J. reine angew. Math. **528** (2000), 149—181

Journal für die reine und angewandte Mathematik


# A Siegel-Weil formula for exceptional groups

By *Wee Teck Gan* at Princeton

**Abstract.** We obtain a Siegel-Weil formula for the dual pair $G_2 \times F_4$ in the quaternionic form of $E_8$ over $\mathsf{Q}$, with $G_2$ split and $F_4$ anisotropic. This identifies the integral over $F_4$ of a theta function on $E_8$ as an Eisenstein series on $G_2$. The Fourier coefficients of the theta integral are related to an embedding problem involving cubic forms.

## Introduction

If $\big(Sp(U), O(V)\big)$ is a dual reductive pair in $Sp(U \otimes V)$, the classical Siegel-Weil formula expresses the integral over $O(V)$ of a theta function on $Sp(U \otimes V)$ as a special value of a corresponding Eisenstein series on $Sp(U)$, at least when some convergence conditions are satisfied so that both sides of the identity are well-defined. In its greatest generality, this formula was established by Kudla and Rallis in [KR1] and [KR2]. Later, in [KR3], they proved a regularized Siegel-Weil formula, even when the convergence conditions are not satisfied. Coupled with the Rankin-Selberg method, the Siegel-Weil formula has been an important tool in the study of special values of automorphic $L$-functions (cf. for example [GrK] and [HK]). With the discovery of Rankin-Selberg integrals involving Eisenstein series on exceptional groups, it is natural to ask for an extension of the Siegel-Weil formula to more general dual reductive pairs. In this paper, we take the first step towards such a formula in exceptional groups, by considering the simplest possible case where one member of the dual pair is anisotropic.

To be precise, let $H$ be the quaternionic form of $E_8$ over $\mathsf{Q}$. Hence $H$ has $\mathsf{Q}$-rank 4, and is split at all finite primes $p$. There is a dual reductive pair $G_2 \times G$ in $H$. Here, $G_2$ is split, whereas $G$ is anisotropic of type $F_4$; it is the automorphism group of the exceptional Jordan algebra of $3 \times 3$ hermitian matrices with coefficients in the $\mathsf{Q}$-algebra of Cayley's octonions. In [G], we have constructed an automorphic realization

$$\theta\colon \Pi \hookrightarrow L^2\big(H(\mathsf{Q})\backslash H(\mathsf{A})\big)$$

of the global minimal representation $\Pi = \hat{\bigotimes_v} \Pi_v$ of $H(\mathsf{A})$, and studied the lifting of non-trivial automorphic forms from $G$ to $G_2$ via the theta kernel. Here, we consider the theta lift of the trivial representation of $G(\mathsf{A})$, i.e. the theta integral

$$I(\theta(f))(g) = \int_{G(\mathrm{Q})\backslash G(\mathrm{A})} \theta(f)(gh)\, dh$$

for $f \in \Pi$, which defines an automorphic form on $G_2$.

For us, the Siegel-Weil formula consists of the following two statements about $I(\theta(f))$:

(i) $I(\theta(f))$ is a linear combination of theta functions, for a suitable $f$.

(ii) $I(\theta(f))$ is an Eisenstein series $E(f)$.

The first statement is more arithmetic in nature. In the classical case, it is almost immediate and follows from unravelling the definition. This is because we have a good notion of Fourier coefficients for Siegel modular forms and a good model for the Weil representation. In the exceptional case, it is not even clear a priori what a theta function on $G_2$ means. It turns out that the main problem here is to formulate a refined theory of Fourier coefficients for certain automorphic forms $\varphi$ on $G_2$. Such a theory has now been developed in [GGS], with the result that one can attach to $\varphi$ a collection of complex numbers $\{c_A(\varphi)\}$, indexed by cubic rings $A$ over $\mathrm{Z}$ with $A \otimes \mathrm{R} \cong \mathrm{R}^3$ and well-defined up to a universal scaling. The numbers $c_A(\varphi)$ are the refined Fourier coefficients of $\varphi$. With this notion of Fourier coefficients, our first result is that $I(\theta(f))$ is indeed a linear combination of theta functions. These theta functions are so-called because their $A$-th Fourier coefficients count the number of embeddings of $A$ into certain integral models of the exceptional Jordan algebra, an embedding problem studied in [GG] and [GG2].

The proof of the above arithmetic result is based on a careful study of the Fourier coefficients (in the usual sense) of $\theta(f)$. This is possible even though we do not have a good model for the minimal representation, but requires a very careful setting up of notations. The first ten sections of the paper systematically introduce these notations and the basic objects of interest, as well as recall and refine various results needed from the literature. Though they contain hardly anything new, they are quite necessary, especially the results of Section 6 on integral models. After this long preparation, we construct in Section 11 some theta functions on $G_2$, and show in Section 12 that $I(\theta(f))$ is a linear combination of these theta functions.

The second statement in the Siegel-Weil formula is more a result about automorphic forms, and occupies the rest of the paper. We study in Section 13 a particular Eisenstein series on $G_2$, and in Section 14, we identify the non-cuspidal part of the theta integral with this Eisenstein series. Finally, we show in the final section that the cuspidal part of the theta integral vanishes.

**Acknowledgments.** It is a pleasure to thank Professor Benedict H. Gross for his encouragement, and for many stimulating discussions during the course of this work. Thanks are also due to Professor Gordan Savin for his invaluable assistance, and Professor Steve Rallis for pointing out the relevance of the paper [KV].

**Notations.** We let $\hat{\mathrm{Z}} = \prod_p \mathrm{Z}_p$ denote the profinite completion of $\mathrm{Z}$, and let $\hat{\mathrm{Q}} = \hat{\mathrm{Z}} \otimes \mathrm{Q}$. The adele ring of $\mathrm{Q}$ is denoted by $\mathrm{A}$.

If $V$ is a vector space over $\mathsf{Q}$, we shall often let $\mathsf{V}$ denote a lattice in $V$. If $v \in \mathsf{V}$, the content $c(v)$ of $v$ (relative to $\mathsf{V}$) is the largest positive integer $e$ for which $v \in e \cdot \mathsf{V}$. This notion makes sense locally as well: if $v \in \mathsf{V} \otimes \mathsf{Z}_p$, its content $c_p(v)$ is the largest $p$-power such that $v \in c_p(v) \cdot (\mathsf{V} \otimes \mathsf{Z}_p)$. Moreover, $c(v) = \prod_p c_p(v)$ for $v \in \mathsf{V}$. Say that $v$ is primitive if it has content 1.

Finally, if $G$ is an algebraic group over $\mathsf{Q}$, $\underline{G}$ will denote an integral model of $G$, i.e. a group scheme over $\mathsf{Z}$ with generic fiber $G$.

## 1. A counting problem

We begin by recalling a counting problem which was studied in [GG] and [GG2], and which motivates the investigations of this paper. The material below is largely taken from [GG2], §1.

Let $R$ be Coxeter's order in the $\mathsf{Q}$-algebra of Cayley's octonions [GG], p. 265, and let $J_3$ be the free abelian group of $3 \times 3$ Hermitian symmetric matrices with entries in $R$. An element of $J_3$ looks like

$$X = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix} \tag{1.1}$$

with $a, b, c \in \mathsf{Z}$, and $x, y, z \in R$. The determinant map

$$\mathrm{N}\colon J_3 \to \mathsf{Z}, \tag{1.2}$$
$$\mathrm{N}(X) = abc + \mathrm{Tr}(xyz) - a \cdot \mathrm{N}(x) - b \cdot \mathrm{N}(y) - c \cdot \mathrm{N}(z)$$

gives a natural cubic form on $J_3$. The identity matrix $I$ and the matrix

$$E = \begin{pmatrix} 2 & \alpha & \bar{\alpha} \\ \bar{\alpha} & 2 & \alpha \\ \alpha & \bar{\alpha} & 2 \end{pmatrix} \tag{1.3}$$

with $\alpha = \frac{1}{2}(-1 + e_1 + e_2 + \cdots + e_7) \in R$ both satisfy $\mathrm{N}(I) = \mathrm{N}(E) = 1$. Hence the triples

$$\begin{cases} J_I = (J_3, \mathrm{N}, I), \\ J_E = (J_3, \mathrm{N}, E) \end{cases}$$

are pointed cubic spaces over $\mathsf{Z}$.

Using $I$ and $E$, one can define symmetric bilinear forms $T_I$ and $T_E$ on $J_3$, and quadratic maps $\#_I$ and $\#_E$ from $J_3$ to itself [EG]. For example, we have:

$$T_I(X, Y) = \frac{1}{2} \cdot \mathrm{Tr}(XY + YX), \tag{1.4}$$

where $\mathrm{Tr}\colon J_3 \to \mathsf{Z}$ is the trace form, and $XY$ denotes usual matrix multiplication. The lattices $(J_3, T_I)$ and $(J_3, T_E)$ are both unimodular. The 5-tuples

$$\begin{cases} J_I = (J_3, \mathrm{N}, I, \#_I, T_I), \\ J_E = (J_3, \mathrm{N}, E, \#_E, T_E) \end{cases}$$

are cubic norm structures over $\mathsf{Z}$ (cf. [KMRT], §38 and [GG2], §2), which give rise to isomorphic Jordan algebras over $\mathsf{Z}[1/2]$. In particular,

$$F = J_I \otimes \mathsf{Q} \cong J_E \otimes \mathsf{Q}$$

is an exceptional Jordan algebra over $\mathsf{Q}$.

Let $A$ be an arbitrary order in an étale cubic algebra $k$ over $\mathsf{Q}$, with norm map $\mathsf{N}$ and trace map Tr. The triple $(A, \mathsf{N}, 1)$ is a pointed cubic space over $\mathsf{Z}$, which gives rise to a cubic norm structure $(A, \mathsf{N}, 1, \#, T)$, where

$$\begin{cases} a^{\#} = a^{-1} \cdot \mathsf{N}(a) & \text{for } a \neq 0, \\ T(a,b) = \mathrm{Tr}(a \cdot b). \end{cases}$$

In [GG], we counted the number of embeddings

$$\begin{cases} A \hookrightarrow J_I, \\ A \hookrightarrow J_E \end{cases}$$

of pointed cubic spaces over $\mathsf{Z}$, when $A$ is the *maximal* order of integral elements in $k$. It was shown in [GG], Lemma 2 that any such embedding is also an embedding of the corresponding cubic norm structures. Let $N(A, J_I)$ and $N(A, J_E)$ denote the number of such embeddings, and let $\Gamma_I$ and $\Gamma_E$ be the finite groups of automorphisms of $J_I$ and $J_E$ (as pointed cubic spaces or cubic norm structures). Then what we computed in [GG] is the number

$$N_A = \frac{N(A, J_I)}{\#\Gamma_I} + \frac{N(A, J_E)}{\#\Gamma_E}. \tag{1.5}$$

This is zero unless $A \otimes \mathsf{R} \cong \mathsf{R}^3$, in which case it is given in terms of the value of the zeta function $\zeta_A(s)$ of $A$ at $s = -3$.

In the final section of [GG], we speculated that the numbers $N_A$, for $A$ maximal, occur as the Fourier coefficients of an automorphic form on $G_2$. We further speculated that there is another automorphic form on $G_2$ whose Fourier coefficients are the numbers $M_A = N(A, J_I) - N(A, J_E)$. This paper began as an attempt to uncover and understand these automorphic forms.

## 2. The group $G$ and the genus of $J_I$

Let $L = \mathrm{Aut}(F, \mathrm{N})$ be the algebraic group of linear maps on $F$ which preserves the cubic form N. It is a quasi-simple simply-connected linear algebraic group over $\mathsf{Q}$ of type $E_6$, which has real rank 2 and is split over $\mathsf{Q}_p$ for all primes $p$. Let $G = \mathrm{Aut}(F, \mathrm{N}, I)$ be the algebraic subgroup of $L$ which fixes $I$. It is anisotropic of type $F_4$ over $\mathsf{R}$, and is split over $\mathsf{Q}_p$ for all $p$. Moreover, $G$ is the automorphism group of the Jordan algebra $F$.

The pointed cubic spaces $J_I$ and $J_E$ are globally inequivalent [EG], but are isomorphic over $\mathsf{Q}$ and $\mathsf{Z}_p$ for all $p$. We say that they are in the same genus. The equivalence classes of pointed cubic spaces in the genus of $J_I$ are parametrized by the double coset space $G(\mathsf{Q})\backslash G(\hat{\mathsf{Q}})/K_I$, where $K_I$ is the open compact subgroup of $G(\hat{\mathsf{Q}})$ which stabilizes the lattice $J_3 \otimes \hat{\mathsf{Z}}$. The bijection is explicitly constructed as follows. For $\alpha \in G(\hat{\mathsf{Q}})$, let $J_\alpha$ be the lattice in $F = J_3 \otimes \mathsf{Q}$ which is uniquely determined by the requirement that

$$J_\alpha \otimes \mathsf{Z}_p = \alpha \cdot (J_3 \otimes \mathsf{Z}_p). \tag{2.1}$$

Then $I \in J_\alpha$ and the cubic form N takes integer values on $J_\alpha$. The pointed cubic space corresponding to $\alpha$ is the triple $(J_\alpha, \mathrm{N}, I)$, and the 5-tuple $(J_\alpha, \mathrm{N}, I, \#, T)$ is the corresponding cubic norm structure. Here we have used $\#$ and $T$ in place of $\#_I$ and $T_I$ for simplicity. It was shown in [Gr] that

$$\# G(\mathsf{Q})\backslash G(\hat{\mathsf{Q}})/K_I = 2, \tag{2.2}$$

so that $J_I$ and $J_E$ represent the two distinct equivalence classes in the genus of $J_I$. In the sequel, it will be more convenient for us to work with the representative $(J_\alpha, \mathrm{N}, I)$ (for suitable $\alpha$) in place of $J_E = (J_3, \mathrm{N}, E)$.

