$$s^2\,Y(s)+2s\,Y(s)+\,Y(s)-2s-3=\frac{4}{s+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s)=\frac{4}{(s+1)^3}+\frac{2s+3}{(s+1)^2}\,.$$

First write

$$\frac{2s+3}{(s+1)^2}=\frac{2(s+1)+1}{(s+1)^2}=\frac{2}{s+1}+\frac{1}{(s+1)^2}\,.$$

We note that

$$\mathcal{L}^{-1}\left[\frac{4}{\xi^3}+\frac{2}{\xi}+\frac{1}{\xi^2}\right]=2\,t^2+2+t\,.$$

So based on the *translation property* of the Laplace transform, the solution of the IVP is

$$y(t)=2\,t^2e^{-t}+t\,e^{-t}+2\,e^{-t}.$$

25. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s)-s\,y(0)-y'(0)+\,Y(s)=\mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2\,Y(s)+\,Y(s)=\mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}[f(t)]&=\int_0^\infty f(t)\,e^{-st}dt\\&=\int_0^1 t\,e^{-st}dt\\&=\frac{1}{s^2}-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}\,.\end{aligned}$$

Solving for the transform,

$$Y(s)=\frac{1}{s^2(s^2+1)}-e^{-s}\frac{s+1}{s^2(s^2+1)}\,.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}$$

and

$$\frac{s}{s^2(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

We find, by inspection, that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = t - sin\,t.$$

Referring to *Line* 13, in Table 6.2.1,

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Let

$$\mathcal{L}[\,g(t)\,] = \frac{s+1}{s^2(s^2+1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+1} - \frac{1}{s^2+1}.$$

Then $g(t) = 1 + t - cos\,t - sin\,t$. It follows, therefore, that

$$\mathcal{L}^{-1}\left[e^{-s}\cdot\frac{s+1}{s^2(s^2+1)}\right] = u_1(t)[\,1 + (t-1) - cos(t-1) - sin(t-1)\,].$$

Combining the above, the solution of the IVP is

$$y(t) = t - sin\,t - u_1(t)[\,1 + (t-1) - cos(t-1) - sin(t-1)\,].$$

26. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2\,Y(s) + \;4\,Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}[f(t)] &= \int_0^\infty f(t)\,e^{-st}dt \\ &= \int_0^1 t\,e^{-st}dt + \int_1^\infty e^{-st}dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}.\end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+4)} - e^{-s}\frac{1}{s^2(s^2+4)}.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right].$$

We find that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}\,t - \frac{1}{8}\,sin\,t\,.$$

Referring to *Line* 13 , in Table 6.2.1 ,

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

It follows that

$$\mathcal{L}^{-1}\left[e^{-s}\cdot\frac{1}{s^2(s^2+4)}\right] = u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\,sin(t-1)\right].$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{4}\,t - \frac{1}{8}\,sin\,t - u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\,sin(t-1)\right].$$

$28(a)$. Assuming that the conditions of Theorem 6.2.1 are satisfied,

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt \\ &= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st}f(t)\right]dt \\ &= \int_0^\infty \left[-t\,e^{-st}f(t)\right]dt \\ &= \int_0^\infty e^{-st}[-tf(t)]dt\,. \end{aligned}$$

(b). Using *mathematical induction*, suppose that for some $k \geq 1$,

$$F^{(k)}(s) = \int_0^\infty e^{-st}\left[(-t)^k f(t)\right]dt\,.$$

Differentiating both sides,

$$\begin{aligned} F^{(k+1)}(s) &= \frac{d}{ds}\int_0^\infty e^{-st}\Big[(-t)^k f(t)\Big]dt \\ &= \int_0^\infty \frac{\partial}{\partial s}\Big[e^{-st}(-t)^k f(t)\Big]dt \\ &= \int_0^\infty \Big[-t\,e^{-st}(-t)^k f(t)\Big]dt \\ &= \int_0^\infty e^{-st}\Big[(-t)^{k+1} f(t)\Big]dt\,. \end{aligned}$$

29. We know that

$$\mathcal{L}\big[e^{at}\big] = \frac{1}{s-a}\,.$$

Based on Prob. 28 ,

$$\mathcal{L}\big[-t\,e^{at}\big] = \frac{d}{ds}\left[\frac{1}{s-a}\right].$$

Therefore,

$$\mathcal{L}\big[t\,e^{at}\big] = \frac{1}{(s-a)^2}.$$

31. Based on Prob. 28 ,

$$\begin{aligned} \mathcal{L}[(-t)^n] &= \frac{d^n}{ds^n}\mathcal{L}[1] \\ &= \frac{d^n}{ds^n}\left[\frac{1}{s}\right]. \end{aligned}$$

