Selected Solutions

Sect 2.4, #6: Find an integer n less than or equal to 70 such that $d(n) = 12$.

Proof: We know that $d(n)$ is multiplicative and that $d(p^a) = a+1$. So we must construct a number $n = p_1^{a_1} \cdots p_k^{a_k}$ such that $(a_1 + 1) \cdots (a_k + 1) = 12$.

There are several ways to write the number 12 as a product:

$$12 = 2 \cdot 6 = 3 \cdot 4 = 2 \cdot 2 \cdot 3.$$

For each of these products, there is a corresponding list of a_i:

$$\begin{aligned} 12 = (11+1) &\implies a_1 = 11 \\ 2 \cdot 6 = (1+1) \cdot (5+1) &\implies a_1 = 1, a_2 = 5 \\ 3 \cdot 4 = (2+1) \cdot (3+1) &\implies a_1 = 2, a_2 = 3 \\ 2 \cdot 2 \cdot 3 = (1+1) \cdot (1+1) \cdot (2+1) &\implies a_1 = a_2 = 1, a_3 = 2. \end{aligned}$$

Since we are looking for small values of n, we should take the primes as small as possible. We will make a list of these small values of n that give $d(n) = 12$.

$$\begin{aligned} 2^{11} &= 2048 \\ 2 \cdot 3^5 &= 486 \\ 3 \cdot 2^5 &= 96 \\ 2^2 \cdot 3^3 &= 108 \\ 3^2 \cdot 2^3 &= 72 \\ 2 \cdot 3 \cdot 5^2 &= 150 \\ 2 \cdot 5 \cdot 3^2 &= 90 \\ 3 \cdot 5 \cdot 2^2 &= 60 \end{aligned}$$

Therefore, $n = 60$ is such an integer.

Comments: I gave a full explanation here to demonstrate a logical pattern of thought to follow when trying to get values of n such that $d(n)$ has a specific value. There is a general notion of thinking clearly about these types of problems so that you arrive at your solution in an efficient and organized manner. Random guessing will sometimes give you the solution quickly, but you should not rely on luck.

Sect 2.4, #7: Find an integer n such that $\sigma(n) = 546$.

Proof: Although the problem asked you to find some n that works, we will find *all* n. Observe that σ is a multiplicative function, that

$$\sigma(p^a) = 1 + p + \cdots + p^a = \frac{p^{a+1} - 1}{p - 1},$$

and that $546 = 2 \cdot 3 \cdot 7 \cdot 13$.

There are 16 factors of 546:

$$1, 2, 3, 6, 7, 13, 14, 21, 26, 39, 42, 78, 91, 182, 273, 546.$$

Our task is to find p^a such that $\sigma(p^a)$ is one of these values. From this list, we will try to construct values of n by picking appropriate combinations of factors.

We will begin with the $a = 1$ case. Here, we are looking for primes p such that $\sigma(p)$ is one of the above factors. But since $\sigma(p) = p + 1$, we only need to check if the above factors are one more than a prime. A quick check yields:

$$\sigma(2) = 3, \sigma(5) = 6, \sigma(13) = 14, \sigma(41) = 42, \sigma(181) = 182.$$

We will now work with the $a > 1$ cases. There is no quick check as in the previous case, so we simply use brute force. However, this does not mean that we can't be logical about it. We will start with $a = 2$ and write down the values of $\sigma(p^a)$ for increasing values of primes until we pass 546. Then we will increase a by one and repeat. We will obtain a list of calculations that isn't very long.

$$\sigma(2^2) = 7, \sigma(3^2) = 13, \sigma(5^2) = 31, \sigma(7^2) = 57, \sigma(11^2) = 133$$
$$\sigma(13^2) = 183, \sigma(17^2) = 307, \sigma(19^2) = 381$$
$$\sigma(2^3) = 15, \sigma(3^3) = 40, \sigma(5^3) = 156, \sigma(7^3) = 400$$
$$\sigma(2^4) = 31, \sigma(3^4) = 121$$
$$\sigma(2^5) = 63, \sigma(3^5) = 364$$
$$\sigma(2^6) = 127, \sigma(2^7) = 255, \sigma(2^8) = 511$$

Of these, only $\sigma(4) = 7$ and $\sigma(9) = 13$ gives us a factor of 546.