Fix $\alpha \in G(\hat{\mathsf{Q}})$ and let $A$ be an order in an étale cubic $\mathsf{Q}$-algebra $k$. For any $\mathsf{Z}$-algebra $B$, let $S(A \otimes B, \alpha)$ denote the set of morphisms of cubic norm structures $A \otimes B \to J_\alpha \otimes B$. We have:

**Proposition 2.3.** (i) *$S(A \otimes \mathsf{Q}_p, \alpha)$ is non-empty, and $G(\mathsf{Q}_p)$ acts transitively on $S(A \otimes \mathsf{Q}_p, \alpha)$.*

(ii) *$S(A \otimes \mathsf{R}, \alpha)$ is non-empty if and only if $A \otimes \mathsf{R} \cong \mathsf{R}^3$, in which case $G(\mathsf{R})$ acts transitively on $S(A \otimes \mathsf{R}, \alpha)$.*

(iii) *$S(A \otimes \mathsf{Q}, \alpha)$ is non-empty if and only if $A \otimes \mathsf{R} \cong \mathsf{R}^3$, in which case $G(\mathsf{Q})$ acts transitively on $S(A \otimes \mathsf{Q}, \alpha)$.*

*Proof.* (i) is [J], Thm. 10, p. 389; (ii) and (iii) follow by a standard cohomological argument [GG], §5. □

We shall show that for each $\alpha \in G(\hat{\mathsf{Q}})$, there is an automorphic form $\varphi_\alpha$ on $G_2$ whose Fourier coefficients are the numbers $\# S(A, J_\alpha)$. This exploits the fact that $G \times G_2$ is a dual reductive pair in the quaternionic form of $E_8$ over $\mathsf{Q}$. We review the construction of this dual pair in the following section.

## 3. The dual pair $G_2 \times G$ in $H$

Let $H$ be the (unique) algebraic group over $\mathsf{Q}$ of type $E_8$, which has real rank 4, and is split over $\mathsf{Q}_p$ for all $p$. We begin with a construction of the Lie algebra $\mathfrak{h}$ of $H$.

Let $V = \langle e_1, e_2, e_3 \rangle$ be a 3-dimensional vector space over $\mathbf{Q}$, and let

$$V^* = \mathrm{Hom}(V, \mathbf{Q})$$

with dual basis $\{e_1^*, e_2^*, e_3^*\}$. Let $\mathfrak{sl}(V)$ be the Lie algebra of endomorphisms of $V$ with trace zero. We identify $\mathfrak{sl}(V)$ with $\mathfrak{sl}_3$ using the basis $\{e_1, e_2, e_3\}$, and write $e_{ij}$ for the endomorphism $e_j^* \otimes e_i$ of $V$.

The Lie algebra $\mathfrak{h}$ can be described elegantly using a $\mathbf{Z}/3\mathbf{Z}$-grading:

$$\mathfrak{h} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1 \tag{3.1}$$

where

$$\begin{cases} \mathfrak{h}_0 = \mathfrak{sl}(V) \oplus \mathrm{Lie}(L), \\ \mathfrak{h}_1 = V \otimes F, \\ \mathfrak{h}_{-1} = V^* \otimes F^*. \end{cases} \tag{3.2}$$

We refer the reader to [Ru] and [S2] for the definition of the Lie brackets in $\mathfrak{h}$. Using the pairing $T = T_I$ of (1.4), we identify $F^* = \mathrm{Hom}(F, \mathbf{Q})$ with $F$ in (3.2). The algebraic group $H$ is the automorphism group of the Lie algebra $\mathfrak{h}$. It has Lie algebra $\mathfrak{h}$, and contains the algebraic subgroup $(SL_3 \times L)/\Delta\mu_3$, whose Lie algebra is $\mathfrak{h}_0$. Indeed (3.1) gives the decomposition of $\mathfrak{h}$ as a representation of this subgroup, while the grading is induced by the action of its center $\mu_3$.

As discussed in [S2], the Killing form of $\mathfrak{h}$ restricts to a Killing form $\langle -, - \rangle$ on $\mathfrak{h}_0$, and induces a nondegenerate $\mathfrak{h}_0$-invariant pairing

$$\langle -, - \rangle : \mathfrak{h}_1 \times \mathfrak{h}_{-1} \to \mathbf{Q}. \tag{3.3}$$

One can normalize the Killing form on $\mathfrak{h}$ so that on $\mathfrak{sl}(V)$, we have:

$$\langle a, b \rangle = \mathrm{Tr}(ab) \tag{3.4}$$

and the pairing (3.3) is given by

$$\langle v \otimes X, v^* \otimes Y \rangle = \langle v, v^* \rangle \cdot T(X, Y). \tag{3.5}$$

We can now construct the dual pair $G_2 \times G \hookrightarrow H$. Let $I \in J_3$ be the identity matrix. The stabilizer in $L$ of $V \otimes I \subset \mathfrak{h}_1$ is the group $G$ by definition (cf. §2). The centralizer of $G$ in $H$ contains $SL(V)$ and has Lie algebra equal to

$$\mathfrak{g}_2 = \mathfrak{sl}(V) \oplus (V \otimes I) \oplus (V^* \otimes I). \tag{3.6}$$

From the Lie brackets, one can check that this is the Lie algebra of the split group $G_2$. Hence the connected component of the centralizer of $G$ is the split group $G_2$, and we have obtained an explicit embedding $G_2 \times G \hookrightarrow H$. Since $G_2 \times G$ is a maximal subgroup in $H$ [GrS], p. 210, $G_2$ is in fact the full centralizer of $G$, so that $G_2 \times G$ is a dual reductive pair in $H$.

## 4. The Heisenberg parabolic of $H$

The group $H$ has a maximal parabolic subgroup $P = M \cdot N$ known as the Heisenberg parabolic. The Levi subgroup $M$ has connected center $Z \cong \mathsf{G}_m$, and its derived group is simply-connected of type $E_7$ and relative rank 3 [GrS]. The unipotent radical $N$ is a 57-dimensional Heisenberg group with 1-dimensional center $Z_2$. We fix an embedding $P \hookrightarrow H$ by describing $\mathsf{p} = \mathrm{Lie}(P)$ as a subalgebra of $\mathsf{h}$ using (3.1) and (3.2).

Let

$$h = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \in \mathsf{sl}(V)$$

and for $i \in \mathsf{Z}$, set

$$\mathsf{h}(i) = \{x \in \mathsf{h} : [h, x] = i \cdot x\}.$$

Then we have:

$$\text{(4.1)} \qquad \begin{cases} \mathsf{h}(0) = \mathsf{t} \oplus \mathrm{Lie}(L) \oplus (e_2 \otimes F) \oplus (e_2^* \otimes F), \\ \mathsf{h}(1) = \mathsf{Q}e_{12} \oplus (e_1 \otimes F) \oplus (e_3^* \otimes F) \oplus \mathsf{Q}e_{23}, \\ \mathsf{h}(2) = \mathsf{Q}e_{13} \end{cases}$$

where $\mathsf{t}$ denotes the diagonal torus of $\mathsf{sl}(V)$. The Lie algebra $\mathsf{p} = \mathsf{m} \oplus \mathsf{n}$ of $P$ is given by:

$$\mathsf{p} = \mathsf{h}(0) \oplus \mathsf{h}(1) \oplus \mathsf{h}(2),$$

with

$$\text{(4.2)} \qquad \begin{cases} \mathsf{m} = \mathsf{h}(0), \\ \mathsf{n} = \mathsf{h}(1) \oplus \mathsf{h}(2), \\ \mathsf{z}_2 = \mathsf{h}(2). \end{cases}$$

We can choose a maximal split torus $T_0$ of $H$, containing the diagonal torus of $SL_3$ and a positive system of roots such that $\mathsf{h}(2) = \mathsf{Q}e_{13}$ is the root subspace of $\mathsf{h}$ corresponding to the highest root $\alpha_0$. Then the diagonal torus of $SL(V)$ serves as the maximal split torus of $G_2$, with highest root $\delta$. Note that $\delta$ is simply the restriction of $\alpha_0$ to the diagonal torus of $SL(V)$.

Let

$$\text{(4.3)} \qquad W = \mathsf{h}(-1) = \mathsf{Q}e_{21} \oplus (e_1^* \otimes F) \oplus (e_3 \otimes F) \oplus \mathsf{Q}e_{32}.$$

We shall denote an element of $W$ as $(a, B, C, d)$. Using the exponential map, and the pairings in (3.4) and (3.5), we have:

$$\text{(4.4)} \qquad W \cong \mathrm{Hom}(\mathsf{h}(1), \mathsf{Q}) \cong \mathrm{Hom}(N, \mathsf{G}_a).$$

Consider the action of $M$ on $W$. The center $Z$ acts by a fundamental character, and we fix an isomorphism of $Z$ with $\mathsf{G}_m$ so that $Z$ acts on $W$ by the identity character. The derived group of $M$ acts by the 56-dimensional minuscule representation, described for example in [GrS], Ch. 2, §3. A highest weight vector of $W$ with respect to $T_0$ is the vector $e_{21} = (1,0,0,0)$. The stabilizer of the line $\mathsf{Q}e_{21}$ is a maximal parabolic subgroup $Q$ of $M$ with Lie algebra $\mathsf{t} \oplus \mathrm{Lie}(L) \oplus (e_2 \otimes F)$. Let $\Omega$ be the $M$-orbit of $e_{21}$; it is a subvariety of the affine space $W$. Then we have [G], Lemma 2.7:

**Lemma 4.5.** *The element $(a,B,C,d) \in W$, with $a \neq 0$, lies in $\Omega$ if and only if*

$$\begin{cases} C = B^{\#}/a, \\ d = N(B)/a^2. \end{cases}$$

## 5. The Heisenberg parabolic of $G_2$

From the description of the embeddings $G_2 \times G \hookrightarrow H$ and $P \hookrightarrow H$, we see that

$$(G_2 \times G) \cap P = P_2 \times G \tag{5.1}$$

where $P_2 = L_2 \cdot U_2$ is a maximal parabolic subgroup of $G_2$, known as the Heisenberg parabolic. On the level of Lie algebras, we have:

$$\begin{cases} \mathsf{l}_2 = \mathsf{t} \oplus \mathsf{Q}(e_2 \otimes I) \oplus \mathsf{Q}(e_2^* \otimes I), \\ \mathsf{u}_2 = \mathsf{u}_2(1) \oplus \mathsf{u}_2(2), \\ \mathsf{u}_2(1) = \mathsf{Q}e_{12} \oplus \mathsf{Q}(e_1 \otimes I) \oplus \mathsf{Q}(e_3^* \otimes I) \oplus \mathsf{Q}e_{23} \subset \mathsf{h}(1), \\ \mathsf{u}_2(2) = \mathsf{z}_2 = \mathsf{Q}e_{13}. \end{cases} \tag{5.2}$$

Thus $U_2 \subset N$ is a Heisenberg group of dimension 5 with center $Z_2$, and the Levi subgroup $L_2$ is isomorphic to $GL_2$. We have already fixed an isomorphism of $Z$ with $\mathsf{G}_m$ in the previous section; we now fix an isomorphism of $L_2$ with $GL_2$ which restricts to the given isomorphism of their centers. As a result, the modulus character $\delta_{P_2}$ of $P_2$ is given by: $\delta_{P_2}(\gamma) = (\det\gamma)^{-3}$, for $\gamma \in GL_2$.

Let

$$\begin{aligned} V_2 &= \mathsf{g}_2 \cap \mathsf{h}(-1) \\ &= \mathsf{Q}e_{21} \oplus \mathsf{Q}(e_1^* \otimes I) \oplus \mathsf{Q}(e_3 \otimes I) \oplus \mathsf{Q}e_{32}. \end{aligned} \tag{5.3}$$

We identify the element

$$a \cdot e_{21} + b \cdot \frac{1}{3} e_1^* \otimes I + c \cdot \frac{1}{3} e_3 \otimes I + d \cdot e_{32} \in V_2$$

with $(a,b,c,d)$. The pairings (3.4) and (3.5), and the exponential map, give an identification

$$V_2 \cong \mathrm{Hom}\big(\mathsf{u}_2(1), \mathsf{Q}\big) \cong \mathrm{Hom}(U_2, \mathsf{G}_a). \tag{5.4}$$

Now consider the action of $L_2$ on $V_2$. If we identify $V_2$ with the space of binary cubic forms over Q by sending $(a,b,c,d) \in V_2$ to $q(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, then the action of $L_2$ on $V_2$ can be idenitifed with the following action of $GL_2$ on binary cubic forms:

$$(\gamma \cdot q)(x,y) = \det \gamma^{-1} \cdot q(Ax + Cy, Bx + Dy)$$

for

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2.$$

Now we have [Wr]:

**Proposition 5.5.** *For any field extension $E$ of $\mathsf{Q}$, the $L_2(E)$-orbits on $V_2 \otimes E$ are in bijection with the isomorphism classes of cubic $E$-algebras. The generic orbits correspond to étale cubic algebras, and are represented by those binary cubic forms $q$ whose discriminant $\Delta(q)$ is non-zero.*

**Example.** Every $L_2$-orbit has a representative $q = (a,b,c,d)$ with $a \neq 0$, and for such a $q$, the corresponding $E$-algebra is $k = E[x]/(ax^3 - bx^2 + cx - d)$.

