tower property $\theta_{2r-2k}(\pi \otimes \mu) \neq 0$ for $k \leq 2\rho_{m,r}$ and so the theorem follows. Hence for the rest of the proof, we assume that $\theta_{2r-2}(\pi \otimes \mu) = 0$ for all μ and the (incomplete) L -function is non-zero holomorphic at $s = \rho_{m,r}$.

Now by the same reasoning as the proof of Theorem 7.7, there is a global K -finite factorizable standard section $\Phi = \otimes' \Phi_v$ and vectors $f_1 = \otimes' f_{1,v}$ and $f_2 = \otimes' f_{1,v}$ so that

$$\prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v)$$

is either non-holomorphic or non-zero holomorphic at $s = \rho_{m,r}$ for S as in the proof of Theorem 7.7. Moreover we may assume that $\Phi_v(-, \rho_{m,r}) \in \sigma_v^+$ for all v 's at which $\pi_v \cong \pi_v \otimes \det$. Then let

$$T = \{v \in S : \Phi_v(-, \rho_{m,r}) \notin \sigma_v^+\},$$

and define

$$\mu = \begin{cases} \text{sign}(T) & \text{if } |T| \text{ is even} \\ \text{sign}(T \cup \{v_0\}) & \text{if } |T| \text{ is odd,} \end{cases}$$

where v_0 is a place at which $\pi_{v_0} \cong \pi_{v_0} \otimes \det$. Then μ is an automorphic character.

Then by invoking the inner product formula for the 2nd term range, for suitable choice of $f'_i \in \pi \otimes \mu$ we have

$$\begin{aligned} & \sum_i \langle \theta_{2r}(f'_1, \phi_{1,i}), \theta_{2r}(f'_2, \phi_{2,i}) \rangle \\ &= \frac{1}{d_m^S(\rho_{m,r})} \text{Val}_{s=\rho_{m,r}} \left(L^S\left(s + \frac{1}{2}, \pi \otimes \mu\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \mu_v \cdot \Phi_v, \pi_v \otimes \mu_v) \right) \end{aligned}$$

where $\varphi = \sum_i \sigma(\phi_{1,i} \otimes \overline{\phi_{2,i}}) \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$ is such that $\Phi_\varphi^{(m,r)} = \Phi$ by Lemma 7.6. Notice that in the right hand side, the product

$$L^S\left(s + \frac{1}{2}, \pi \otimes \mu\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \mu_v \cdot \Phi_v, \pi_v \otimes \mu_v)$$

is holomorphic by Lemma 6.8 at $s = \rho_{m,r}$, and thus non-zero holomorphic at $s = \rho_{m,r}$ because the product of the local factors is non-zero and $L^S\left(s + \frac{1}{2}, \pi \otimes \mu\right) = L^S\left(s + \frac{1}{2}, \pi\right)$. Therefore the theorem has been proven. \square

The above two theorems are viewed as non-vanishing results “up to disconnectedness”. Indeed, if we use the “dual pair” $(\text{SO}(V), \text{Sp}(2r))$ instead of the usual $(\text{O}(V), \text{Sp}(2r))$, the above theorems are restated as follows:

Corollary 7.9. *Let τ be a cuspidal automorphic representation of $\text{SO}(V, \mathbb{A})$ with $\dim V = m$.*

- (a) *Assume $m \geq 2r + 2$. Further assume that the (incomplete) standard L -function $L^S(s, \tau)$ has a pole at $s = \frac{m-2r}{2}$. Then $\theta_{2r}(\tau) \neq 0$.*
- (b) *Assume $m = 2r + 1$. Further assume that the (incomplete) standard L -function $L^S(s, \tau)$ does not vanish at $s = \frac{1}{2}$. Then $\theta_{2r}(\tau) \neq 0$.*
- (c) *Assume m is even with $r + 1 \leq m \leq 2r$ such that $\tau^c \not\cong \tau$. Assume that $L^S\left(1 + \frac{2r-m}{2}, \tau\right) \neq 0$. (A pole is allowed.) Then $\theta_{2r}(\tau) \neq 0$. (Here τ^c is the representation obtained by conjugating τ via the outer automorphism on $\text{SO}(V)$.)*

Remark 7.10. If we assume τ is generic, then automatically $L^S\left(1 + \frac{2r-m}{2}, \tau\right) \neq 0$ for $m = 2r$, and so part (c) of this corollary implies $\theta_m(\tau) \neq 0$ (if $\tau^c \not\cong \tau$). This is a part of the main theorem of [GRS], though their theorem applies even when $\tau^c \cong \tau$.

