MATH. 20D, MIDTERM I

You have **45 minutes** for this exam. Please write legibly and show all working. **No calculators are allowed.** Write your name, ID number and your TA's name below.

Name:

ID Number:

TA's name:

(1) (30 points) Solve the following initial value problems:

(a) $ty' + y^2 = 0$, $y(1) = 1$.

(b) $(2y - \sin(t)) \cdot y' = 2t + y \cdot \cos(t)$, $y(0) = 1$.

(c) $y' = t^2 + 1$, $y(0) = 1$.

(a) $\frac{dy}{dt} = -\frac{y^2}{t} \Rightarrow -\frac{dy}{y^2} = \frac{dt}{t} \Rightarrow y^{-1} = \ln t + c$

$y(1) = 1 \Rightarrow c = 1$, so $\boxed{y = \frac{1}{1+\ln t}}$

(b) $(2y - \sin t)\,dy - (2t + y\cos t)\,dt = 0.$

This is exact, since $\frac{\partial}{\partial t}(2y - \sin t) = -\cos t = \frac{\partial}{\partial y}(-2t - y\cos t)$

So can find f st

$$\begin{cases} \frac{\partial f}{\partial y} = 2y - \sin t & - (1) \\ \frac{\partial f}{\partial t} = -2t - y\cos t & - (2) \end{cases}$$

$(1) \Rightarrow f = y^2 - y\sin t + g(t)$

so $\frac{\partial f}{\partial t} = -y\cos t + g'(t) = -y\cos t - 2t$

so $g'(t) = -2t$, ie $g(t) = -t^2$.

 ~~Pg. 2~~

So soln is $\boxed{y^2 - y\sin t - t^2 = c}$. $y(0) = 1 \Rightarrow \boxed{c = 1}$

(c) $\frac{dy}{dt} = t^2 + 1 \Rightarrow y = \frac{t^3}{3} + t + c$. $y(0) = 1 \Rightarrow \boxed{c = 1}$

(2) (30 points) A tank contains 1 gallon of water and 5 oz of salt. Water with a salt concentration of $1 + \sin(t) \cdot e^{-t/2}$ oz/gallon flows into the tank at a rate of 1/2 gallon per minute, and the mixture in the tank flows out at the same rate.

(a) Set up a differential equation which describes the change in the quantity $Q(t)$ of salt in the tank at time t.

(b) Solve the differential equation in (a) to find an explicit expression for $Q(t)$.

(c) How much salt remains in the tank after a long time?

(a) $\frac{dQ}{dt} = \frac{1}{2}(1 + \sin t\, e^{-\frac{t}{2}}) - \frac{1}{2}Q$, $Q(0) = 5$

(b) $\frac{dQ}{dt} + \frac{1}{2}Q = \frac{1}{2}(1 + \sin t\, e^{-t/2})$

Integrating factor is $e^{\frac{t}{2}}$.

So $\frac{d}{dt}(Q e^{t/2}) = \frac{1}{2}(e^{\frac{t}{2}} + \sin t)$

$\Rightarrow Q e^{\frac{t}{2}} = e^{\frac{t}{2}} - \frac{1}{2}\cos t + C$

$\Rightarrow Q(t) = 1 - \frac{1}{2}e^{-\frac{t}{2}}\cos t + C e^{-\frac{t}{2}}$

$Q(0) = 5 \Rightarrow C = 9/2$

So $Q(t) = 1 - \frac{1}{2}e^{-\frac{t}{2}}\cos t + \frac{9}{2}e^{-\frac{t}{2}}$

(c) As $t \to \infty$, $Q(t) \to 1$.

(3) (30 points) Consider the following differential equation

$$y'(t) = (y-2)^3 \cdot (e^y - 1)^2.$$

Answer the following questions without solving the differential equation. You should justify your answers.

(a) Find all the equilibrium solutions.

(b) Decide if the equilibrium solutions are stable, unstable or semi-stable.

(c) If $y(3) = \frac{3}{2}$, what is the value of $\lim_{t\to\infty} y(t)$?

(a) $y' = 0 \Rightarrow \boxed{y=2}$ or $e^y = 1$, ie $\boxed{y=0}$

(b) Since $y'(t) = (y-2)^3(e^y-1)^2$, & $(e^y-1)^2 \geq 0$

the sign of $y'(t)$ is the same as the sign of $(y-2)^3$.

So if $y > 2$, $y' > 0$

if $0 < y < 2$, $y' < 0$

if $y < 0$, $y' < 0$

So $y = 0$ is semi-stable

$y = 2$ is unstable.

(c) If $y(3) = 3/2$, ~~the~~ & $0 < 3/2 < 2$, $\cancel{y(t) \to}$

$y(t) \to 0$ as $t \to \infty$