The inclusion $U_2 \hookrightarrow N$ induces a restriction map $\mathrm{Hom}(N, \mathsf{G}_a) \to \mathrm{Hom}(U_2, \mathsf{G}_a)$ given by

$$\tau\colon (a,B,C,d) \mapsto \big(a, \mathrm{Tr}(B), \mathrm{Tr}(C), d\big) \tag{5.6}$$

which is equivariant with respect to the action of $L_2$. For any extension field $E$ of $\mathsf{Q}$, and any *generic* $q \in V_2 \otimes E$, with corresponding étale algebra $k$, let

$$\Omega(q) = \{x \in \Omega(E) \colon \tau(x) = q\}.$$

Then we have:

**Proposition 5.7.** *$\Omega(q)$ is in natural bijection with the set of embeddings $k \hookrightarrow F \otimes E$ of cubic norm structures. Moreover, if $E = \mathsf{Q}$ or $\mathsf{R}$, both sets are empty unless $A \otimes \mathsf{R} \cong \mathsf{R}^3$.*

*Proof.* Without loss of generality, assume that $q = (a,b,c,d)$ with $a \neq 0$, so that $k$ is as given in the above example. To give an embedding of $k$ into $F \otimes E$, it is necessary and sufficient to specify the image $X$ of $x$, which can be any element satisfying:

$$\begin{cases} \mathrm{Tr}(X) = b/a, \\ \mathrm{Tr}(X^\#) = c/a, \\ \mathrm{N}(X) = d/a. \end{cases}$$

On the other hand, by Lemma 4.5, the map

$$X \mapsto \big(a, a \cdot X, a \cdot X^\#, a \cdot \mathrm{N}(X)\big)$$

sets up a bijection between the set of such elements $X$ and $\Omega(q)$. $\square$

## 6. Integral models

In this section, we fix $\alpha \in G(\hat{\mathsf{Q}})$ and use it to carry out the constructions of Sections 2–5 over $\mathsf{Z}$. In particular, we construct integral models $\underline{L}$, $\underline{H}$, $\underline{G}$, $\underline{G_2}$, $\underline{P}$ and $\underline{P_2}$. We stress that these models depend a priori on the choice of $\alpha$. This choice of $\alpha$ will be fixed from this section to Section 11, and we shall suppress $\alpha$ in many of the notations for simplicity.

Let $J = J_\alpha$ be the lattice in $F$ determined by $\alpha$ in (2.1). As we have noted before, the symmetric bilinear module $(J, T)$ is unimodular. Let $\underline{L}$ be the group scheme of automorphisms of the lattice $J$ which preserves the cubic form $\mathsf{N}$; it is a group over $\mathsf{Z}$ in the sense of [Gr]. In other words, it is a smooth group scheme over $\mathsf{Z}$, with generic fiber $L$ and whose closed fibers are reductive [Gr], §4. For different choices of $\alpha$, the integral models $\underline{L}$ are isomorphic, by the strong approximation theorem. Let $\underline{G}$ be the subgroup scheme of $\underline{L}$ fixing $I$; it is also a group over $\mathsf{Z}$ [Gr], §4, so that $\underline{G}(\mathsf{Z}_p)$ is a hyperspecial maximal compact subgroup of $G(\mathsf{Q}_p)$ for all $p$.

Let $\mathsf{V}$ be the lattice spanned by $\{e_1, e_2, e_3\}$ in $V$. Using $\mathsf{V}$ and $J$, we can give a model of $\underline{H}$ over $\mathsf{Z}$, with good reduction [Gr] over $\mathsf{Z}/p\mathsf{Z}$ for all $p$. More precisely, the lattice

$$\mathsf{H} = \mathrm{Lie}\big((SL(\mathsf{V}) \times \underline{L})/\Delta\mu_3\big) \oplus (\mathsf{V} \otimes J) \oplus (\mathsf{V}^* \otimes J) \tag{6.1}$$

is a Lie algebra over $\mathsf{Z}$, and one can check that over $\mathsf{Z}_p$, it is the Chevalley order of $\mathfrak{h}$. Let $\underline{H}$ be the schematic closure of $H$ in $GL(\mathsf{H})$. The results of [BT], §3.2 and [B], §4 imply that $\underline{H}$ is smooth over $\mathsf{Z}$ with Lie algebra $\mathsf{H}$, and $K_\alpha = \underline{H}(\hat{\mathsf{Z}})$ is a hyperspecial maximal compact subgroup of $H(\hat{\mathsf{Q}})$. In particular, we have the Iwasawa decomposition: $H(\hat{\mathsf{Q}}) = P(\hat{\mathsf{Q}}) \cdot K_\alpha$.

The smooth integral model $\big(SL(\mathsf{V}) \times \underline{L}\big)/\Delta\mu_3$ acts faithfully on $\mathsf{H}$ and thus embeds as a closed subgroup of $\underline{H}$ [BT], §1.2.5. Hence, we have an embedding $\underline{G} \hookrightarrow \underline{L} \hookrightarrow \underline{H}$, which realizes $\underline{G}$ as the pointwise stabilizer of the sublattice $\mathsf{V} \otimes I \subset \mathsf{H}$ in $\underline{L}$. In particular, the schematic closure of $G$ in $\underline{H}$ is the smooth model $\underline{G}$.

Similarly, one can show that the schematic closure of $G_2$ in $\underline{H}$ is the Chevalley model $\underline{G_2}$. It is the closed subgroup scheme of $\underline{H}$ which fixes $\mathrm{Lie}(\underline{G})$ pointwise, and has Lie algebra

$$\mathsf{sl}(\mathsf{V}) \oplus (\mathsf{V} \otimes I) \oplus (\mathsf{V}^* \otimes I) \subset \mathsf{H}.$$

It is important to note that the closed subgroup scheme $\underline{G_2}$ of $\underline{H}$ is independent of the choice of $\alpha$, by which we mean that for any $\mathsf{Z}$-algebra $B$, the subgroup $\underline{G_2}(B)$ of $G_2(B \otimes \mathsf{Q})$ is independent of the choice of $\alpha$. In particular, we obtain a specific hyperspecial maximal compact subgroup of $G_2(\hat{\mathsf{Q}})$, and we have the dual pair $\underline{G_2} \times \underline{G} \hookrightarrow \underline{H}$.

Now let $\underline{P} = \underline{M} \cdot \underline{N}$ be the schematic closure of $P$ in $\underline{H}$. The results of [GrS], p. 173 imply that $\underline{M}$ is a group over $\mathsf{Z}$, so that $\underline{M}(\hat{\mathsf{Z}}) = M(\hat{\mathsf{Q}}) \cap K_\alpha$ is a hyperspecial maximal compact subgroup of $M(\hat{\mathsf{Q}})$. Let

$$\begin{aligned} \mathsf{W} &= \mathsf{H} \cap \mathfrak{h}(-1) \\ &= \mathsf{Z}e_{21} \oplus (e_1^* \otimes J) \oplus (e_3 \otimes J) \oplus \mathsf{Z}e_{32}. \end{aligned} \tag{6.2}$$

Then $\underline{M}$ acts on $\mathsf{W}$ and

$$\mathsf{W} \cong \mathrm{Hom}(\mathsf{H}(1), \mathsf{Z}) \cong \mathrm{Hom}(\underline{N}, \mathsf{G}_a). \tag{6.3}$$

**Proposition 6.4.** *For any $n$, $\underline{M}(\mathsf{Z}_p)$ acts transitively on the set*

$$\{v \in (\mathsf{W} \otimes \mathsf{Z}_p) \cap \Omega(\mathsf{Q}_p) : c_p(v) = p^n\}.$$

*Proof.* By the Iwasawa decomposition, $\underline{M}(\mathsf{Z}_p)$ acts transitively on the lines in $\Omega(\mathsf{Q}_p)$. On the other hand, it fixes the lattice $\mathsf{W} \otimes \mathsf{Z}_p$ and so preserves content. □

Now let $\underline{P_2} = \underline{L_2} \cdot \underline{U_2}$ be the schematic closure of $P_2$ in $\underline{G_2}$. Then $\underline{L_2} \cong \underline{GL_2}$ and $\underline{U_2} \subset \underline{N}$. Further, $\mathrm{Hom}(\underline{U_2}, \underline{G_a}) \subset \mathrm{Hom}(U_2, G_a)$ is the sublattice

$$\mathsf{V}_2 = \mathsf{Z}e_{21} \oplus \mathsf{Z} \cdot \left(\frac{1}{3}e_1^* \otimes I\right) \oplus \mathsf{Z} \cdot \left(\frac{1}{3}e_3 \otimes I\right) \oplus \mathsf{Z}e_{32} \subset V_2, \tag{6.5}$$

which can be identified with the lattice of binary cubic forms over $\mathsf{Z}$. Now we have [GGS], Prop. 4.2 and Prop. 5.2:

**Proposition 6.6.** *There is a natural bijection between $\mathsf{V}_2$ and the set of pairs $(A, \{\alpha, \beta\})$, where $A$ is a cubic ring over $\mathsf{Z}$ and $\{1, \alpha, \beta\}$ is a good basis of $A$, i.e. a basis with $\alpha\beta \in \mathsf{Z}$. For $(a, b, c, d) \in \mathsf{V}_2$, the corresponding pair $(A, \{\alpha, \beta\})$ is given by*:

$$A = \mathsf{Z} \cdot 1 + \mathsf{Z} \cdot \alpha + \mathsf{Z} \cdot \beta$$

*with multiplication rules*

$$\begin{cases} \alpha\beta = -ad, \\ \alpha^2 = -ac + b\alpha - a\beta, \\ \beta^2 = -bd + d\alpha - c\beta. \end{cases}$$

*This induces a bijection between the $\underline{L_2}(\mathsf{Z})$-orbits on $\mathsf{V}_2$ and isomorphism classes of cubic rings over $\mathsf{Z}$. The ring $A$ is Gorenstein if and only if the corresponding $\underline{L_2}(\mathsf{Z})$-orbit is primitive.*

**Example.** Suppose that $(a, b, c, d) \in \mathsf{V}_2$ satisfies $a \neq 0$ and the corresponding $\mathsf{Q}$-algebra

$$k = \mathsf{Q}[x]/(ax^3 - bx^2 + cx - d)$$

is étale. Then $A$ is the order in $k$ with good basis 1, $\alpha = a \cdot x$ and $\beta = a \cdot x^{\#}$. In particular,

$$\begin{cases} \beta = a^{-1} \cdot \alpha^{\#}, \\ \mathsf{N}\alpha = a^2 d. \end{cases} \tag{6.7}$$

The natural restriction map $\tau \colon \mathrm{Hom}(N, \mathsf{G}_a) \to \mathrm{Hom}(U_2, \mathsf{G}_a)$ sends the lattice $\mathsf{W}$ to $\mathsf{V}_2$. For $q \in \mathsf{V}_2$, let

$$\Omega(q, \alpha) = \{v \in \mathsf{W} \cap \Omega(\mathsf{Q}) : \tau(v) = q\}. \tag{6.8}$$

We remind the reader that $\mathsf{W}$ depends on the choice of $\alpha \in G(\hat{\mathsf{Q}})$ which explains the appearance of $\alpha$ on the left hand side of (6.8). Moreover, if $q' = \gamma \cdot q$ for $\gamma \in \underline{L_2}(\mathsf{Z})$, then $\Omega(q', \alpha) = \gamma \cdot \Omega(q, \alpha)$. Now we have the following important observation:

**Proposition 6.9.** *Suppose the $\underline{L_2}(\mathsf{Z})$-orbit of $q$ corresponds to the étale cubic ring $A$. Then the sets $S(A, \alpha)$ and $\Omega(q, \alpha)$ are in natural bijection.*

*Proof.* By Proposition 5.7, both sets are empty unless $A \otimes \mathsf{R} \cong \mathsf{R}^3$. Let $\{1, \alpha, \beta\}$ be the good basis of $A$ determined by $q = (a, b, c, d)$. Without loss of generality, assume that $a \neq 0$. Given $j \in S(A, \alpha)$, $j$ is completely determined by $B = j(\alpha) \in J$ and $C = j(\beta) \in J$. By (6.7), $B$ and $C$ satisfy:

$$\begin{cases} C = a^{-1} \cdot B^{\#}, \\ \mathrm{N}(B) = a^2 d. \end{cases}$$

Hence the element $v = (a, B, C, d)$ lies in $\mathsf{W} \cap \Omega(\mathsf{Q})$ by Lemma 4.5, and $\tau(v) = q$. This also shows that if $S(A, \alpha)$ is not empty, neither is $\Omega(q, \alpha)$.

