APPLIED ALGEBRA:
SYLLABUS

The references for this course are:

(i) G. James and M. Liebeck: Representations and Characters of Groups
(ii) J. P. Serre: Linear representations of finite groups, GTM 42
(iii) B. Sagan: Representations, combinatorial algorithms and symmetric functions
(iv) W. Fulton and J. Harris: Representation theory, GTM 129
(v) I. G. MacDonald: Symmetric fucntions and Hall polynomials.

1. **Representation Theory of Finite Groups**

The first part of the course concerns the basic theory of representations of finite groups and covers the following topics:

- Group actions, orbits and stabilizers;
- Representations of finite groups, simple examples and basic notions: subrepresentations, irreducibility, quotients and complete reducibility;
- Linear algebraic constructions: tensor products, duals, Hom's, Sym^k and $\wedge^k$
- G-linear Hom's and Schur's lemma;
- Classification and construction of irreducible representations of finite abelian groups;
- Maschke's theorem (complete reducibility theorem), and the canonical decomposiiton of a representation into isotypic summands.
- the regular representation and its decomposition into irreducibles;
- Group algebra $\mathbb{C}[G]$ and its isomorphism with a product of matrix algebras;
- Number of irreducible representations = number of conjugacy classes;
- Characters of representations and class functions;
- inner product on the space of class functions and orthonormality of irreducible characters;
- a representation is determined by its character;
- the character table, row and column orthogonality;
- Representations of $G \times H$;
- Restriction of representations and induction;
- Frobenius reciprocity;
- construction and classification of irreducible characters/representations of various groups, such as dihedral groups, finite Heisenberg groups, symmetric groups S_n and alternating groups A_n for $n \leq 5$.
- Computation of character tables for above groups. Completing a character table given certain rows.

2. **Representation Theory of S_n**

The second part of the course concerns the construction and classification of irreducible representations of the symmetric groups S_n from a combinatorial approach. In particular, the following topics are covered:

- Partitions, Young diagrams, Tableaux, Tabloids;
- Dominance and lexicographic order;
- Construction of the standard modules M^λ associated to a partition λ;
- Construction of the Specht modules S^λ as a submodule of M^λ;
- Showing that the Specht modules are irreducible and exhaust the irreducible representations of S_n;
- dimension of S^λ and the number of standard λ-tableaux;
- Semistandard generalized tableaux of shape λ and content μ;
- Decomposition of M^λ into irreducibles and semistandard generalized tableaux;
- Branching rules from S_n to S_{n-1}, and induction from S_{n-1} to S_n.

3. **Symmetric Functions**

The last part of the course is on the theory of symmetric functions, as in the first chapter of the classic book of MacDonald ("Symmetric Functions and Hall Polynomials"). Topics include:

- the ring Λ_n of symmetric polynomials in n variables with $\mathbb{Z}$-coefficients;
- the ring Λ of symmetric fucitons as an inverse limit of the Λ_n's;
- various bases of Λ: the monomial symmetric functions m_λ, the elementaryu symmetric functions e_λ, the complete symmetric functions h_λ and the power symmetric polynomials p_λ;
- the generating functions $E(t)$, $H(t)$ and $P(t)$ for the above symmetric functions;
- the involution $|omega$ on Λ;
- relations of above bases via determinantal identities;
- skew-symmetric polynomials and the Schur functions s_λ
- the Jacobi-Trudi identity;
- Inner product on Λ and orthonormal basis;
- Cauchy identity and inner product;
- Transition matrices between different bases of Λ;
- Relation with representation theory of symmetric groups: the characteristic map and its many properties;
- Frobenius' formula for the character table of symmetric groups.