8. On the lowest occurrence conjecture

Once the non-vanishing theorems in the previous section have been proven, the natural question to ask is whether the L -function conditions are necessary. In this last section, we will examine this issue. First we show that there is at least an example in which the L -function condition is necessary. Namely, we construct a cuspidal automorphic representation τ on $\text{SO}(4)$ so that $L^S(s, \tau)$ vanishes at $s = 1$ and $\theta_4(\tau) = 0$. After that, following [GJS], we will consider the non-vanishing of theta lifts by using the notation of the lowest occurrence as in [GJS].

8.1. An example. We will construct an example as mentioned above. Let $V = D$ be a non-split quaternion division algebra over F . Then first consider the similitude group $\text{GSO}(D)$. It is well-known that $\text{GSO}(D) \cong (D^\times \times D^\times)/\mathbb{G}_m$ where \mathbb{G}_m is embedded as $\{(t, t^{-1}); t \in \mathbb{G}_m\}$. Hence, a cuspidal automorphic representation $\tilde{\tau}$ of $\text{GSO}(D)(\mathbb{A})$ is identified with a cuspidal automorphic

representation $\tau_1 \boxtimes \tau_2$ of $D^\times(\mathbb{A}) \times D^\times(\mathbb{A})$ so that τ_1 and τ_2 have the same central character. Now assume

$$\tau_1 \boxtimes \tau_2 = \tau_1 \boxtimes \mathbf{1},$$

i.e. τ_1 has the trivial central character and τ_2 is the trivial character $\mathbf{1}$ on $D^\times(\mathbb{A})$. This is certainly a cuspidal automorphic representation since $\mathrm{GSO}(D)$ is anisotropic. Then let τ be an irreducible constituent of $\tau_1 \boxtimes \tau_2|_{\mathrm{SO}(D, \mathbb{A})}$. (Here the restriction refers to the restriction of functions.) Then one sees

$$L^S(s, \tau) = L^S(s + \frac{1}{2}, \tau_1) \cdot L^S(s - \frac{1}{2}, \tau_1).$$

(See the second paragraph of the proof of Theorem 5.4 in [Sch].) Hence if we choose τ_1 to be such that $L^S(\frac{1}{2}, \tau_1) = 0$, then we have $L^S(1, \tau) = 0$.

Now we have

Proposition 8.1. *Let τ be the cuspidal automorphic representation on $\mathrm{SO}(D, \mathbb{A})$ constructed above. Then $\theta_4(\tau) = 0$. Equivalently, if π is a cuspidal automorphic representation on $\mathrm{O}(D)$ such that $\pi|_{\mathrm{SO}(D, \mathbb{A})}$ (restriction of functions) contains τ , then for any automorphic character μ on $\mu_2(F) \backslash \mu_2(\mathbb{A})$, we have $\theta_4(\pi \otimes \mu) = 0$.*

Proof. To prove the proposition, it suffices to show that the global theta lift of $\tau \boxtimes \tau_2 = \tau_1 \boxtimes \mathbf{1}$ from $\mathrm{GSO}(D)$ to $\mathrm{GSp}(4)$ is zero. For this, we note that if $\theta(\tau_1 \boxtimes \tau_2)$ is nonzero, it must have a nonzero Bessel model associated to a pair (E, χ) where E is a quadratic field extension of F and χ is an automorphic character of E^\times (cf. [PT] for the definition of Bessel models). By [PT, Theorem 3], the Bessel model of $\theta(\tau_1 \boxtimes \tau_2)$ with respect to the pair (E, χ) is nonzero if and only if both τ_1 and τ_2 have nonzero period integrals against the character χ^{-1} of the maximal torus $E^\times \hookrightarrow D^\times$. Since τ_2 is the trivial representation of D^\times , the (E, χ) -Bessel model of $\theta(\tau_1 \boxtimes \tau_2)$ is nonzero if and only if χ is trivial and τ_1 has nonzero period integral over E^\times . By a well-known result of Waldspurger [W], this holds if and only if

$$L(\frac{1}{2}, \tau_1) \cdot L(\frac{1}{2}, \tau_1 \otimes \chi_E) \neq 0$$

where χ_E is the quadratic character associated to E/F by class field theory. To summarize, we have shown that a necessary condition for the nonvanishing of $\theta(\tau_1 \boxtimes \mathbf{1})$ is the nonvanishing of $L(\frac{1}{2}, \tau_1)$. Since we are assuming that $L(\frac{1}{2}, \tau_1) = 0$, we conclude that $\theta(\tau_1 \boxtimes \mathbf{1}) = 0$. \square