Conversely, if $(a, B, C, d) \in \Omega(q, \alpha)$, then we define $j: A \to J$ by setting $j(\alpha) = B$ and $j(\beta) = C$. One checks that $j \otimes \mathsf{Q}$ is an embedding of cubic norm structures, so that $j \in S(A, \alpha)$. This shows that if $\Omega(q, \alpha)$ is not empty, neither is $S(A, \alpha)$. Moreover, it is easy to see that the two maps we have defined are inverses to each other. This proves the proposition. □

## 7. The maximal compact subgroup of $H(\mathsf{R})$

The integral models constructed in the last section furnish maximal compact subgroups for the groups of $\hat{\mathsf{Q}}$-points. In this section, we describe a maximal compact subgroup $K$ of the connected real Lie group $H(\mathsf{R})$. As a compact Lie group,

$$K \cong (SU_2^* \times E_7)/\Delta\mu_2 \tag{7.1}$$

where the $*$ is there to distinguish this $SU_2$ from other $SU_2$ which may appear later. We fix the embedding $K \hookrightarrow H(\mathsf{R})$ by describing its Lie algebra $\mathfrak{k} = \mathfrak{su}_2^* \oplus \mathfrak{e}_7$ as a subalgebra of $\mathfrak{h} \otimes \mathsf{R}$ using (3.1) and (3.2).

Let $\mathfrak{so}_3 \subset \mathfrak{sl}(V \otimes \mathsf{R})$ be the Lie algebra of $3 \times 3$ skew symmetric matrices, and write $e_i \wedge e_j$ for $e_{ij} - e_{ji} \in \mathfrak{sl}_3(\mathsf{R})$. Let $\mathfrak{g} \subset \mathrm{Lie}(L(\mathsf{R}))$ be the Lie algebra of $G(\mathsf{R})$; it is the annihilator of the subspace $(V \otimes I) \otimes \mathsf{R} \subset \mathfrak{h} \otimes \mathsf{R}$. Then $\mathfrak{so}_3 \oplus \mathfrak{g}$ is the Lie algebra of a maximal compact subgroup of $(SL_3 \times_{\mu_3} L)(\mathsf{R})$. The Lie algebra of $K$ can now be described as a subalgebra of $\mathfrak{h}$ as follows. Write $e(i) \otimes X$ for $e_i \otimes X - e_i^* \otimes X$, with $X \in F_{\mathsf{R}} = F \otimes \mathsf{R}$. Then

$$\mathfrak{k} = \mathfrak{so}_3 \oplus \mathfrak{g} \oplus (e(1) \otimes F_{\mathsf{R}}) \oplus (e(2) \otimes F_{\mathsf{R}}) \oplus (e(3) \otimes F_{\mathsf{R}}). \tag{7.2}$$

The $\mathfrak{su}_2^*$ in $\mathfrak{k} = \mathfrak{su}_2^* \oplus \mathfrak{e}_7$ is the subspace spanned by:

$$\begin{cases} e_1 \wedge e_2 + e(3) \otimes I, \\ e_2 \wedge e_3 + e(1) \otimes I, \\ e_3 \wedge e_1 + e(2) \otimes I, \end{cases} \tag{7.3}$$

whereas the subalgebra $\mathfrak{e}_7$ is spanned by $\mathfrak{g}$ and elements of the form

$$\begin{cases} -\mathrm{Tr}(Z) e_1 \wedge e_2 + e(3) \otimes Z, \\ -\mathrm{Tr}(X) e_2 \wedge e_3 + e(1) \otimes X, \\ -\mathrm{Tr}(Y) e_3 \wedge e_1 + e(2) \otimes Y, \end{cases} \tag{7.4}$$

for $X, Y, Z \in F_{\mathsf{R}}$.

From the above description, one sees that

$$K \cap (G_2(\mathsf{R}) \times G(\mathsf{R})) = K_{G_2} \times G(\mathsf{R}) \tag{7.5}$$

where $K_{G_2} = (SU_2^* \times SU_2)/\Delta\mu_2$ is a maximal compact subgroup of $G_2(\mathsf{R})$, and $SU_2 \subset E_7$. Similarly, $K \cap M(\mathsf{R})$ is a maximal compact subgroup of $M(\mathsf{R})$ with connected component $(U_1 \times E_6)/\Delta\mu_3$. Its Lie algebra is

$$\mathfrak{u}_1 \oplus \mathfrak{e}_6 = \mathfrak{g} \oplus e(2) \otimes F_{\mathsf{R}} \tag{7.6}$$

where $\mathfrak{u}_1$ is the 1-dimensional space $\mathsf{R} \cdot e(2) \otimes I$.

Let

$$\begin{cases} z_0 = e_3 \wedge e_1, \\ z_2 = e(2) \otimes I. \end{cases}$$

These elements are both contained in $\mathrm{Lie}(G_2(\mathsf{R}))$, and were denoted by $z_1$ and $z_2$ respectively in [GrW], p. 114–115. Indeed, if $G_2^{\lambda} \simeq SL_2$ denotes the rank one subgroup of $G_2$ corresponding to a root $\lambda$, then $z_0$ spans the Lie algebra of $K_{G_2} \cap G_2^{\delta}$, where we recall that $\delta$ is the highest root of $G_2$. On the other hand, $z_2$ spans the Lie algebra of $K_{G_2} \cap L_2(\mathsf{R})$. Writing $z_0$ and $z_2$ in terms of the decomposition $\mathfrak{k} = \mathfrak{su}_2^* \oplus \mathfrak{e}_7$, we see that

$$\begin{cases} z_0 = \frac{1}{2} h - \frac{1}{2} z, \\ z_2 = \frac{3}{2} h + \frac{1}{2} z \end{cases} \tag{7.7}$$

for $z \in \mathfrak{e}_7$ and $h \in \mathfrak{su}_2^*$ given by

$$h = \frac{1}{2} e_3 \wedge e_1 + \frac{1}{2} e(2) \otimes I.$$

## 8. The minimal representation of $H(\mathsf{R})$

We now recall some properties of the minimal representation of $H(\mathsf{R})$. The minimal representation was constructed in [GrW] as a continuation of quaternionic discrete series representations. It is a smooth Fréchet representation $\Pi_\infty$ of moderate growth in the sense of [C] and [W], and is canonically associated to its underlying Harish-Chandra module $(\Pi_\infty)_K$. We shall say that $\Pi_\infty$ is the Casselman-Wallach globalization of $(\Pi_\infty)_K$.

If $\lambda$ denotes the highest weight of the 56-dimensional minuscule representation of the compact $E_7$, the $K$-type decomposition of $(\Pi_\infty)_K$ is

$$(\Pi_\infty)_K = \bigoplus_{n \geqq 0} \mathrm{Sym}^{8+n}(\mathsf{C}^2) \otimes \pi(n\lambda) \tag{8.1}$$

where $\pi(n\lambda)$ denotes the irreducible representation of compact $E_7$ with highest weight $n\lambda$. In particular, the minimal $K$-type is the 9-dimensional representation $\mathrm{Sym}^8(\mathsf{C}^2) \otimes \mathsf{C}$ of $(SU_2^* \times E_7)/\Delta\mu_2$. Let $\{v_{-8}, v_{-6}, \ldots, v_8\}$ be a basis of $\mathrm{Sym}^8(\mathsf{C}^2)$ such that

$$h(v_i) = i \cdot v_i. \tag{8.2}$$

Then we see from (7.7) that

$$\begin{cases} z_0(v_i) = \dfrac{1}{2} i \cdot v_i, \\ z_2(v_i) = \dfrac{3}{2} i \cdot v_i. \end{cases} \tag{8.3}$$

It was shown in [GrW], Prop. 13.6 that there is an embedding

$$\Pi_\infty \hookrightarrow \mathrm{Ind}_{P(\mathsf{R})}^{H(\mathsf{R})} \delta_P^{5/29}. \tag{8.4}$$

From (8.3), one sees that $\mathrm{Sym}^8(\mathsf{C}^2) \otimes \mathsf{C}$ (and hence $\Pi_\infty$) occurs in the smooth induced representation with multiplicity one, contrary to what was remarked in the introduction of [GrW]. The unique subspace of the degenerate principal series affording the minimal $K$-type consists of functions of the form

$$f(k) = \langle v_0, kv \rangle, \quad k \in K \quad \text{and} \quad v \in \mathrm{Sym}^8(\mathsf{C}^2), \tag{8.5}$$

where $\langle \cdot, \cdot \rangle$ is a $K$-invariant inner product on $\mathrm{Sym}^8(\mathsf{C}^2)$, which is unique up to scaling.

In [HPS], Thm. 5.4, the decomposition of the restriction of $\Pi_\infty$ to $G_2(\mathsf{R}) \times G(\mathsf{R})$ is completely determined. The only case of their results we need here is:

**Proposition 8.6.** *The $G(\mathsf{R})$-invariant subspace of $\Pi_\infty$ is a quaternionic discrete series representation $\pi$ of $G_2(\mathsf{R})$. The minimal $K_{G_2}$-type of $\pi$ is the minimal $K$-type of $\Pi_\infty$.*

We now describe an important multiplicity one result for $\pi$. The group $\mathsf{V}_2 \otimes \mathsf{R} \cong \mathrm{Hom}(U_2(\mathsf{R}), \mathsf{R})$ can be identified with the character group $\mathrm{Hom}(U_2(\mathsf{R}), S^1)$ via $f \mapsto e^{2\pi i f}$. Recall that $L_2(\mathsf{R})$ has two generic orbits on $\mathsf{V}_2 \otimes \mathsf{R}$, determined by the sign of the discriminant $\Delta$. The following proposition is a special case of a result of Wallach [W2]:

**Proposition 8.7.** *Let $\chi \in \mathrm{Hom}(U_2(\mathsf{R}), S^1)$, and let $\mathrm{Hom}_{U_2(\mathsf{R})}(\pi, \mathsf{C}(\chi))$ denote the space of continuous linear forms $l$ on $\pi$ which satisfies $l(g \cdot v) = \chi(g) \cdot l(v)$ for $g \in U_2(\mathsf{R})$. Then*

$$\dim \mathrm{Hom}_{U_2(\mathsf{R})}(\pi, \mathsf{C}(\chi)) = \begin{cases} 1, & \text{if } \Delta(\chi) > 0; \\ 0, & \text{if } \Delta(\chi) < 0. \end{cases}$$

## 9. The minimal representation of $H(\mathsf{Q}_p)$

In this section, we fix $\alpha \in G(\hat{\mathsf{Q}})$ as in Section 6. In particular, this determines a hyperspecial maximal compact subgroup $K_p$ of $H(\mathsf{Q}_p)$. The minimal representation $\Pi_p$ has a 1-dimensional space of vectors fixed by $K_p$, and is the unique irreducible submodule of $\mathrm{Ind}_{P(\mathsf{Q}_p)}^{H(\mathsf{Q}_p)} \delta_P^{5/29}$ [S]. As in the real case, we identify $\Pi_p$ with this subspace of the degenerate principal series. Let $\Gamma_p$ be the $K_p$-spherical vector in $\Pi_p$ normalized by $\Gamma_p(1) = 1$.

Fix a non-trivial additive character $\psi_p$ of $\mathsf{Q}_p$ with conductor $\mathsf{Z}_p$. Then we identify $\mathrm{Hom}(N(\mathsf{Q}_p), \mathsf{Q}_p)$ with $\mathrm{Hom}(N(\mathsf{Q}_p), S^1)$ via $f \mapsto \psi_p \circ f$. Hence the character group of $N(\mathsf{Q}_p)$ is identified with $W \otimes \mathsf{Q}_p$. We have the following important multiplicity one result [MS], Thm. 6.1 (the restriction $p \neq 2$ there has recently been removed by Savin):

**Proposition 9.1.** (i) *For non-trivial* $\Lambda \in \mathrm{Hom}(N(\mathsf{Q}_p), S^1)$,

$$\dim \mathrm{Hom}_{N(\mathsf{Q}_p)}(\Pi_p, \mathsf{C}(\Lambda)) = \begin{cases} 1, & \text{if } \Lambda \in \Omega(\mathsf{Q}_p); \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *For* $\Lambda \in \Omega(\mathsf{Q}_p)$, *let* $M_\Lambda(\mathsf{Q}_p) \subset M(\mathsf{Q}_p)$ *be the stabilizer of* $\Lambda$. *Then for any* $L_\Lambda \in \mathrm{Hom}_{N(\mathsf{Q}_p)}(\Pi_p, \mathsf{C}(\Lambda))$, *we have*

$$m \cdot L_\Lambda = \delta_P(m)^{-5/29} \cdot L_\Lambda$$

*for any* $m \in M_\Lambda(\mathsf{Q}_p)$.

For the rest of this section, we consider the value of $L_\Lambda(\Gamma_p)$, where $L_\Lambda$ is any non-zero element of $\mathrm{Hom}_{N(\mathsf{Q}_p)}(\Pi_p, \mathsf{C}(\Lambda))$.

**Lemma 9.2.** *Let* $\Lambda$ *and* $\Lambda'$ *be elements of* $\Omega(\mathsf{Q}_p)$.

(i) *If* $\Lambda \notin \mathsf{W} \otimes \mathsf{Z}_p$, *then* $L_\Lambda(\Gamma_p) = 0$.

(ii) *If* $\Lambda$ *and* $\Lambda'$ *in* $\mathsf{W} \otimes \mathsf{Z}_p$ *have the same content, then* $L_\Lambda(\Gamma_p) \neq 0$ *if and only if* $L_{\Lambda'}(\Gamma_p) \neq 0$.

(iii) *For any* $\Lambda$, *there exists* $\Lambda' \in \mathsf{Q}_p^\times \cdot \Lambda$ *such that* $L_{\Lambda'}(\Gamma_p) \neq 0$.