Therefore, the only numbers we need to work with are

$$\sigma(2) = 3, \sigma(5) = 6, \sigma(4) = 7, \sigma(9) = 13, \sigma(13) = 14, \sigma(41) = 42, \sigma(181) = 182.$$

We will start with the large factors on the list and work our way to smaller factors, keeping our factors in decreasing order:

$$\begin{aligned} 546 &= 182 \cdot 3 = \sigma(181) \cdot \sigma(2) = \sigma(362) \\ &= 42 \cdot 13 = \sigma(41) \cdot \sigma(9) = \sigma(369) \\ &= 14 \cdot 39 = 14 \cdot 13 \cdot 3 = \sigma(13) \cdot \sigma(2) \cdot \sigma(9) = \sigma(234) \\ &= 13 \cdot 42 = 13 \cdot 7 \cdot 6 = \sigma(9) \cdot \sigma(4) \cdot \sigma(5) = \sigma(180) \\ &= 13 \cdot 7 \cdot 3 \cdot 2 \leftarrow \text{Impossible!} \end{aligned}$$

Therefore, $\sigma(n) = 546$ when $n = 180, 234, 362, 369$.

Comments: Once again, the purpose here is to help you think logically about this problem. To give a reference for the amount of time it should take for this problem, the $a = 1$ case should take no more than a couple minutes. The $a > 1$ case should take no more than about 10 minutes (there are only a couple dozen σ values to compute). When doing problems like this one or the previous one, you should try to avoid computing σ for composite values. By restricting to primes and powers of primes you will be able to work very quickly.

Web Problem #2c: Find the function g defined by

$$g(n) = \sum_{d|n} \mu(d).$$

Proof: A quick calculation suggests that

$$g(n) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We know by Theorem 2.15 that g is a multiplicative function. Therefore, we only need to compute its values at 1 and powers of primes (p^a where $a \geq 1$).

$$g(1) = \mu(1) = 1$$

$$g(p^a) = \mu(1) + \mu(p) + \mu(p^2) + \cdots + \mu(p^a) = 1 + (-1) + 0 + \cdots + 0 = 0$$

Therefore, if $n > 1$, then if we write out the prime factorization of n, we get

$$g(n) = g(p_1^{a_1}) \cdots g(p_k^{a_k}) = 0.$$

Web Problem #2d: For any positive integer n, show that there exist n consecutive integers on μ which takes the value 0.

Proof: Let $p_1, p_2, \cdots, p_n$ be n distinct primes and consider the following system of congruences:

$$\begin{aligned} x &\equiv 0 \pmod{p_1^2} \\ x + 1 &\equiv 0 \pmod{p_2^2} \\ &\vdots \\ x + (n-1) &\equiv 0 \pmod{p_n^2} \end{aligned}$$

Since each of the moduli are relatively prime to each other, by the Chinese Remainder Theorem there is a unique solution

$$x \equiv b \pmod{p_1^2 p_2^2 \cdots p_n^2}.$$

It does not matter what the value of b is. The point is that such a number exists. This means that there exists a number x such that $x, x+1, \cdots x+(n-1)$ are all divisible by the square of a prime. But by the definition of μ, this means that

$$\mu(x) = \mu(x+1) = \cdots = \mu(x + (n-1)) = 0.$$

Web Problem #3: Find all n such that $\sigma(n) = 28$.

Proof: After #7 from Section 2.4, this is easy. 28 factors as $2^2 \cdot 7$. The factors of 28 are

$$1, 2, 4, 7, 14, 28.$$

Following the same procedure as before (I will leave out the details), we arrive at the following values that will give factors of 28:

$$\sigma(3) = 4, \sigma(4) = 7, \sigma(13) = 14.$$

There is only one combination that works:

$$28 = 4 \cdot 7 = \sigma(3) \cdot \sigma(4) = \sigma(12).$$

So the only value of n such that $\sigma(n) = 28$ is $n = 12$.