8.2. On the lowest occurrence conjecture. Now we will consider the non-vanishing problem following [GJS]. First as in [GJS], we define the notion of lowest occurrence. Let π be a cuspidal automorphic representation of $\mathrm{O}(V, \mathbb{A})$, where $\dim V = m$. Then define

$$\mathrm{FO}_\psi(\pi) := \min\{r : \theta_{\psi, 2r}(\pi) \neq 0\},$$

namely $\mathrm{FO}_\psi(\pi)$ is the rank of first occurrence of the theta lift to the symplectic tower. Then we also define

$$\mathrm{LO}_\psi(\pi) := \min_\mu \{\mathrm{FO}_\psi(\pi \otimes \mu)\},$$

where μ runs through all the automorphic characters on $\mu_2(F) \backslash \mu_2(\mathbb{A})$, namely $\mathrm{LO}_\psi(\pi)$ is the rank of first occurrence “up to disconnectedness”, and following [GJS] we call it the rank of “lowest occurrence”. Note that until this point, we have been fixing the additive character ψ , but in this section we consider $\mathrm{FO}_\psi(\pi)$ and $\mathrm{LO}_\psi(\pi)$ varying as ψ varies.

Next recall that for each irreducible admissible representation π_v of $\mathrm{O}(V, F_v)$, Lapid and Rallis ([LR]) have defined the local L -factor $L_v(s, \pi_v)$ in such a way that it satisfies a number of expected

properties which determine it uniquely. Hence it makes sense to consider the complete (standard) L -function $L(s, \pi)$ of π . Then define

$$\text{Pole}(\pi) := \begin{cases} \max\{s_0 \in \{1, 2, \dots, \frac{m}{2} - 1\} : L(s, \pi) \text{ has a pole at } s = s_0\} & \text{if } m \text{ is even} \\ \max\{s_0 \in \{\frac{3}{2}, \frac{5}{2}, \dots, \frac{m}{2} - 1\} : L(s, \pi) \text{ has a pole at } s = s_0\} & \text{if } m \text{ is odd,} \end{cases}$$

whenever it exists. Now if $\text{Pole}(\pi)$ does not exist, *i.e.* the complete L -function $L(s, \pi)$ does not have a pole in the above set, then define

$$\text{Zero}(\pi) := \begin{cases} \min\{s_0 \in \{1, 2, \dots, \frac{m}{2}\} : L(s, \pi) \text{ does not vanish at } s = s_0\} & \text{if } m \text{ is even} \\ \min\{s_0 \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{m}{2} - 1\} : L(s, \pi) \text{ does not vanish at } s = s_0\} & \text{if } m \text{ is odd.} \end{cases}$$

If $\text{Zero}(\pi)$ is not defined in this way, namely the complete L -function $L(s, \pi)$ vanishes at all the s_0 in the above set, then we define

$$\text{Zero}(\pi) := \begin{cases} \frac{m}{2} + 1 & \text{if } m \text{ is even} \\ \frac{m}{2} & \text{if } m \text{ is odd,} \end{cases}$$

so that whenever $\text{Pole}(\pi)$ does not exist, $\text{Zero}(\pi)$ exists.

However, as we have seen in the previous section, our non-vanishing theorems are stated in terms of the incomplete L -function $L^S(s, \pi)$. So similarly to $\text{Pole}(\pi)$ and $\text{Zero}(\pi)$, let us define its incomplete analogue. Namely for a finite set S of places,

$$\text{Pole}^S(\pi) := \begin{cases} \max\{s_0 \in \{1, 2, \dots, \frac{m}{2} - 1\} : L^S(s, \pi) \text{ has a pole at } s = s_0\} & \text{if } m \text{ is even} \\ \max\{s_0 \in \{\frac{3}{2}, \frac{5}{2}, \dots, \frac{m}{2} - 1\} : L^S(s, \pi) \text{ has a pole at } s = s_0\} & \text{if } m \text{ is odd,} \end{cases}$$

whenever it exists, and when it does not exist we define $\text{Zero}^S(\pi)$ in the analogous way as $\text{Zero}(\pi)$. Those definitions make sense for any finite S , and so $\text{Pole}^S(\pi) = \text{Pole}(\pi)$ and $\text{Zero}^S(\pi) = \text{Zero}(\pi)$ if S is empty, though in practice we only consider the case where S contains all the ‘‘bad’’ places. Clearly,

$$\text{Pole}(\pi) \geq \text{Pole}^S(\pi) \quad \text{and} \quad \text{Zero}(\pi) \leq \text{Zero}^S(\pi),$$

whenever both sides exist, because each local factor $L(s, \pi_v)$ never has a zero.