*Proof.* (i) If $\Lambda \notin \mathsf{W} \otimes \mathsf{Z}_p$, there exists $n \in \underline{N}(\mathsf{Z}_p) = N(\mathsf{Q}_p) \cap K_p$ such that $\Lambda(n) \neq 1$. Hence,

$$L_\Lambda(\Gamma_p) = L_\Lambda(n \cdot \Gamma_p) = \Lambda(n) \cdot L_\Lambda(\Gamma_p)$$

which implies that $L_\Lambda(\Gamma_p) = 0$.

(ii) By Proposition 6.4, there exists $m \in \underline{M}(\mathsf{Z}_p) = M(\mathsf{Q}_p) \cap K_p$ such that $\Lambda' = m \cdot \Lambda$. Suppose that $L_\Lambda(\Gamma_p) \neq 0$. By the multiplicity one result of Proposition 9.1, $m \cdot L_\Lambda$ is a non-zero multiple of $L_{\Lambda'}$. Moreover, $m \cdot L_\Lambda(\Gamma_p) = L_\Lambda(\Gamma_p) \neq 0$.

(iii) Since $\Pi_p$ is generated by $\Gamma_p$ as a representation of $H(\mathsf{Q}_p)$, $L_\Lambda(g \cdot \Gamma_p) \neq 0$ for some $g \in H(\mathsf{Q}_p)$. By the Iwasawa decomposition, we may assume $g \in M(\mathsf{Q}_p)$. The stabilizer of the line $\mathsf{Q}_p^\times \cdot \Lambda$ in $M(\mathsf{Q}_p)$ is a maximal parabolic subgroup $Q_\Lambda(\mathsf{Q}_p)$ of $M(\mathsf{Q}_p)$; so by Iwasawa decomposition again, we may assume $g \in Q_\Lambda(\mathsf{Q}_p)$. Then $g = z \cdot m$ for $z \in Z(\mathsf{Q}_p)$ and $m \in M_\Lambda(\mathsf{Q}_p)$. Hence by Proposition 9.1(ii), $L_\Lambda(z \cdot \Gamma_p) \neq 0$ for some $z \in Z(\mathsf{Q}_p)$, i.e. $L_{z^{-1}\cdot\Lambda}(\Gamma_p) \neq 0$. Since $z^{-1} \cdot \Lambda \in \mathsf{Q}_p^\times \cdot \Lambda$, this proves the proposition. □

The lemma implies the existence of a $p$-power $e_p$ with the following properties:

$$(9.3) \qquad \begin{cases} L_\Lambda(\Gamma_p) = 0 \text{ unless } \Lambda \in \mathsf{W} \otimes \mathsf{Z}_p \text{ has content divisible by } e_p; \\ L_\Lambda(\Gamma_p) \neq 0 \text{ for any } \Lambda \in \mathsf{W} \otimes \mathsf{Z}_p \text{ with content } e_p. \end{cases}$$

Moreover, we have:

**Lemma 9.4.** *The quantity $e_p$ is independent of the choice of $\alpha \in G(\hat{\mathsf{Q}})$.*

*Proof.* Suppose that $\alpha$ and $\beta$ are in $G(\hat{\mathsf{Q}})$. Let $\Lambda \in \mathsf{W}_\alpha \otimes \mathsf{Z}_p$ have content $e_p(\alpha)$, so that $L_\Lambda(\Gamma_p^\alpha) \neq 0$. Now $\beta\alpha^{-1} \cdot \Gamma_p^\alpha$ is a $K_p^\beta$-spherical vector in $\Pi_p$, and $\beta\alpha^{-1} \cdot \Lambda$ is an element of $\mathsf{W}_\beta \otimes \mathsf{Z}_p$ of content $e_p(\alpha)$. Since $\beta\alpha^{-1} \cdot L_\Lambda$ is a non-zero multiple of $L_{\beta\alpha^{-1}\cdot\Lambda}$ and

$$\beta\alpha^{-1} \cdot L_\Lambda(\beta\alpha^{-1} \cdot \Gamma_p^\alpha) = L_\Lambda(\Gamma_p^\alpha) \neq 0,$$

we deduce that $e_p(\beta)$ divides $e_p(\alpha)$, and the result follows by symmetry. □

## 10. The automorphic theta module

Let $\Pi = \Pi_\infty \otimes \hat{\Pi}$, with $\hat{\Pi} = \bigotimes_p \Pi_p$, be the global minimal representation of $H(\mathsf{A}) = H(\mathsf{R}) \times H(\hat{\mathsf{Q}})$. We identify $\Pi$ with a subspace of $\mathrm{Ind}_{P(\mathsf{A})}^{H(\mathsf{A})} \delta_P^{5/29}$, and let $\Pi_K = (\Pi_\infty)_K \otimes \hat{\Pi}$ be the subspace of $K$-finite vectors.

Let $\mathscr{A}(H)$ denote the space of automorphic forms on $H$ defined as in [GGS], §7. This is the usual definition, except that we do not require the functions to be $K$-finite; instead we let $\mathscr{A}(H)_K$ be the subspace of $K$-finite functions in $\mathscr{A}(H)$. In [G], we have shown that if $f \in \Pi_K$ and $f_s \in \mathrm{Ind}_{P(\mathsf{A})}^{H(\mathsf{A})} \delta_P^s$ is the unique standard section extending $f$, then the Eisenstein series $E(-, f_s)$ is holomorphic at $s = \frac{5}{29}$, and gives an embedding

$N(\mathsf{Q})$ with the $\mathsf{Q}$-vector space $W$. For $\Lambda \in W$, define $\Theta_\Lambda \colon \Pi \to \mathsf{C}$ by:

$$\Theta_\Lambda(f) = \int_{N(\mathsf{Q})\backslash N(\mathsf{A})} \theta(f)(n)\overline{\Lambda(n)}\, dn, \tag{10.2}$$

where $dn$ is the Haar measure giving $N(\mathsf{Q})\backslash N(\mathsf{A})$ volume 1. It is an element of the space $\mathrm{Hom}_{N(\mathsf{A})}(\Pi, \mathsf{C}(\Lambda))$. Proposition 9.1 and the weak approximation theorem imply:

**Proposition 10.3.** *Suppose that* $\Lambda$ *is non-trivial.*

5)5$\Theta$

$\Lambda$

*nonzero* [illegible] $(\mathsf{Q})$.

Set $e = \prod_p e_p$. Then we have the following important proposition:

**Proposition 10.7.** *Suppose that* $\Lambda \in \Omega(\mathsf{Q})$ *and* $\Delta(\tau(\Lambda)) > 0$. *Let* $\theta_\Lambda: \pi \to \mathsf{C}$ *be the linear form defined by*

$$v \mapsto \Theta_\Lambda(v \otimes \Gamma)$$

*with* $\Gamma = \bigotimes_p \Gamma_p$. *Then*

(i) $\theta_\Lambda$ *is zero unless* $\Lambda \in \mathsf{W}$ *and* $c(\Lambda)$ *is divisible by* $e$. *If* $c(\Lambda) = e$, *then* $\theta_\Lambda$ *is non-zero.*

(ii) *If* $\Lambda$ *and* $\Lambda'$ *in* $\Omega(\mathsf{Q}) \cap \mathsf{W}$ *have the same content, and* $\tau(\Lambda') = g \cdot \tau(\Lambda)$ *for* $g \in L_2(\mathsf{R})$, *then*

$$\theta_{\Lambda'} = |\det(g)|^{-5} \cdot (g \cdot \theta_\Lambda).$$

*Proof.* Statement (i) follows from (10.6), (9.3) and the definition of $e$. We now prove (ii). One can choose:

$$\begin{cases} m \in M(\mathsf{Q}) \text{ such that } \Lambda' = m \cdot \Lambda, \text{ by hypothesis;} \\ \hat{m} \in \underline{M}(\hat{\mathsf{Z}}) \text{ such that } \Lambda' = \hat{m} \cdot \Lambda, \text{ by Proposition 6.4;} \\ h \in G(\mathsf{R}) \text{ such that } h \cdot \Lambda' = g \cdot \Lambda, \text{ by Propositions 2.3(ii) and 5.7.} \end{cases}$$

In particular, the elements $m^{-1}\hat{m} \in M(\hat{\mathsf{Q}})$ and $m^{-1}h^{-1}g \in M(\mathsf{R})$ both fix $\Lambda$. Let

$$\gamma = (m^{-1}h^{-1}g, m^{-1}\hat{m}) \in M_\Lambda(\mathsf{R}) \times M_\Lambda(\hat{\mathsf{Q}}).$$

Now we have:

$$\begin{aligned} \theta_{\Lambda'}(v) &= \Theta_{\Lambda'}(v \otimes \Gamma) \\ &= m \cdot \Theta_\Lambda(v \otimes \Gamma) \\ &= |\delta_P(\gamma)|^{5/29} \cdot \big((m \cdot \gamma) \cdot \Theta_\Lambda\big)(v \otimes \Gamma) \quad \text{by Proposition 10.3(ii)} \\ &= |\det(g)|^{-5} \cdot \Theta_\Lambda(g^{-1} \cdot v \otimes \Gamma) \\ &= |\det(g)|^{-5} \cdot (g \cdot \theta_\Lambda)(v). \qquad \square \end{aligned}$$

## 11. Theta functions on $G_2$

We continue to fix $\alpha \in G(\hat{\mathsf{Q}})$, and our goal is to construct some theta functions on $G_2$. Let $\varphi = \varphi_\alpha: \pi \to \mathscr{A}(G_2)$ be defined by

$$(11.1) \qquad v \mapsto \theta(v \otimes \Gamma)|_{G_2(\mathsf{A})}.$$

Then $\varphi$ is $G_2(\mathsf{R}) \times G_2(\hat{\mathsf{Z}})$-equivariant, and is a modular form of weight 4 and level 1 in the sense of [GGS]. Note that $\varphi$ depends only on the double coset $G(\mathsf{Q}) \cdot \alpha \cdot K_I$, where $K_I$ is the stabilizer of $J_3 \otimes \hat{\mathsf{Z}}$ in $G(\hat{\mathsf{Q}})$ (cf. Section 3).

We first review the notion of Fourier coefficients for such forms, as developed in [GGS]. For generic $\chi \in \mathsf{V}_2 \subset V_2$, the linear form $l_\chi: \pi \to \mathsf{C}$ defined by

$$l_\chi(v) = \int_{U_2(\mathsf{Q})\backslash U_2(\mathsf{A})} \theta(v \otimes \Gamma)(u)\overline{\chi(u)}\, du \tag{11.2}$$

$$= \int_{\underline{U_2}(\mathsf{Z})\backslash U_2(\mathsf{R})} \varphi(v)(u)\overline{\chi(u)}\, du$$

is an element of $\mathrm{Hom}_{U_2(\mathsf{R})}(\pi, \mathsf{C}(\chi))$ and thus by Proposition 8.7, is zero unless $\Delta(\chi) > 0$. Fix an arbitrary $\chi_0$ with $\Delta(\chi_0) > 0$, and a non-zero element $l_0$ in the 1-dimensional space $\mathrm{Hom}_{U_2(\mathsf{R})}(\pi, \mathsf{C}(\chi_0))$. For any $g \in L_2(\mathsf{R})$, with $\chi = g \cdot \chi_0$, the element

$$l_\chi^0 = \frac{1}{|\det(g)|^5} \cdot (g \cdot l_0)$$

is a non-zero element of $\mathrm{Hom}_{U_2(\mathsf{R})}(\pi, \mathsf{C}(\chi))$, and is independent of the choice of $g$. Hence there is a constant $c_\chi(\varphi) \in \mathsf{C}$ so that

$$l_\chi = c_\chi(\varphi) \cdot l_\chi^0.$$

The number $c_\chi(\varphi)$ depends only on the $\underline{L_2}(\mathsf{Z})$-orbit of $\chi$, and hence we denote it by $c_A(\varphi)$, where $A$ is the corresponding cubic ring over $\mathsf{Z}$. The collection $\{c_A(\varphi)\}$ indexed by those cubic rings $A$ with $A \otimes \mathsf{R} \cong \mathsf{R}^3$ are the Fourier coefficients of $\varphi$.

We now compute some Fourier coefficients of the modular form $\varphi$ defined in (11.1). For $\chi_A \in \mathsf{V}_2$, whose $\underline{L_2}(\mathsf{Z})$-orbit corresponds to the cubic ring $A$, Proposition 10.7(i) implies that

$$l_{\chi_A} = \sum_{\Lambda \in e\cdot \mathsf{W} \cap \Omega(\chi_A, \alpha)} \theta_\Lambda.$$

Hence $l_{\chi_A} = 0$ unless $e \cdot \mathsf{W} \cap \Omega(\chi_A, \alpha)$ is non-empty. Assuming that this is the case, we must have $\Delta(\chi_A) > 0$ and $e$ divides $c(\chi_A)$. Now suppose that $A = \mathsf{Z} + e \cdot A_0$ with $A_0$ Gorenstein. Then $\chi_A = e \cdot \chi_{A_0}$ in $\mathsf{V}_2$, with $\chi_{A_0}$ primitive. Moreover,

$$e \cdot \mathsf{W} \cap \Omega(\chi_A, \alpha) = e \cdot \Omega(\chi_{A_0}, \alpha),$$

so that

$$l_{\chi_A} = \sum_{\Lambda \in e\cdot \Omega(\chi_{A_0}, \alpha)} \theta_\Lambda.$$

Since every element of $\Omega(\chi_{A_0}, \alpha)$ is primitive, it follows by Proposition 10.7(ii) that $\theta_\Lambda$ is independent of $\Lambda \in e \cdot \Omega(\chi_{A_0}, \alpha)$, and by Proposition 6.9

$$\#\Omega(\chi_{A_0}, \alpha) = \#S(A_0, \alpha).$$

Hence,

$$l_{\chi_A} = \#S(A_0, \alpha) \cdot \theta_\Lambda,$$

for any $\Lambda \in e \cdot \Omega(\chi_{A_0}, \alpha)$. Note that this holds as well when $\Omega(\chi_{A_0}, \alpha)$ is empty, since both sides of the equation are zero.

