MATH. 104A, HOMEWORK 3 (DUE 10/18)

Do the following questions from the textbook:

Sect. 2.3: # 2, 3 ; Sect. 2.5: 3, 8, 12

In addition, do the following problems:

(1) Find all integer solutions to the linear Diophantine equation:

$$3x + 4y + 5z = 15.$$

(2) Let

$$C_r^n = \frac{n!}{r! \cdot (n-r)!}.$$

Show that C_r^n is an integer (Hint: show that for any prime p, more powers of p divide the numerator than the denominator).

(3a) Suppose that

$$f(x) = x^k + a_{k-1}x^{k-1} + ... + a_1 x + a_0$$

is a polynomial with integer coefficients whose leading coefficient is 1. If α is a root of $f(x)$ and α is not an integer, show that α must be irrational. (Hint: suppose by contradiction that $\alpha = \frac{m}{n}$ is rational, with m and n relatively prime. Clear denominators and try to obtain a contradiction.)

(b) Using (a), show that $\sqrt{2} + \sqrt{5}$ is irrational.

(4a) Show by mathematical induction that the product of n integers of the form $4k + 1$ is itself of this form.

(b) Using (a), show that there are infinitely many primes of the form $4k - 1$. (Hint: Imitate Euclid's proof that there are infinitely many primes. If there are only finitely many primes p_1,...,p_r of the form $4k - 1$, consider the number $N = 4p_1...p_r - 1$.)