General Comments

- The proof of web problem A can be found in the book. See Theorem 2.12.
- Web problems B and C are both true, and it's easiest to use the unique prime factorizations and the alternative characterizations of the GCD and LCM.
- Web problem D is false. Take $n = 3$. Note that $n = 1$ is not a counterexample because 1 is not a prime.

Selected Solution

Sect 5.3, #1: Find all solutions to the equation $x^2 + 2y^2 = w^2$ with $x > 0$, $y > 0$, and $w > 0$.

Proof: Let (x, y, w) be a solution. Without loss of generality, we may assume that $(x, y, w) = 1$ (to get all solutions, we take multiples of these *primitive* solutions). Notice that if $d \mid (x, y)$ then $d^2 \mid w^2 = x^2 + 2y^2$, so that $(x, y) = 1$. Similarly, $(x, w) = (y, w) = 1$.

Therefore, there can be at most one even number among x, y, and w.

Suppose that all of them are odd. Then, noting that odd squares are 1 modulo 4, we have $x^2 + 2y^2 \equiv 3 \mod 4$ and $w^2 \equiv 1 \mod 4$, contradicting $x^2 + 2y^2 = w^2$.

Suppose that x is even. Then w^2 is even, implying that w is even, contradicting that there is only one even value among the three variables. Similarly, w cannot be even.

The only case remaining is when y is even. We can rewrite our equation as $(w+x)(w-x) = 2y^2$. Since x and w are both odd, their sum and difference are both even. Rearranging the terms and dividing by 4, we get

$$\frac{1}{2}\left(\frac{w+x}{2}\right)\left(\frac{w-x}{2}\right) = \left(\frac{y}{2}\right)^2,$$

where both fractions on the left are integers. Notice that by looking at all four possibilities, one of $w + x$ and $w - x$ is divisible by 4 and the other is not. Therefore, we have two cases.

Case 1, $4 \mid w + x$: We have

$$\left(\frac{w+x}{4}\right)\left(\frac{w-x}{2}\right) = \left(\frac{y}{2}\right)^2,$$

where both fractions on the left are integers. We want to show that they are relatively prime. Let d be the GCD of the two integers on the left. Then d divides all integer linear combinations of the two, so in particular,

$$d \mid 2 \cdot \left(\frac{w+x}{4}\right) + \left(\frac{w-x}{2}\right) \implies d \mid w$$
$$d \mid 2 \cdot \left(\frac{w+x}{4}\right) - \left(\frac{w-x}{2}\right) \implies d \mid x.$$

So $d \mid (w, x) = 1$ and we have shown the two numbers are relatively prime.

By Theorem 2.12, we must have

$$\frac{w+x}{4} = u^2 \text{ and } \frac{w-x}{2} = v^2$$

for some integers u and v. Solving for our original variables, we get

$$\begin{aligned} x &= 2u^2 - v^2 \\ y &= 2uv \\ w &= 2u^2 + v^2 \end{aligned}$$

Since $y > 0$, we need both u and v to have the same sign, and we can take them both to be positive. Since $x > 0$, we have $\sqrt{2}u > v > 0$. Finally, since x and w are odd, v must be odd. This completes the first case.

Case 2, $4 \mid w-x$**:** The proof is essentially the same as above, so it will not be typed out completely. When we use Theorem 2.12, we get

$$\frac{w+x}{2} = u^2 \text{ and } \frac{w-x}{4} = v^2$$

for some integers u and v. We solve for the original variables to get

$$\begin{aligned} x &= u^2 - 2v^2 \\ y &= 2uv \\ w &= u^2 + 2v^2 \end{aligned}$$

The conditions are on u and v are that $u > \sqrt{2}v > 0$ and v is odd.