Since $\Lambda$ has content $e$, Proposition 10.7(i) says that $\theta_\Lambda \neq 0$. So we can write

$$\theta_\Lambda = c_\Lambda \cdot l^0_{\chi_A}$$

for some $c_\Lambda \neq 0$. Suppose now that $\chi_{A'}$ is another element of $\mathsf{V}_2$ of content $e$, with $A' = \mathsf{Z} + e \cdot A'_0$. Choose $g \in L_2(\mathsf{R})$ so that $\chi_{A'} = g \cdot \chi_A$. Then

$$l^0_{\chi_{A'}} = |\det(g)|^{-5} \cdot (g \cdot l^0_{\chi_A}).$$

By Proposition 10.7(ii), for any $\Lambda' \in \Omega(\chi_{A'}, \alpha)$, we deduce that $c_{\Lambda'} = c_\Lambda \neq 0$. Hence, by rescaling the map $\varphi$, we can assume that $c_\Lambda = 1$. We have thus shown:

**Theorem 11.3.** *Fix $\alpha \in G(\hat{\mathsf{Q}})$, and let $\varphi_\alpha$ be the modular form of weight 4 and level 1 defined by (11.1). The Fourier coefficient $c_A(\varphi_\alpha)$ is zero unless e divides the content of A. For any Gorenstein $A_0$, let $A = \mathsf{Z} + e \cdot A_0$, so that A has content e. Then*

$$c_A(\varphi_\alpha) = \#S(A_0, \alpha).$$

**Corollary 11.4.** *There is a modular form $\varphi_I$ on $G_2$, of weight 4 and level 1, such that $c_A(\varphi_I) = 0$ unless e divides the content of A, and for any $A = \mathsf{Z} + e \cdot A_0$ of content e,*

$$c_A(\varphi_I) = N(A_0, J_I).$$

*Similarly, there is a form $\varphi_E$ such that for $A = \mathsf{Z} + e \cdot A_0$,*

$$c_A(\varphi_E) = N(A_0, J_E).$$

## 12. The theta integral

In this section, we take $\alpha = 1 \in G(\hat{\mathsf{Q}})$ in the constructions of the previous sections. In particular, $\underline{G}(\hat{\mathsf{Z}}) = K_I$.

For $f \in \Pi$, we define the theta integral

$$I\big(\theta(f)\big)(g) = \int_{G(\mathsf{Q})\backslash G(\mathsf{A})} \theta(f)(gh)\,dh, \quad \text{for } g \in G_2(\mathsf{A}), \tag{12.1}$$

where $dh$ is the Tamagawa measure on $G(\mathsf{A})$. Then $I\big(\theta(f)\big)$ is an element of $\mathscr{A}(G_2)$. Moreover, it is not difficult to see that the modular form

$$v \mapsto I\big(\theta(v \otimes \Gamma)\big)$$

is equal to

$$\frac{1}{\#\Gamma_I} \cdot \varphi_I + \frac{1}{\#\Gamma_E} \cdot \varphi_E.$$

There is a unique (up to scaling) automorphic form $\Phi$ on $G$ which is right-invariant under $G(\mathsf{R}) \times K_I$, and which is orthogonal to the constant functions. We set:

$$J\big(\theta(f)\big) = \int_{G(\mathsf{Q})\backslash G(\mathsf{A})} \theta(f)(gh)\Phi(h)\,dh. \tag{12.2}$$

Then $J\big(\theta(f)\big)$ is an element of $\mathscr{A}(G_2)$. Moreover,

$$v \mapsto J\big(\theta(v \otimes \Gamma)\big)$$

is equal to $\varphi_I - \varphi_E$, whence

**Proposition 12.3.** *There is a modular form of weight* 4 *and level* 1 *on* $G_2$, *whose* $A$*-th Fourier coefficient is zero unless* $A$ *has content divisible by* $e$. *If* $A = \mathsf{Z} + e \cdot A_0$ *has content* $e$, *then its* $A$*-th Fourier coefficient is equal to*

$$N_{A_0} = \frac{N(A_0, J_I)}{\#\Gamma_I} + \frac{N(A_0, J_E)}{\#\Gamma_E}.$$

*Similarly, there is a modular form of weight* 4 *and level* 1 *whose* $A$*-th Fourier coefficient, for* $A$ *of content* $e$, *is equal to* $M_{A_0} = N(A_0, J_I) - N(A_0, J_E)$.

We have thus confirmed the speculations of [GG], §8. In fact, we shall see after Theorem 15.5 that $e = 1$.

## 13. Eisenstein series on $G_2$

The rest of the paper is devoted to the study of $I\big(\theta(f)\big)$. We shall show that $I\big(\theta(f)\big)$ is an Eisenstein series concentrated along the Borel subgroup of $G_2$, and the goal of this section is to investigate this Eisenstein series.

We begin with some results on the structure of $G_2$. Let $\alpha$ be the short simple root, $\beta$ the long simple root, and $\delta$ the highest root (relative to the diagonal torus of $SL_3$ and the positive system of roots we've chosen before). We have:

$$(13.1)\qquad \begin{aligned}\langle \beta, \alpha^\vee \rangle &= -3,\\ \langle \alpha, \beta^\vee \rangle &= -1.\end{aligned}$$

For any positive root $\lambda$, let $G_2^\lambda$ be the corresponding rank one subgroup, and let

$$h_\lambda \colon SL_2 \to G_2^\lambda$$

be the corresponding homomorphism. Let

$$w_\lambda = h_\lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that $w_\lambda$ represents the simple reflection in $\lambda$ in the Weyl group $W_{G_2}$ of $G_2$. Note that if $\lambda$ is a long root of $G_2$, then it is the restriction of a root for $H$. Hence $\lambda^\vee$ is also a coroot of $H$, and $w_\lambda \in H(\mathsf{Q})$ represents a reflection in the Weyl group $W$ of $H$.

Our description in Sections 3 and 7 shows that $K_\lambda = K_{G_2} \cap G_2^\lambda(\mathsf{R})$ is a maximal compact subgroup of $G_2^\lambda(\mathsf{R})$. It is a one-dimensional compact torus, with Lie algebra $\mathfrak{k}_\lambda$. For example,

$$(13.2)\qquad \begin{aligned}\mathfrak{k}_\alpha &= \mathsf{R} z_2,\\ \mathfrak{k}_\delta &= \mathsf{R} z_0,\end{aligned}$$

where $z_0$ and $z_2$ are defined in Section 7. Further, we let

$$u_\lambda(t) = h_\lambda \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

and

$$k_\lambda(\theta) = h_\lambda \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\lambda.$$

Recall that $P_2 = L_2 \cdot U_2$ is the Heisenberg parabolic subgroup of $G_2$. The derived group of $L_2$ is $G_2^\alpha$, and the modulus character is given by:

$$(13.3)\qquad \delta_{P_2} = 3\delta = 9\alpha + 6\beta.$$

Moreover, the coset representatives for $P_2 \backslash G_2 / P_2$ can be chosen to be: 1, and

$$(13.4)\qquad \begin{aligned} w_1 &= w_\beta,\\ w_2 &= w_\beta w_\alpha w_\beta,\\ w_3 &= w_\beta w_\alpha w_\beta w_\alpha w_\beta.\end{aligned}$$

For $w \in W_{G_2}$, let $\Phi_w = \{\lambda \in \Phi^+ : w \cdot \lambda < 0\}$. Then we have:

$$\Phi_{w_1} = \{\beta\}; \tag{13.5}$$
$$\Phi_{w_2} = \{\beta, \alpha + \beta, 3\alpha + 2\beta\};$$
$$\Phi_{w_3} = \{\beta, \alpha + \beta, 3\alpha + 2\beta, 2\alpha + \beta, 3\alpha + \beta\}.$$

We are now ready to study a particular Eisenstein series on $G_2$. Let

$$I = \hat{\bigotimes_v} I_v = \mathrm{Ind}_{P_2(\mathsf{A})}^{G(\mathsf{A})} \delta_{P_2}^{5/3} \tag{13.6}$$

be the smooth degenerate principal series. For $f \in I$, the sum

$$E(g, f) = \sum_{\gamma \in P_2 \backslash G_2} f(\gamma g) \tag{13.7}$$

converges absolutely and locally uniformly in $g$, and defines an Eisenstein series in $\mathscr{A}(G_2)$. For our purposes, we shall not need to consider an arbitrary $f \in I$, but only those described below. Recall that the minimal representation $\Pi$ of $H$ is a submodule of $\mathrm{Ind}_{P(\mathsf{A})}^{H(\mathsf{A})} \delta_P^{5/29}$. By restriction, we obtain a $G_2(\mathsf{A})$-intertwining map:

$$\mathrm{Res}\colon \Pi \to I, \tag{13.8}$$
$$f \mapsto f|_{G_2}.$$

We shall only be interested in those $f$ in $\mathrm{Res}(\Pi)$. Following [KR1], we call these *Siegel-Weil sections*. For $v$ finite, the local induced representation $I_v$ is irreducible [M], Theorem 3.1(i), whereas for $v$ real, $I_v$ has $\pi$ (cf. Proposition 8.6) as a submodule. Hence, $f = \bigotimes_v f_v$ is Siegel-Weil if and only if $f_\infty \in \pi$.

Now the main result in this section is:

**Proposition 13.9.** *If $f$ is Siegel-Weil, the constant term $E_{P_2}(g, f)$, when regarded as an automorphic form on $L_2$, is contained in the direct sum of the one-dimensional representation $|\det|^{-5}$ and the automorphic representation of $GL_2$ generated by the holomorphic Eisenstein series $E_{12}$ of level* 1 *and weight* 12, *with central character* $|\det|^{-3}$.

The rest of this section is devoted to the proof of the proposition. First, note that

$$I \subset \mathrm{Ind}_{P_0(\mathsf{A})}^{G_2(\mathsf{A})} \delta_{P_0}^{\frac{1}{2}} \cdot \chi$$

where $P_0$ is the Borel subgroup of $G_2$, and $\chi =$

and $E^{L_2}$ refers to Eisenstein series on $L_2$. We need to determine the induced representation of $L_2$ in which $M(w_1,\chi)(f)|_{L_2}$ and $M(w_2,\chi)(f)|_{L_2}$ lie. If

$$M(w_i,\chi)(f)|_{L_2} \in \mathrm{Ind}_{B(\mathsf{A})}^{L_2(\mathsf{A})}(\delta_B^{\frac{1}{2}+s_i}\cdot|\det|^{t_i}),$$

where $B$ is the Borel subgroup of $L_2$, then $s_i$ satisfies:

$$\left(\delta_{P_0}^{\frac{1}{2}}\cdot w_i(\chi)\right)\circ\alpha^\vee = (\delta_B^{\frac{1}{2}+s_i})\circ\alpha^\vee.$$

This implies, for example, that

$$\begin{cases} s_1 = \dfrac{11}{2}, \\ s_2 = 5. \end{cases}$$

Similarly, one finds that $t_1 = -3$. Hence, the function $M(w_1,\chi)(f)|_{L_2}$ lies in the induced representation giving rise to the twist of $E_{12}$ by $|\det|^{-3}$. This induced representation has a unique irreducible submodule, whose real component is a discrete series representation. Thus the content of the proposition is that $M(w_1,\chi)(f)|_{L_2}$ lies in this discrete series submodule and the last two terms on the right hand side of (13.10) vanish when $f$ is Siegel-Weil.

Henceforth, assume that $f$ is Siegel-Weil. To complete the proof of the proposition, it remains to show that

$$M(w_2,\chi)(f) = M(w_3,\chi)(f) = 0,$$

so that

$$E_{P_2}(g,f) = f(g) + E^{L_2}\left(g, M(w_1,\chi)(f)|_{L_2}\right). \tag{13.11}$$

For this, it suffices to show that, when $v$ is real,

$$M_v(w_\alpha, w_\beta\chi)\circ M_v(w_\beta,\chi)(f) = 0 \tag{13.12}$$

for all $f$ in the minimal $K_{G_2}$-type $\mathrm{Sym}^8(\mathsf{C}^2)\otimes\mathsf{C}$. From the value of $s_1$ computed above, one sees that, considered as an intertwining operator for the rank 1 group $G_2^\alpha \cong SL_2$, $M_v(w_\alpha, w_\beta\chi)$ is the standard intertwining operator from the principal series representation of $SL_2(\mathsf{R})$ with the 11-dimensional irreducible representation as a quotient to its dual. Hence we have:

**Lemma 13.13.** *Suppose that $v$ is real, and $f \in M_v(w_\beta,\chi)(I_v)$ is $K_{G_2}$-finite. Then*

$$M_v(w_\alpha, w_\beta\chi)(f) = 0$$

*if and only if, for all $\varphi$ in the $K_{G_2}$-type of $f$, $\varphi|_{L_2}$ is orthogonal to the weight spaces of $K_\alpha = K_{G_2}\cap L_2(\mathsf{R})$ of weight between $-10$ and $10$.*

In view of this lemma, we need to consider $M_v(w_\beta,\chi)(f)|_{K_\alpha}$ for $f$ in the minimal $K_{G_2}$-type $\mathrm{Sym}^8(\mathsf{C}^2)\otimes\mathsf{C}$. Recall from (8.5) that

$$f(k) = \langle v_0, kv_f \rangle$$

for some vector $v_f \in \mathrm{Sym}^8(\mathbf{C}^2)$. By (8.3) and (13.2), the weights of $K_\alpha$ on $\mathrm{Sym}^8(\mathbf{C}^2)$ are $-12, -9, \ldots, 12$. Hence, by Lemma 13.13, it suffices to show:

**Lemma 13.14.**

$$M_v(w_\beta, \chi)(f)(k) = \langle v', kv_f \rangle$$

*with* $v' \in \mathbf{C}(v_8 + v_{-8})$.

