## MATH. 20D, MIDTERM 2

You have **45 minutes** for this exam. Please write legibly and show all working. **No calculators are allowed.** Write your name, ID number and your TA's name below.

**Name:**

**ID Number:**

**TA's name:**

**(1)** (30 points) Consider the initial value problem, where $b$ is a constant:

$$y'' - by' + (b-1)y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

(a) Solve the above initial value problem, taking care to consider possibly different cases according to the value of $b$.

Hint: note that $b^2 - 4b + 4 = (b-2)^2$. Also, recall the formula for solving a quadratic equation $x^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

(b) Suppose now that $b = 2$, but the initial conditions are replaced by

$$y(0) = \alpha \quad \text{and} \quad y'(0) = 1.$$

Find the solution to the initial value problem in this case. Describe how the solution behave as $t \to \infty$, taking care to distinguish the different possibilities for different values of $\alpha$.

a) The characteristic eqn is: $r^2 - br + (b-1) = 0$

$$\parallel$$

$$(r-1)(r-(b-1))$$

This has 2 roots $r = 1$ or $r = b-1$.

Case 1: $b \neq 2$ (so that $b-1 \neq 1$).

Then the 2 roots are different. So the general soln is:

$$y(t) = c_1 e^t + c_2 e^{(b-1)t}.$$

$$\left.\begin{aligned} y(0)=0 &\Rightarrow c_1 + c_2 = 0 \\ y'(0)=1 &\Rightarrow c_1 + (b-1)c_2 = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= -1/_{b-2} \\ c_2 &= 1/_{b-2} \end{aligned}$$

Case 2: $b = 2$ (so that the 2 roots are equal).

General soln is $y(t) = c_1 e^t + c_2 t e^t$.

Initial conditions $\Rightarrow$ $c_1 = 0$, $c_2 = 1$.

1 b) If $b = 2$, so that general sol$^n$ is

$$y(t) = c_1 e^t + c_2 t e^t$$

but $y(0) = \alpha$, $y'(0) = 1$, then

$$c_1 = \alpha \quad \& \quad c_2 = 1 - \alpha$$

So $$y(t) = \alpha e^t + (1-\alpha) t e^t$$

If $\alpha \geq 1$, $y(t) \longrightarrow \infty$ as $t \to \infty$, because the coefficient $(1-\alpha)$ of $te^t$ is $\geq 0$.

if $\alpha < 1$, then $y(t) \longrightarrow -\infty$ as $t \to \infty$

**(2)** (30 points) Find the general solution to the following non-homogeneous equation:

$$y'' - 2y' + 2y = g(t). \longleftarrow (*)$$

when

(a) $g(t) = t$.

(b) $g(t) = 4\cos(2t)$.

Consider first the associated homogeneous eqn:

$$y'' - 2y' + 2y = 0.$$

Characteristic eqn: $r^2 - 2r + 2 = 0$

$$\Rightarrow (r-1)^2 + 1 = 0 \Rightarrow r = 1 \pm i.$$

So general sol^n is $y(t) = c_1 e^t \cos t + c_2 e^t \sin t$

(a) To find a particular sol^n, try $y(t) = at + b$.

Then $y' = a$, $y'' = 0$.

Sub. into $(*)$: $-2a + 2(at+b) = t$.

i.e. $2at + (2b - 2a) = t$.

$$\Rightarrow a = \tfrac{1}{2}, \quad b = \tfrac{1}{2}$$

So general sol^n of $(*)$ is $y(t) = c_1 e^t \cos t + c_2 e^t \sin t + \tfrac{1}{2}(t+1)$

(b) Try $y(t) = a\cos 2t + b \sin 2t$

$y' = -2a \sin 2t + 2b \cos 2t$

$y'' = -4a \cos 2t - 4b \sin 2t$

$y'' - 2y' + 2y = (-4a - 4b + 2a)\cos 2t + (-4b + 4a + 2b)\sin 2t$

$= (-2a - 4b)\cos 2t + (4a - 2b)\sin 2t$

So $\begin{cases} 2a + 4b = -4 \\ 4a - 2b = 0 \end{cases} \Rightarrow \begin{array}{l} a = -2/5 \\ b = -4/5 \end{array}$

General sol^n is: $y(t) = c_1 e^t \cos t + c_2 e^t \sin t - \frac{1}{5}(2\cos 2t + 4 \sin 2t)$

**(3)** (30 points) Consider the differential equation

$$x^2y'' - 3xy' + 4y = 0$$

(a) (5 points) Show that $y_1(x) = x^2$ is a solution.

(b) (10 points) Suppose that $y_2(x)$ is another solution which satisfies $y_2(1) = 0$ and $y_2'(1) = 1$. Find the Wronskian $W(y_1, y_2)(x)$ as a function of $x$.

(c) (3 points) Does the Wronskian you found in (c) vanish somewhere? If so, how do you reconcile that with the fact that the Wronskian of two solutions of a given 2nd order linear ODE is either identically zero or nowhere vanishing?

(d) (12 points) Find a second solution of the given ODE.

(a) $x^2(2) - 3x(2x) + 4x^2 = 0.$

(b) $W(y_1, y_2) = C e^{+\int \frac{3}{x} dx}$ by Abel's Theorem

$= C x^3$

On the other hand. $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$.

At $x=1$, $W(y_1, y_2)(1) = y_1(1) y_2'(1) - y_2(1) y_1'(1)$

$= 1$

So $C = 1$. & $\boxed{W(y_1, y_2) = x^3}$

(c) $W(y_1, y_2)$ vanishes at $x=0$. But note that since the eqn is $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$.

$x=0$ is a point of discontinuity for the coefficient of $y'$ & $y$.

Thus the non-vanishing of $W(y_1, y_2)$ only holds when $x \neq 0$.

(d) One can use the method of reduction of order here, but I'll do it by a different way. From (b), we know

$y_1 y_2' - y_2 y_1' = W(y_1, y_2) = x^3$.

So $x^2 y_2' - 2x y_2 = x^3$, ie $y_2' - \frac{2}{x} y_2 = x$.

Introducing the integrating factor. $e^{-\int \frac{2}{x} dx} = x^{-2}$ we have

$(x^{-2} y_2)' = x^{-1}$. $\Rightarrow x^{-2} y_2 = \ln x$

$\Rightarrow \boxed{y_2 = x^2 \ln x}$