MATH. 140A PRACTICE FINAL (FALL 2008)

(1) (30 points) Decide if the following statements are true or false, giving justification for your answers.

(a) The intersection of any collection of open sets in a metric space is open.

(b) A countable union of countable sets is countable.

(c) Let $f : X \longrightarrow Y$ be a continuous function between two metric spaces X and Y and suppose that $E \subset Y$ is compact. Then $f^{-1}(E)$ is also complete.

(d) If a subset E of a metric space is bounded, so is its closure $\overline{E}$.

(e) If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = w_0 \in \mathbb{C}$, then it converges absolutely at any $w \in \mathbb{C}$ such that $|w| < |w_0|$.

(f) If $\sum_n a_n$ and $\sum_n b_n$ converges, so does $\sum_n a_n b_n$.

(2a) (10 points) Give the definition of the following phrases:

(i) the metric space X is compact;

(ii) $\limsup_n a_n = \alpha$;

(iii) $f : X \longrightarrow Y$ is uniformly continuous.

(b) (10 points) Show that if $K_1 \supset K_2 \supset K_3 \supset ......$ is a decreasing sequence of compact sets, then $\cap_n K_n$ is nonempty.

(c) (10 points) If X is compact, show that X contains a dense subset which is at most countable.

(3a) (15 points) Consider the sequence a_n defined recursively by: $a_1 = 3$ and

$$a_{n+1} = \sqrt{7 + 6a_n} \quad \text{for } n \geq 1.$$

Show that the sequence a_n is convergent and find its limit.

(b) (15 points) Find the radius of convergence of the power series $\sum_n \frac{2^n}{n^2} z^n$.

(4) (30 points) Let $E \subset \mathbb{R}$ and suppose $f : E \to \mathbb{R}$ is uniformly continuous. Let (x_n) and (y_n) be 2 sequences in E for which $\lim x_n = \lim y_n$ exists in $\mathbb{R}$.

(a) Prove that $\lim f(x_n)$ exists. (Caution: You may NOT assume that $\lim x_n \in E$.)

(b) Prove that $\lim f(x_n) = \lim f(y_n)$.

(c) Show by example that (a) may fail if f is assumed only to be continuous, but not uniformly continuous.

(5a) (5 points) State the Heine-Borel theorem.

(b) (10 points) Show that if $f : X \longrightarrow Y$ is continuous and X is compact, then $f(X)$ is compact.

(c) (15 points) Show that a subset $E \subset \mathbb{R}^n$ is compact if and only if every real valued continuous function on E is bounded.