Therefore,

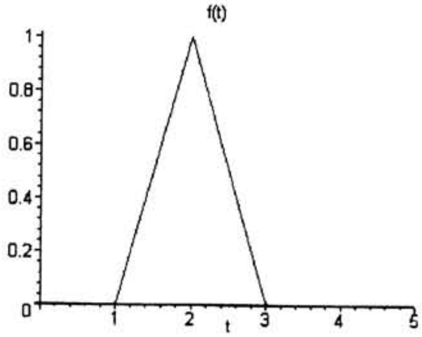
$$\begin{aligned} \mathcal{L}[t^n] &= (-1)^n\frac{(-1)^n n!}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}}\,. \end{aligned}$$

33. Using the *translation property* of the Laplace transform,

$$\mathcal{L}\big[e^{at}\,sin\,bt\big] = \frac{b}{(s-a)^2+b^2}\,.$$

Therefore,

6.



7. Using the Heaviside function, we can write

$$f(t) = (t-2)^2\, u_2(t)\,.$$

The Laplace transform has the property that

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Hence

$$\mathcal{L}\Big[\,(t-2)^2\, u_2(t)\Big] = \frac{2\,e^{-2s}}{s^2}.$$

9. The function can be expressed as

$$f(t) = (t-\pi)[u_\pi(t) - u_{2\pi}(t)].$$

Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t-\pi)\, u_\pi(t) - (t-2\pi)\, u_{2\pi}(t) - \pi\, u_{2\pi}(t)\,.$$

It follows that

$$\mathcal{L}[\,f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

10. It follows directly from the *translation property* of the transform that

$$\mathcal{L}[\,f(t)] = \frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s}.$$

11. Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t-2)\, u_2(t) - \, u_2(t) - (t-3)\, u_3(t) - u_3(t)\,.$$

It follows that

$$\mathcal{L}[f(t)] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

12. It follows directly from the *translation property* of the transform that

$$\mathcal{L}[f(t)] = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

13. Using the fact that $\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] = t^3 e^{2t}.$$

15. First consider the function

$$G(s) = \frac{2(s-1)}{s^2 - 2s + 2}.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2 + 1}.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = 2\, e^t \cos t.$$

Hence

$$\mathcal{L}^{-1}\left[e^{-2s} G(s)\right] = 2\, e^{(t-2)} \cos(t-2)\, u_2(t).$$

16. The *inverse transform* of the function $2/(s^2 - 4)$ is $f(t) = \sinh 2t$. Using the *translation property* of the transform,

$$\mathcal{L}^{-1}\left[\frac{2\, e^{-2s}}{s^2 - 4}\right] = \sinh 2(t-2) \cdot u_2(t).$$

17. First consider the function

$$G(s) = \frac{(s-2)}{s^2 - 4s + 3}.$$

Completing the square in the denominator,

$$G(s) = \frac{(s-2)}{(s-2)^2 - 1}\,.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = e^{2t}\,cosh\,t\,.$$

Hence

$$\mathcal{L}^{-1}\left[\frac{(s-2)e^{-s}}{s^2-4s+3}\right] = e^{2(t-1)}cosh\,(t-1)\,u_1(t)\,.$$

18. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}\,.$$

It follows from the *translation property* of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}\right] = u_1(t) + u_2(t) - u_3(t) - u_4(t)\,.$$

19(a). By definition of the Laplace transform,

$$\mathcal{L}[\,f(ct)] = \int_0^\infty e^{-st} f(ct)dt\,.$$

Making a change of variable, $\tau = ct$, we have

$$\begin{aligned}\mathcal{L}[\,f(ct)] &= \frac{1}{c}\int_0^\infty e^{-s(\tau/c)} f(\tau)d\tau \\ &= \frac{1}{c}\int_0^\infty e^{-(s/c)\tau} f(\tau)d\tau\,.\end{aligned}$$

Hence $\mathcal{L}[\,f(ct)] = \frac{1}{c}\,F\left(\frac{s}{c}\right)$, where $s/c > a$.

(b). Using the result in Part (a),

$$\mathcal{L}\left[f\left(\frac{t}{k}\right)\right] = k\,F(ks).$$

Hence

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k}\,f\left(\frac{t}{k}\right).$$

(c). From Part (b),

$$\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Note that $as + b = a(s + b/a)$. Using the fact that $\mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)]_{s \to s-c}$,

$$\mathcal{L}^{-1}[F(as + b)] = e^{-bt/a} \frac{1}{a} f\left(\frac{t}{a}\right).$$