*Proof.* The proof of this lemma is a direct computation. For $x \in \mathbf{R}$, let $x = \cot\theta$, with $0 < \theta < \pi$, Then, for $k \in K_\alpha$, we have:

$$\begin{aligned} M_v(w_\beta, \chi)(f)(k) &= \int_{-\infty}^{\infty} f\big(w_\beta u_\beta(x)k\big)\,dx \\ &= \int_0^\pi f\left(h_\beta\begin{pmatrix} \sin\theta & -\cos\theta \\ 0 & \operatorname{cosec}\theta \end{pmatrix} k_\beta(\theta)k\right) \operatorname{cosec}^2\theta\,d\theta \\ &= \int_0^\pi \sin^3\theta \cdot \langle k_\beta(-\theta)v_0, kv_f\rangle\,d\theta. \end{aligned}$$

Hence, we need to evaluate $k_\beta(-\theta)v_0$. Now $\{v_i\}$ is a basis of eigenvectors for $K_\delta$, but not for $K_\beta$. However, $K_\delta$ and $K_\beta$ are contained in $SL_3 \subset G_2$, and are in fact conjugate in $K \cap SL_3 = SO_3$. Suppose that $\mathbf{C}^2 = \langle e_1, e_2 \rangle$ is the standard representation of $\mathfrak{su}_2 \cong \mathrm{Lie}(SO_3)$ such that $\{e_1, e_2\}$ is a basis of eigenvectors for $\mathfrak{k}_\delta$. Then, with $e_i$ suitably normalized, a basis of eigenvectors for $\mathfrak{k}_\beta$ is given by:

$$f_1 = e_1 + ie_2,$$

$$f_2 = e_1 - ie_2.$$

Now $v_0 = e_1^4 e_2^4 = 2^{-8}(f_1^2 - f_2^2)^4 \in \mathrm{Sym}^8(\mathbf{C}^2)$. This allows us to compute the action of $k_\beta(\theta)$ on $v_0$, and an easy exercise in calculus gives:

$$M_v(w_\beta, \chi)f(k) = \frac{2}{35}\langle v_8 + v_{-8}, kv_f\rangle$$

as required. $\square$

The above lemma implies that $M(w_1, \chi)(f)|_{L_2}$ lies in the discrete series submodule and

$$M(w_2, \chi)(f) = M(w_3, \chi)(f) = 0$$

for $f \in \mathrm{Res}(\Pi)$, and the proposition is proved.

## 14. The Eisenstein component

In Section 12, we have introduced the theta integral $I\big(\theta(f)\big)$ for $f \in \Pi$. In particular, $I\big(\theta(f)\big)$ is an element of $\mathscr{A}(G_2)$, and we can decompose $I\big(\theta(f)\big)$ into its Eisenstein part and cuspidal part:

$$I(\theta(f)) = I(\theta(f))_{\mathrm{Eis}} + I(\theta(f))_{\mathrm{cusp}}. \tag{14.1}$$

In this section, we show:

**Theorem 14.2.** *For any* $f \in \Pi$,

$$I(\theta(f))_{\mathrm{Eis}} = E(\mathrm{Res}(f)).$$

It suffices to prove the theorem for $f \in \Pi_K$, and we begin by recalling some results about the constant term $\theta(f)_P$ of $\theta(f)$ along $P$. When $f \in \Pi_K$, it was shown in [G], §6 thas30.6762 2csho

$a \in Z \cong \mathrm{G}_m$, regarded as a subrepresentation of a suitable degenerate principal series, and let

$$\theta_M : \Pi_M \to \mathscr{A}(M)$$

be the automorphic realization of $\Pi_M$ given by the Eisenstein series in the integral above. Then we need to show:

**Lemma 14.5.** *For any $f \in \Pi_M$,*

$$I(\theta_M(f))(g) = E(g, f|_{L_2}), \quad g \in L_2(\mathrm{A}).$$

*Proof.* One can check easily that both sides have the same constant term. Hence,

$$I(\theta_M(f)) = E(-, f|_{GL_2}) + \phi(f),$$

for some cusp form $\phi(f)$, and we need to show that $\phi(f)$ is zero for all $f$.

Let $U$ be the unipotent radical of the Borel subgroup of $L_2$, and $\psi = \prod_v \psi_v$ an arbitrary non-trivial character of $\mathrm{Q}\backslash\mathrm{A}$. Then we claim that:

$$\dim \mathrm{Hom}_{G(\mathrm{A}) \times U(\mathrm{A})}(\Pi_M, \mathrm{C} \otimes \mathrm{C}(\psi)) \leqq 1.$$

To prove the claim, it suffices to show that the corresponding multiplicity one statement holds for each local place. For $v$ real, any such linear form factors through the projection map onto $(\Pi_M)_\infty^{G(\mathrm{R})}$. By [GrS], Section 3, Proposition 3.2, the $G(\mathrm{R})$-invariant subspace of $(\Pi_M)_\infty$ is, as a representation of $GL_2$, the discrete series representation with minimal weight $\pm 12$. Hence, the result follows by the uniqueness of the Whittaker functional. For $p$ a finite prime, it follows from [MS], Theorem 1.1 that

$$(\Pi_{M_p})_{U(\mathrm{Q}_p), \psi_p} \cong C_c^\infty(\omega)$$

where

$$\omega = \{X \in J_3 \otimes \mathrm{Q}_p : \mathrm{Tr}(X) = 1, X^\# = 0\}.$$

But $G(\mathrm{Q}_p)$ acts transitively on $\omega$ [J], Chap. IX, Thm. 10, so that up to scaling, there is a unique $G(\mathrm{Q}_p)$-invariant linear functional on $C_c^\infty(\omega)$. This proves the claim.

Now suppose that $\phi(f)$ is non-zero for some $f$. The two linear forms

$$\begin{cases} \lambda_\psi : f \mapsto \phi(f)_{U,\psi}(1), \\ \mu_\psi : f \mapsto E(1, f|_{GL_2})_{U,\psi} \end{cases}$$

on $\Pi_M$ are non-zero elements of $\mathrm{Hom}_{G(\mathrm{A}) \times U(\mathrm{A})}(\Pi_M, \mathrm{C} \otimes \mathrm{C}(\psi))$. Hence by the multiplicity one result above, they are non-zero multiples of each other, say $\lambda_\psi = c_\psi \mu_\psi$. Since the non-trivial characters of $U(\mathrm{Q})\backslash U(\mathrm{A})$ are in one orbit under the maximal torus of $L_2$, $c_\psi$ is actually independent of $\psi$. This implies that, when restricted to $SL_2$, $\phi(f)$ is a linear combination of a constant function and an Eisenstein series. With this contradiction, the lemma is proved. □

We have thus shown statement (ii) above. Now we complete the proof of the theorem by proving (i). Let $P_1 = L_1 \cdot U_1$ be the other maximal parabolic subgroup of $G_2$. As in Sections 4 and 5, one can find a maximal parabolic subgroup $P'$ of $H$, with Levi subgroup $M' \cong (GL_2 \times L)/\Delta\mu_3$, and such that

$$(G_2 \times G) \cap P' = P_1 \times G.$$

It remains to show that $I(\theta(f))_{P_1}$ has no cuspidal components, when regarded as an automorphic form on $L_1$. One shows that:

$$I(\theta(f))_{P_1}(g) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \theta(f)_{P'}(gg')\, dg'.$$

But using the results in [G], one can check that, when pulled back to $GL_2 \times L$, $\theta(f)_{P'}$ is the product of an Eisenstein series on $GL_2$ and a constant function on $L$. Hence, $I(\theta(f))_{P_1}$ is also an Eisenstein series on $L_1$. This proves (i), and hence the theorem.

## 15. The cuspidal component

We now study the cuspidal component $I(\theta(f))_{\mathrm{cusp}}$ of $I(\theta(f))$, and our goal is to show that $I(\theta(f))_{\mathrm{cusp}}$ is identically zero.

We begin with the following criterion for the non-vanishing of a non-generic cusp form on $G_2$:

**Proposition 15.1.** *The following are equivalent*:

(i) *$F$ is a non-generic cusp form on $G_2$*;

(ii) $F_{U_1 \cap U_2} = 0$;

(iii) $F_{U'} = 0$, *where $U'$ is the commutator subgroup of $U_1$*.

*In particular, if $F$ is a non-generic cusp form on $G_2$, satisfying $F_{U_2,\chi} = 0$ for any $\chi \in V_2$ with $\Delta(\chi) \neq 0$, then $F = 0$.*

*Proof.* The equivalence of (i) and (ii) follows immediately from the definition of a non-generic cusp form. Moreover, since $U' \subset U_1 \cap U_2$, it is clear that (iii) implies (ii). To show that (ii) implies (iii), consider the Fourier expansion of $F_{U'}$ along $U_1/U'$. The fact that $F_{U_1 \cap U_2} = 0$ implies that $F_{U_1,\psi} = 0$ for any $\psi$ such that $\psi|_{U_1(\mathbb{A}) \cap U_2(\mathbb{A})} = 1$. But the non-constant Fourier coefficients along $U_1/U'$ are permuted transitively by $L_1(\mathbb{Q})$, the Levi factor of $P_1(\mathbb{Q})$. Hence, all the Fourier coefficients of $F_{U'}$ are zero.

Now suppose that $F$ is a non-generic cusp form, so that $F_{U'} = 0$. This implies that $F_{U_2,\chi} = 0$ for all $\chi$ satisfying $\Delta(\chi) = 0$. Hence, by hypothesis, $F_{U_2,\chi} = 0$ for all $\chi \in V_2$. By [G], Lemma 9.1, this implies that $F = 0$. $\square$

Now we show the following multiplicity one result:

**Proposition 15.2.** *Suppose that $\chi \in V_2$ satisfies $\Delta(\chi) > 0$. For $\mathsf{Q}_v = \mathsf{R}$ or $\mathsf{Q}_p$,*

$$\dim \operatorname{Hom}_{G(\mathsf{Q}_v)\times U_2(\mathsf{Q}_v)}\big(\Pi_v, \mathsf{C} \otimes \mathsf{C}(\chi)\big) = 1.$$

*Proof.* For $v$ finite, this is similar to the proof of the analogous statement in the proof of Lemma 14.5, and follows from [GrS], Ch. 6, Lemma 2.9, which in turn relies on the multiplicity one result of Proposition 9.1.

It remains to treat the archimedean case. As we remarked earlier, this is a very special case of a recent result of Wallach, but as his paper [W2] has yet to appear in print, we provide a proof below. Lemma 10.5 implies that the relevant space of linear forms is non-zero. Hence, it suffices to show that

$$\dim \operatorname{Hom}_{U_2(\mathsf{R})}\big(\pi, \mathsf{C}(\chi)\big) \leqq 1.$$

We already know that $\pi$ is a quotient of the degenerate principal series representation $\operatorname{Ind}_{P_2(\mathsf{R})}^{G_2(\mathsf{R})}\delta_{P_2}^{-2/3}$. Thus the desired inequality is a consequence of the inequality

$$\dim \operatorname{Hom}_{U_2(\mathsf{R})}\big(\operatorname{Ind}_{P_2(\mathsf{R})}^{G_2(\mathsf{R})}\delta_{P_2}^{-2/3}, \mathsf{C}(\chi)\big) \leqq 1.$$

Now Bruhat's theory reduces this to bounding the dimension of a certain space of distributions on $G_2(\mathsf{R})$, which transform via $\delta_{P_2}^{-2/3}$ under left translation by $P_2(\mathsf{R})$, and via $\chi$ under right translation by $U_2(\mathsf{R})$. Since there is only a one-dimensional space of such distributions on the open orbit $P_2(\mathsf{R})w_3U_2(\mathsf{R})$, it remains to show that there is no such non-zero distribution supported on the complement of $P_2(\mathsf{R})w_3U_2(\mathsf{R})$.

For this, it appears a priori that we are unable to apply the usual estimates in Bruhat's theory, because the double coset space $P_2(\mathsf{R})\backslash G_2(\mathsf{R})/U_2(\mathsf{R})$ has uncountably many representatives. Fortunately, an extension of Bruhat's theory by Kolk and Varadarajan [KV] allows us to proceed as usual. Hence, for $\mu \in P_2(\mathsf{R})\backslash G(\mathsf{R})/U_2(\mathsf{R})$, and $k \geqq 0$ an integer, let

$$N_{\mu,k} = \dim \operatorname{Hom}_{U_{2,\mu}(\mathsf{R})}\big(\operatorname{Sym}^k V_\mu \otimes \mathsf{C}, \mathsf{C}(\chi)\big),$$

where

$$U_{2,\mu}(\mathsf{R}) = U_2(\mathsf{R}) \cap \mu^{-1}P_2(\mathsf{R})\mu,$$

and $V_\mu$ is the representation of $U_{2,\mu}(\mathsf{R})$ on $\operatorname{Lie}(G_2)/\big(\operatorname{Lie}(U_2) + \operatorname{Lie}(\mu^{-1}P_2\mu)\big)$ by the adjoint action. Then, by [KV], Theorem 3.15, we need to show that $N_{\mu,k} = 0$ if $\mu \neq w_3$. For this, it suffices to show that $\chi$ is non-trivial when restricted to $U_{2,\mu}(\mathsf{R})$ if $\mu \neq w_3$.

