MATH. 104A, HOMEWORK 8 (DUE 11/27 MON)

Do the following questions from the textbook:

Misc. Exercises (Pg. 108): # 1

(1) Find all different solutions modulo 60 of the quadratic congruence:

$$x^2 + 26x + 33 \equiv 0(\mod 60).$$

(Hint: $60 = 3 \cdot 4 \cdot 5$).

(2a) We showed in class that Euler's function ϕ is multiplicative. Determine the function

$$F_\phi(n) = \sum_{d|n} \phi(d).$$

(b) Deduce from (a) that

$$\sum_{d|n} \phi(d) = n.$$

(3) Let $\left(\frac{a}{p}\right)$ denote the Legendre symbol. Compute the value of

$$\left(\frac{5}{11}\right), \quad \left(\frac{7}{11}\right), \quad \left(\frac{8}{11}\right).$$

(4) Let $p \geq 7$ be a prime number. Show that one of 2, 5 and 10 is a quadratic residue mod p. Deduce from this that there are two consecutive integers which are quadratic residues mod p.

(5) Find all odd primes p such that -2 is a quadratic residue mod p.

(6) Show that there are infinitely many primes of the form $8k+3$. (Hint: if there are only fnitely many such primes $p_1, ...p_r$, consider $Q = (p_1...p_r)^2 + 2$ and use Question 5).

(7) Find all odd primes p such that the Legendre symbol $\left(\frac{7}{p}\right) = -1$.

(8) Show that there are infinitely many primes of the form $5k+4$. (Hint: if there are only finitely many such primes $p_1, ..., p_r$, consider $Q = 5(2p_1...p_r)^2 - 1$).

Questions 6 and 8 are special cases of **Dirichlet's theorem on primes in arithmetic progression**, which says that if $GCD(a, n) = 1$, then there are infinitely many primes which are $\equiv a \bmod n$. The proof of this is beyond the scope of this course, but it may be covered in Math104C.