In any case, using this language, let us state the lowest occurrence conjecture by Ginzburg-Jiang-Soudry [GJS, Conjecture 1.2] as follows.

Conjecture 8.2 (Ginzburg-Jiang-Soudry). *Let π be a cuspidal automorphic representation of $\text{O}(V, \mathbb{A})$ with $\dim V = m$ and assume that $\text{Pole}^S(\pi)$ exists. Then*

$$\text{LO}_\psi(\pi) = \frac{m}{2} - \text{Pole}^S(\pi).$$

Note that in particular, this conjecture implies that for π satisfying the assumption of the conjecture, $\text{LO}_\psi(\pi)$ is independent of the choice of ψ .

Now the non-vanishing theorems we proved in the previous section can be stated as follows, recalling that part (a) is a theorem of Ginzburg-Jiang-Soudry [GJS, Thm. 1.1].

Theorem 8.3. *Let π be a cuspidal automorphic representation of $\text{O}(V, \mathbb{A})$, S a finite set of places containing all the archimedean places and the ramified places.*

(a) *Assume $\text{Pole}^S(\pi)$ exists. Then*

$$\text{LO}_\psi(\pi) \leq \frac{m}{2} - \text{Pole}^S(\pi).$$

(b) Assume $\text{Pole}^S(\pi)$ does not exist. Further assume $\dim V$ is even, and there is a place v such that $\pi_v \cong \pi_v \otimes \det$. Then

$$\text{LO}_\psi(\pi) \leq \frac{m}{2} - 1 + \text{Zero}^S(\pi).$$

If one could show that the \leq is actually $=$ for the first case, then the conjecture by [GJS] would be proven. Also analogously to their conjecture, one might wonder if the same is true for the second case. In what follows, we will investigate this issue.

First of all, to make \leq into $=$ in the above theorem is almost synonymous to the converse of our non-vanishing theorem. However one difficulty is in how to control the analytic behavior of the “bad factors” of the local zeta integrals present in the inner product formula. One possible way to get around this difficulty is to consider the complete L -function and assume the following expected property of the local L -factor.

Conjecture 8.4. Let π_v be an irreducible admissible representation of $O(V, F_v)$, where v can be archimedean or non-archimedean. Then the normalized zeta integral

$$Z_v^*(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v) := \frac{Z_v(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v)}{L_v(s + \frac{1}{2}, \pi_v)}$$

is holomorphic for all $s \in \mathbb{C}$ and for any choice of $f_{i,v}$ and standard Φ_v .

We will take up this issue in our later work. However for the purpose of this paper, we do not need the full strength of this conjecture but the following weaker version suffices.

Conjecture 8.5. Let π_v be an irreducible admissible representation of $O(V, F_v)$, where v can be archimedean or non-archimedean. Then we have the following.

(a) The normalized zeta integral $Z_v^*(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v)$ is holomorphic at all $s \in \mathbb{C}$ such that

$$s + \frac{1}{2} \in \begin{cases} \{1, 2, \dots, \frac{m}{2} - 1\} & \text{if } m \text{ is even} \\ \{\frac{3}{2}, \frac{5}{2}, \dots, \frac{m}{2} - 1\} & \text{if } m \text{ is odd.} \end{cases}$$

(b) The normalized zeta integral $Z_v^*(s, f_{1,v}, f_{2,v}, \Phi_v, \pi_v)$ is holomorphic at all $s \in \mathbb{C}$ such that

$$s + \frac{1}{2} \in \begin{cases} \{1, 2, \dots, \frac{m}{2} + 1\} & \text{if } m \text{ is even} \\ \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{m}{2}\} & \text{if } m \text{ is odd.} \end{cases}$$

The reason we have two slightly different versions is that, as we will see below, when both $\text{Pole}^S(\pi)$ and $\text{Pole}(\pi)$ exist, the first version suffices, and when both $\text{Zero}^S(\pi)$ and $\text{Zero}(\pi)$ exist, the second one does.

Another difficulty lies in the fact that our second term identity is “weak” in the sense that it works only for those φ 's in $\mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))^\circ$. Of course, it is also expected that the second term identity is true in the strong form. Namely, we have

Conjecture 8.6. The weak second term identity (Theorem 5.6) as well as the weak first term identity on the boundary (Proposition 5.3) can be extended to all $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$.

