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MATH 20D - Practice Exam #2
closed book, no calculators, no computers, no notes, ...
each problem is worth the same number of points

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1) a) Find an integrating factor u for $y' - xy = -x$.
   b) Solve the ODE in part a) by multiplying it by the integrating factor u and recognizing the left hand side as $(uy)'$.

2) Solve the separable ODE $y y' + x = 0$ with initial condition $y(2)=2$. Find the interval $a \le x \le b$ where the solution $y(x)$ is defined. If you were to plot the direction field, what would you expect to see?

3) A tank contains 100 gallons of salty water made by dissolving 80 lb. of salt in water. Pure water runs into the tank at the rate of 4 gal./min, and the well-stirred mixture runs out at the same rate. Let $Q(t)$ be the amount of salt in the tank at time t.
   a) Set up the ODE for this situation using $Q'(t)$ = rate in - rate out.
   b) Find the formula for $Q(t)$.
   c) Find the time required for half the salt to leave the tank. Don't compute the decimal approximation here.

4) True - False. Tell whether the following statements are true or false. Give a reason for your answer.
   a) The ODE $(y \cos x + 2x e^y) + (\sin x + x^2 e^y - 1)\, y'(x) = 0$ is exact.

   b) If f and $\dfrac{\partial f}{\partial y}$ are continuous in a rectangle $|x-x_0| \le a$, $|y-y_0| \le b$, then there is some interval on which the initial value problem
   $$y'(x) = f(x,y), \quad f(x_0)=y_0 ,$$
   cannot have 2 distinct solutions.

   c) The logistic equation $y'(x) = ry(1-y/K)$ has two equilibrium solutions $y = K$ and $y = 0$. The first of these is stable while the second is unstable.

5) a) Find a fundamental set of solutions for $y'' + y' - 2y = 0$.

   b) Compute the Wronskian determinant of the fundamental set from part a).

   c) Solve the initial value problem: $y'' + y' - 2y = 0$, $y(0)=2$, $y'(0) = 3$.

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Practice Exam Solutions

1) a) We want $\mu(x)$ so that

$$\mu y' - \mu x y = \mu y' + \mu' y$$

So we want $\frac{d\mu}{dx} = -x\mu$

$$\Rightarrow \frac{d\mu}{\mu} = -x\,dx$$

$$\Rightarrow \ln|\mu| = -\frac{x^2}{2} + c'$$

$$\mu = c\, e^{-x^2/2}$$

So take $\boxed{\mu(x) = e^{-x^2/2}}$

b) $$e^{-\frac{x^2}{2}} y' - e^{-\frac{x^2}{2}} x y = -x e^{-x^2/2}$$

$$\Rightarrow \left(e^{-\frac{x^2}{2}} y\right)' = -x e^{-x^2/2}$$

$$\Rightarrow e^{-\frac{x^2}{2}} y = -\int x e^{-x^2/2}\,dx + c$$, upon integrating

$$y(x) = -e^{x^2/2}\left\{\int x e^{-\frac{x^2}{2}}\,dx + c\right\}$$

$$= -e^{x^2/2}\left\{-e^{-\frac{x^2}{2}} + c'\right\}$$

$$\boxed{y = 1 + c e^{x^2/2}}$$

Check: $y' = c x e^{x^2/2}$

$$y' - xy = c x e^{x^2/2} - x\left(1 + c e^{x^2/2}\right)$$

$$= -x \checkmark$$

2) $y \frac{dy}{dx} + x = 0$, $y(2) = 2$

$\Rightarrow y\,dy = -x\,dx$

$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = c'$

$\Rightarrow x^2 + y^2 = c$, a circle centered at the origin

Here $c \geq 0$

$\Rightarrow y = \pm\sqrt{c - x^2}$

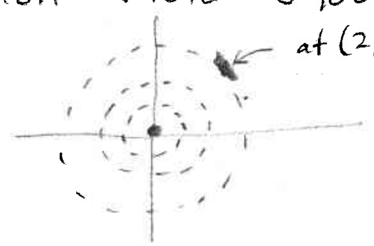
To fit the initial condition $x = 2 \rightarrow y = 2$

$4 + 4 = c \quad \Rightarrow \quad c = 8$

$y = +\sqrt{8 - x^2}$ (choose + since $y = 2 > 0$)

This is only defined if $-\sqrt{8} \leq x \leq \sqrt{8} = 2\sqrt{2}$

The direction field should circle about the origin



at (2,2) $\frac{dy}{dx} = -\frac{x}{y} = -1$

3) a) $Q'(t) = \left(\text{salt rate in}\right) - \left(\text{salt rate out}\right)$

$Q'(t) = 0 - 4 \cdot \frac{Q}{100}$

$\frac{dQ}{dt} = -.04\,Q$

b) $\frac{dQ}{Q} = -.04\,dt$

$\ln|Q| = -.04t + c'$ integrating

$\Rightarrow Q = C e^{-.04t}$ exponentiating

$Q(0) = C = 80 \Rightarrow Q(t) = 80e^{-.04t}$

3 c) $40 = Q(t) = 80e^{-.04t}$

$\frac{1}{2} = e^{-.04t}$

$\ln\frac{1}{2} = -\ln 2 = -.04t$

$t = \frac{\ln 2}{.04} = \frac{\ln 2}{\frac{4}{100}} = \frac{100}{4}\ln 2$

$= 25 \ln 2$ minutes

4) T-F

True a) $M = y\cos x + 2xe^y$, $N = \sin x + x^2e^y - 1$

$M_y = N_x$?

$M_y = \cos x + 2e^y = N_x$ ✓

True b) This is a consequence of the comforting existence and uniqueness theorem on page 144 of our text.

True c) $y' = ry\left(1 - \frac{y}{K}\right)$

Where is $y' = 0$? $y = 0$ & $y = K$

The solution is (see p. 83 of text)

$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$, $y_0 = y(0)$, $0 < y_0 < K$

$y(t) \longrightarrow K$ as $t \to \infty$

So near $y=0$ the solutions for $y>0$ increase. $\Rightarrow$ $y=0$ unstable

And near $y=K$ the solutions for $0 < y_0 < K$ increase to K. So $y_0 = K$ is stable

5) a) The corresponding quadratic

$$r^2 + r - 2 = (r+2)(r-1) = 0$$

$$r = -2, 1$$

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{t}$$

b) $$W(y_1, y_2)(t) = \det \begin{pmatrix} e^{-2t} & e^{t} \\ e^{-2t}(-2) & e^{t} \end{pmatrix}$$

$$= e^{-2t} e^{t} - e^{-2t}(-2) e^{t}$$

$$= e^{-t} + 2e^{-t} = 3e^{-t} \neq 0$$

c) $$y(t) = c_1 e^{-2t} + c_2 e^{t}$$

$$y'(t) = -2c_1 e^{-2t} + c_2 e^{t}$$

$$2 = y(0) = c_1 + c_2$$

$$3 = y'(0) = -2c_1 + c_2$$

Subtract

$$3c_1 = -1 \implies c_1 = -1/3$$

$$c_2 = 2 - c_1 = 2 + 1/3 = 7/3$$

$$\boxed{y(t) = -\frac{1}{3} e^{-2t} + \frac{7}{3} e^{t}}$$