Now we can enumerate the representatives $\mu \neq w_3$ easily, as in [JR]. They are given by the following list:

$$1, w_1, w_1 w_\alpha u_\alpha(t),$$

$$w_2, w_2 w_\alpha u_\alpha(t),$$

for $t \in \mathsf{R}$. It is easy to check that $N_{\mu,k} = 0$, whenever $\mu = 1$, $w_1$ or $w_2$. Hence it remains to show that $N_{\mu,k} = 0$ for the other representatives $\mu$. The proofs are similar for these, and we will just do the more difficult case where $\mu = w_2 w_\alpha u_\alpha(t)$.

Let

$$u(a,b,c,d,z) = u_\beta(a) \cdot u_{\alpha+\beta}(b) \cdot u_{2\alpha+\beta}(c) \cdot u_{3\alpha+\beta}(d) \cdot u_{3\alpha+2\beta}(z)$$

be an element of $U_2(\mathsf{R})$. Under the adjoint action of $L_2(\mathsf{R})$, the representation $U_2(\mathsf{R})/Z_2(\mathsf{R})$ is isomorphic to the representation of $GL_2$ on the space of binary cubic forms over $\mathsf{R}$. This is given by the identification:

$$u(a,b,c,d,z) \mapsto ax^3 - 3bx^2y + 3cxy^2 - dy^3.$$

Using this, we can identify the subgroup $U_{2,\mu}$, where $\mu = w_2 w_\alpha u_\alpha(t)$ for a fixed $t$. Indeed, after a short computation, we see that if $u(a,b,c,d,z)$ lies in $U_{2,\mu}(\mathsf{R})$, then

$$c = 2bt - at^2,$$

$$d = 3bt^2 - 2at^3.$$

Now suppose, without loss of generality, that $\chi = (1, 0, p, q)$ as an element of $V_2 \otimes \mathsf{R}$, so that

$$\chi\big(u(a,b,c,d,z)\big) = \exp\big(2\pi i(a + pc + qd)\big).$$

We need to show that the restriction of $\chi$ to $U_{2,\mu}(\mathsf{R})$ is non-trivial. However, if it is, then

$$a + pc + qd = a(1 - pt^2 - 2qt^3) + b(2pt + 3qt^2)$$

must be identically zero, for all $a$ and $b$. Hence the coefficients must vanish, so that $t \neq 0$, and we deduce that $\Delta(\chi) = 4p^3 + 27q^2 = 0$. Since this contradicts the assumption that $\Delta(\chi) > 0$, we must have $N_{\mu,k} = 0$ for all $k$. This completes the proof of the proposition. □

**Corollary 15.3.** *Suppose that $\chi \in V_2$ satisfies $\Delta(\chi) > 0$. Then*

$$\dim \operatorname{Hom}_{G(\mathsf{A}) \times U_2(\mathsf{A})}\big(\Pi, \mathsf{C} \otimes \mathsf{C}(\chi)\big) \leqq 1.$$

*Assume further that* $\dim \mathrm{Hom}_{G(\mathsf{A})\times U_2(\mathsf{A})}(\Pi, \mathsf{C}\otimes\mathsf{C}(\chi)) = 1$. *Then, for almost all* $p$, *any non-zero element in the* 1*-dimensional space*

$$\mathrm{Hom}_{G(\mathsf{Q}_p)\times U_2(\mathsf{Q}_p)}(\Pi_p, \mathsf{C}\otimes\mathsf{C}(\chi))$$

*is non-zero on the spherical vector of* $\Pi_p$.

We can construct a non-zero element of $\mathrm{Hom}_{G(\mathsf{Q}_p)\times U_2(\mathsf{Q}_p)}(\Pi_p, \mathsf{C}\otimes\mathsf{C}(\chi))$ as follows. Recall that there is a $G_2(\mathsf{A})$-equivariant surjection

$$\mathrm{Res}\colon \Pi \to \pi \otimes \left(\hat{\bigotimes_p} I_p\right) \subset I.$$

For each prime $p$, the integral

$$\varphi \mapsto \int_{U_2(\mathsf{Q}_p)} \varphi(w_3 u)\cdot\overline{\chi(u)}\,du \tag{15.4}$$

is absolutely convergent and defines a non-zero element $l_p$ of $\mathrm{Hom}_{U_2(\mathsf{Q}_p)}(I_p, \mathsf{C}(\chi))$. Hence, $l_p \circ \mathrm{Res}_p$ is a non-zero element of the 1-dimensional space $\mathrm{Hom}_{G(\mathsf{Q}_p)\times U_2(\mathsf{Q}_p)}(\Pi_p, \mathsf{C}\otimes\mathsf{C}(\chi))$. Now suppose that $\chi$ satisfies all the conditions of the above corollary, then for almost all $p$, $l_p$ is non-zero on the spherical vector of $I_p$. For these $p$, we normalize $l_p$ so that it takes value 1 on the spherical vector. Let $l_\infty$ be a non-zero element of the 1-dimensional space $\mathrm{Hom}_{U_2(\mathsf{R})}(\pi, \mathsf{C}(\chi))$. Then the map $l_\chi := \bigotimes_v l_v$ is a non-zero element of

$$\mathrm{Hom}_{U_2(\mathsf{A})}(\mathrm{Res}(\Pi), \mathsf{C}(\chi)),$$

and $l_\chi \circ \mathrm{Res}$ is a non-zero element of $\mathrm{Hom}_{G(\mathsf{A})\times U_2(\mathsf{A})}(\Pi, \mathsf{C}\otimes\mathsf{C}(\chi))$.

Now we can prove the Siegel-Weil formula:

**Theorem 15.5.** $I(\theta(f))_{\mathrm{cusp}} = 0$ *for any* $f \in \Pi$. *In particular*,

$$I(\theta(f)) = E(\mathrm{Res}(f)).$$

*Proof.* First we claim that there is no $f \in \Pi$ such that $I(\theta(f))$ is a non-zero cusp form. If not, then by Proposition 15.1, there is an $f_0 \in \Pi$ and a nondegenerate $\chi \in V_2$ such that

$$\begin{cases} I(\theta(f_0))_{\mathrm{Eis}} = 0 \quad \text{and} \\ I(\theta(f_0))_\chi(1) \neq 0. \end{cases}$$

Then $\chi$ satisfies all the conditions of the previous corollary, since $I(\theta(f_0))_\chi$ vanishes if $\Delta(\chi) < 0$. Hence the linear functional

$$f \mapsto \left(I(\theta(f))_{\mathrm{cusp}}\right)_\chi(1)$$

is a non-zero element of $\mathrm{Hom}_{G(\mathsf{A})\times U_2(\mathsf{A})}(\Pi, \mathsf{C}\otimes\mathsf{C}(\chi))$, and $f_0$ is not in its kernel. By the discussion preceding the theorem, we have another non-zero linear form $l_\chi \circ \mathrm{Res}$. By Corollary 15.3, these linear functionals must be non-zero multiples of each other. But this is not possible, since $l_\chi \circ \mathrm{Res}(f_0) = 0$. With this contradiction, the claim is established.

As a result of the above claim, the projection of $I \circ \theta(\Pi)$ onto the space of Eisenstein series is injective. Hence $I \circ \theta(\Pi)$ is isomorphic to the irreducible representation $\mathrm{Res}(\Pi)$. By [M], Lemma 5.1, the representations $I_p$ are not unitarizable. Hence, the projection of $I \circ \theta(\Pi)$ onto the space of cusp forms must be the zero map. This proves the theorem. □

It follows from Theorem 15.5 and [JR], Theorem 2 (i) that, when $A + \mathsf{Z}\times\mathsf{Z}\times\mathsf{Z}$, the $A$-th Fourier coefficients of the modular forms in Proposition 12.3 are non-zero. Hence, we conclude that the quantity $e$ in Theorem 11.3, Corollary 11.4 and Proposition 12.3 is equal to 1. For more details, we refer the reader to [GGS], §9 and §10.

## References


[B] *A. Borel*, Properties and Linear Representations of Chevalley Groups, Springer Lect. Notes Math. **131** (1970).

[BT] *F. Bruhat* and *J. Tits*, Groupes Reductifs sur un Corps Locale II, Publ. Math. I.H.E.S. **60** (1984), 197–376.

[C] *W. Casselman*, Canonical Extensions of Harish-Chandra Modules to Representations of $G$, Canadian J. Math. **41** (1989), 385–438.

[EG] *N. Elkies* and *B. H. Gross*, The Exceptional Cone and the Leech Lattice, Intern. Math. Res. Notices **14** (1996), 665–698.

[G] *W. T. Gan*, An Automorphic Theta Module for Quaternionic Exceptional Groups, Canadian J. Math. **52** (4) (2000), 737–756.

[Gr] *B. H. Gross*, Groups over $\mathsf{Z}$, Invent. Math. **124** (1996), 263–278.

[GG] *B. H. Gross* and *W. T. Gan*, Commutative Subrings of Certain Non-Associative Rings, Math. Ann. **314** (1999), 265–283.

[GG2] *B. H. Gross* and *W. T. Gan*, Integral Embeddings of Cubic Norm Structures, J. Alg., to appear.

[GGS] *B. H. Gross*, *W. T. Gan* and *G. Savin*, Fourier Coefficients of Modular Forms on $G_2$, preprint 2000.

[GrK] *B. H. Gross* and *S. Kudla*, Heights and the Central Critical Values of Triple Product $L$-functions, Compos. Math. **81** (1992), 143–209.

[GrS] *B. H. Gross* and *G. Savin*, Motives with Galois Group of type $G_2$, Compos. Math. **114**, No. 2 (1998), 153–217.

[GrW] *B. H. Gross* and *N. Wallach*, On quaternionic discrete series representations and their continuations, J. reine angew. Math. **481** (1996), 73–123.

[HK] *M. Harris* and *S. Kudla*, The Central Critical Value of a Triple Product $L$-function, Ann. Math. **133** (1991), 605–672.

[HPS] *J. S. Huang*, *P. Pandzic* and *G. Savin*, New Dual Pair Correspondences, Duke Math. J. **82**, No. 2 (1996), 447–471.

[J] *N. Jacobson*, Structure and Representations of Jordan Algebras, AMS Coll. Publ. **39** (1968).

[JR] *D. H. Jiang* and *S. Rallis*, Fourier Coefficients of Eisenstein Series of the Exceptional Group of Type $G_2$, Pacific J. Math. **181**, No. 2 (1997), 281–314.

[KMRT] *M.-A. Knus*, *A. Merkejev*, *M. Rost* and *J.-P. Tignol*, The Book of Involutions, AMS Coll. Publ. **44** (1998).

[KV] *J. A. C. Kolk* and *V. S. Varadarajan*, On the Transverse Symbol of Vectorial Distributions and Some Applications to Harmonic Analysis, Indag. Math., New Series **7**, No. 1 (1996), 67–96.

[KR1] *S. Kudla* and *S. Rallis*, On the Weil-Siegel formula, J. reine angew. Math. **387** (1988), 1–68.

[KR2] *S. Kudla* and *S. Rallis*, On the Weil-Siegel formula II: Isotropic convergent case, J. reine angew. Math. **391** (1988), 65–84.

[KR3] *S. Kudla* and *S. Rallis*, A Regularized Siegel-Weil Formula, Ann. Math. **140** (1994), 1–80.

[MS] *K. Magaard* and *G. Savin*, Exceptional Θ-Correspondences, Compos. Math. **107** (1997), 89–123.

[M] *G. Muić*, The Unitary Dual of $p$-adic $G_2$, Duke Math. J. **90**, No. 3 (1997), 465–493.
[Ru] *K. Rumelhart*, Minimal Representations of Exceptional $p$-adic Groups, Represent. Th. **1** (1997), 133–181.
[S] *G. Savin*, The Dual Pair $PGL_2 \times G_J$; $G_J$ is the Automorphism Group of the Jordan Algebra, Invent. Math. **118** (1994), 141–160.
[S2] *G. Savin*, A Class of Supercuspidal Representations of $G_2(k)$, Canad. Math. Bull. **42**, No. 3 (1999), 393–400.
[W] *N. Wallach*, $C^\infty$-vectors, in: Representations of Lie Groups and Quantum Groups, Pitman Res. Notes Math. Ser. **311** (1900), 205–270.
[W2] *N. Wallach*, Generalized Whittaker Vectors for Holomorphic and Quaternionic Representations, preprint 2000.
[Wr] *D. Wright*, The Adelic Zeta Function Associated to the Space of Binary Cubic Forms, Math. Ann. **270** (1985), 503–534.

Mathematics Department, Princeton University, Princeton, NJ 08544, U.S.A.
e-mail: wtgan@math.princeton.edu

Eingegangen 29. Juni 1999, in revidierter Fassung 7. Februar 2000