Then we can prove the following

Proposition 8.7. Let π be a cuspidal automorphic representation of $O(V, \mathbb{A})$ with $\dim V = m$.

(a) Assume Conjecture 8.5 (a), and both $\text{Pole}(\pi)$ and $\text{Pole}^S(\pi)$ exist. Then

$$\frac{m}{2} - \text{Pole}(\pi) \leq \text{LO}_\psi(\pi) \leq \frac{m}{2} - \text{Pole}^S(\pi).$$

(b) Assume Conjecture 8.5 (b) and 8.6 and both $\text{Zero}(\pi)$ and $\text{Zero}^S(\pi)$ exist. Further assume that m is even, and there is a place v such that $\pi_v \cong \pi_v \otimes \det$. Then

$$\frac{m}{2} - 1 + \text{Zero}(\pi) \leq \text{LO}_\psi(\pi) \leq \frac{m}{2} - 1 + \text{Zero}^S(\pi).$$

Proof. First let us consider (b). We already know that $\text{LO}_\psi(\pi) \leq \frac{m}{2} - 1 + \text{Zero}^S(\pi)$. Let $r = \text{LO}_\psi(\pi)$ and so there exists μ such that $\theta_{\psi, 2r}(\pi \otimes \mu) \neq 0$, and moreover since $r = \text{LO}_\psi(\pi)$, it is in the space of cusp forms. So there exists $f_1, f_2 \in \pi \otimes \mu$ and $\phi_1, \phi_2 \in \mathcal{S}((V \otimes W^+)(\mathbb{A}))$ such that

$$\langle \theta_{2r}(f_1, \phi_1), \theta_{2r}(f_2, \phi_2) \rangle \neq 0,$$

where $\dim W = 2r$. Notice that we may assume that f_i 's are factorizable. But we do not know if $\sigma(\phi_1 \otimes \phi_2)$ is in $\mathcal{S}((\mathcal{V}^+ \otimes W^+)(\mathbb{A}))^\circ$. (Here σ is as in Proposition 6.3.) However if we assume Conjecture 8.6, by using essentially the same proof as the one for Theorem 6.4, we can obtain

$$\begin{aligned} & \langle \theta_{2r}(f_1, \phi_1), \theta_{2r}(f_2, \phi_2) \rangle \\ &= \frac{1}{\kappa d_m^S(\rho_{m,r})} \sum_i \text{Val}_{s=\rho_{m,r}} \left(L^S\left(s + \frac{1}{2}, \pi \otimes \mu\right) \prod_{v \in S} Z_v(s, f_{1,v}, f_{2,v}, \Phi_{i,v}, \pi_v \otimes \mu_v) \right) \\ &= \frac{1}{\kappa d_m^S(\rho_{m,r})} \sum_i \text{Val}_{s=\rho_{m,r}} \left(L\left(s + \frac{1}{2}, \pi \otimes \mu\right) \prod_{v \in S} Z_v^*(s, f_{1,v}, f_{2,v}, \Phi_{i,v}, \pi_v \otimes \mu_v) \right), \end{aligned}$$

for some standard section $\sum_i \Phi_i$ so that each Φ_i is factorizable. Then if the left hand side is non-zero, then for some i ,

$$\text{Val}_{s=\rho_{m,r}} \left(L\left(s + \frac{1}{2}, \pi \otimes \mu\right) \prod_{v \in S} Z_v^*(s, f_{1,v}, f_{2,v}, \Phi_{i,v}, \pi_v \otimes \mu_v) \right)$$

must be non-zero. By Conjecture 8.5 (b), we know that each product

$$\prod_{v \in S} Z_v^*(s, f_{1,v}, f_{2,v}, \Phi_{i,v}, \pi_v \otimes \mu_v)$$

is holomorphic at $s = \rho_{m,r}$. Also by our assumption, $L(s + \frac{1}{2}, \pi)$, which is the same as $L(s + \frac{1}{2}, \pi \otimes \mu)$, does not have a pole at this point. So all those imply $L(s + \frac{1}{2}, \pi)$ must be non-zero at $s = \rho_{m,r}$. Then we have

$$\text{Zero}(\pi) \leq r - \frac{m}{2} + 1.$$

Hence the proposition follows.

Next we consider (a). For this ‘‘pole range’’, to invoke the inner product formula to the extent we need, we do not have to extend the first term identity as in Proposition 5.1, but we only need the version of Proposition 3.3, which already works for any $\varphi \in \mathcal{S}((\mathcal{V}^+ \otimes W)(\mathbb{A}))$. Then the proposition follows in a similar way as (b). The detail is left to the reader. \square

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