20. First write

$$F(s) = \frac{n!}{\left(\frac{s}{2}\right)^{n+1}}.$$

Let $G(s) = n!/s^{n+1}$. Based on the results in Prob. 19,

$$\frac{1}{2}\mathcal{L}^{-1}\left[G\left(\frac{s}{2}\right)\right] = g(2t),$$

in which $g(t) = t^n$. Hence

$$\mathcal{L}^{-1}[F(s)] = 2\,(2t)^n = 2^{n+1} t^n.$$

23. First write

$$F(s) = \frac{e^{-4(s-1/2)}}{2(s-1/2)}.$$

Now consider

$$G(s) = \frac{e^{-2s}}{s}.$$

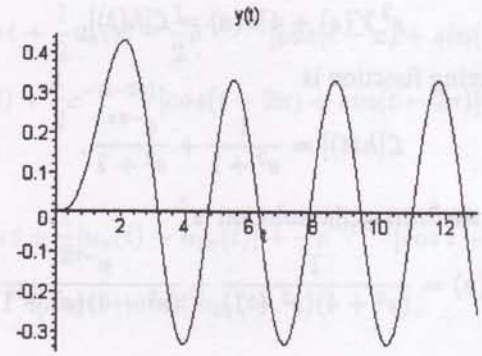
Using the result in Prob. $19(b)$,

$$\mathcal{L}^{-1}[G(2s)] = \frac{1}{2} g\left(\frac{t}{2}\right),$$

in which $g(t) = u_2(t)$. Hence $\mathcal{L}^{-1}[G(2s)] = \frac{1}{2}\, u_2(t/2) = \frac{1}{2}\, u_4(t)$. It follows that

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2} e^{t/2}\, u_4(t).$$

24. By definition of the Laplace transform,



Since there is no *damping term*, the solution follows the forcing function, after which the response is a steady oscillation about $y = 0$.

5. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2\,Y(s) = \mathcal{L}[f(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 3s\,Y(s) + 2\,Y(s) = \mathcal{L}[f(t)].$$

The transform of the forcing function is

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right].$$

Hence

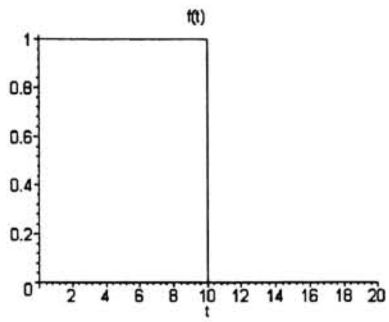
$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}.$$

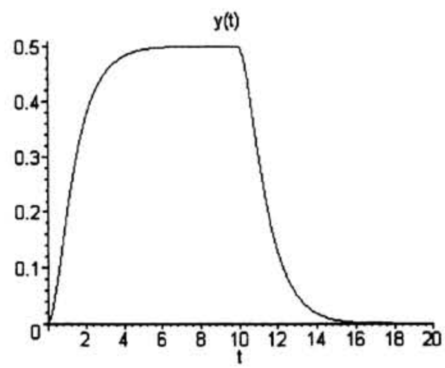
Based on Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \frac{1}{2}\left[1 + e^{-2(t-10)} - 2e^{-(t-10)}\right]u_{10}(t).$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{2}[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}\left[e^{-(2t-20)} - 2e^{-(t-10)}\right]u_{10}(t)\,.$$





The solution increases to a *temporary* steady value of $y = 1/2$. After the forcing ceases, the response decays exponentially to $y = 0$.

6. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - \,y(0)] + 2\,Y(s) = \frac{e^{-2s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - 1 = \frac{e^{-2s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2 + 3s + 2} + \frac{e^{-2s}}{s(s^2 + 3s + 2)}\,.$$

Using partial fractions,

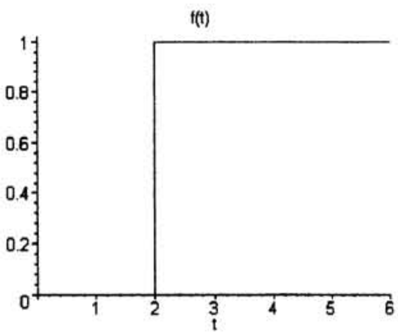
$$\frac{1}{s^2+3s+2}=\frac{1}{s+1}-\frac{1}{s+2}$$

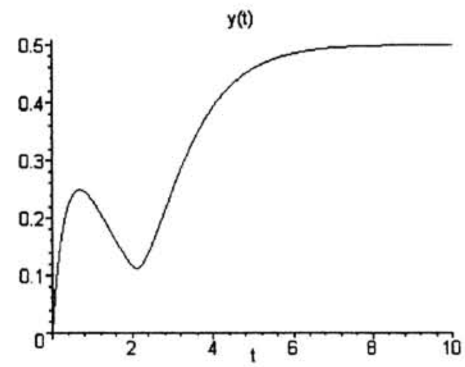
and

$$\frac{1}{s(s^2+3s+2)}=\frac{1}{2}\left[\frac{1}{s}+\frac{1}{s+2}-\frac{2}{s+1}\right].$$

Taking the inverse transform. term-by-term, the solution of the IVP is

$$y(t)=e^{-t}-e^{-2t}+\left[\frac{1}{2}-e^{-(t-2)}+\frac{1}{2}e^{-2(t-2)}\right]u_2(t)\,.$$





Due to the initial conditions, the response has a transient *overshoot*, followed by an exponential convergence to a steady value of $y_s=1/2$.

7. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s)-s\,y(0)-y'(0)+Y(s)=\frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2 Y(s) + Y(s) - s = \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{s^2+1} + \frac{e^{-3\pi s}}{s(s^2+1)}.$$

Using partial fractions,

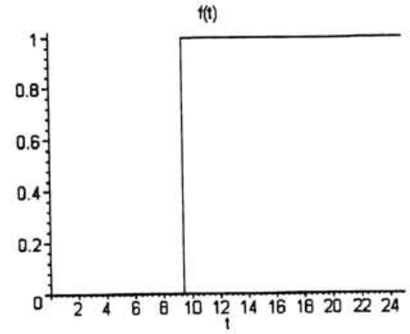
$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

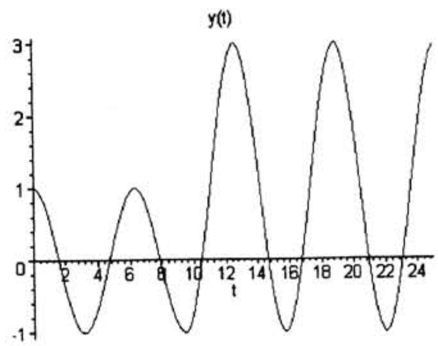
Hence

$$Y(s) = \frac{s}{s^2+1} + e^{-3\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right].$$

Taking the inverse transform, the solution of the IVP is

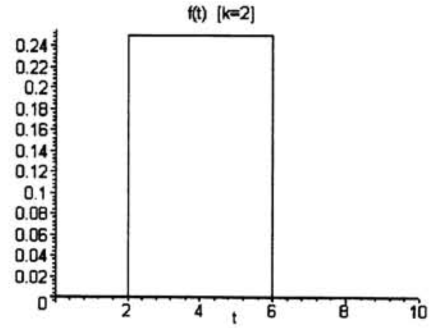
$$\begin{aligned} y(t) &= cos\, t + [1 - cos(t - 3\pi)]u_{3\pi}(t) \\ &= cos\, t + [1 + cos\, t]u_{3\pi}(t). \end{aligned}$$





Due to initial conditions, the solution temporarily oscillates about $y = 0$. After the forcing is applied, the response is a steady oscillation about $y_m = 1$.

$18(a)$.



(b). The forcing function can be expressed as

$$f_k(t) = \frac{1}{2k}[u_{4-k}(t) - u_{4+k}(t)].$$

Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \frac{1}{3}[s\,Y(s) - \,y(0)] + 4\,Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Applying the initial conditions,

$$s^2\,Y(s) + \frac{1}{3}s\,Y(s) + 4\,Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Solving for the transform,

$$Y(s) = \frac{3\,e^{-(4-k)s}}{2ks(3s^2+s+12)} - \frac{3\,e^{-(4+k)s}}{2ks(3s^2+s+12)}.$$

Using partial fractions,

$$\begin{aligned}\frac{1}{s(3s^2+s+12)} &= \frac{1}{12}\left[\frac{1}{s} - \frac{1+3s}{3s^2+s+12}\right] \\ &= \frac{1}{12}\left[\frac{1}{s} - \frac{1}{6}\frac{1+6\left(s+\frac{1}{6}\right)}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right].\end{aligned}$$

Let

$$H(s) = \frac{1}{8k}\left[\frac{1}{s} - \frac{\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right].$$

It follows that

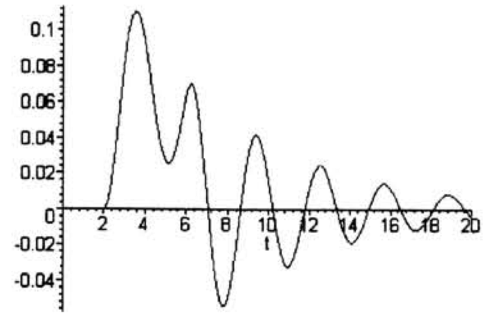
$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{1}{8k} - \frac{e^{-t/6}}{8k}\left[\frac{1}{\sqrt{143}} sin\left(\frac{\sqrt{143}\,t}{6}\right) + cos\left(\frac{\sqrt{143}\,t}{6}\right)\right].$$

Based on Theorem 6.3.1 , the solution of the IVP is

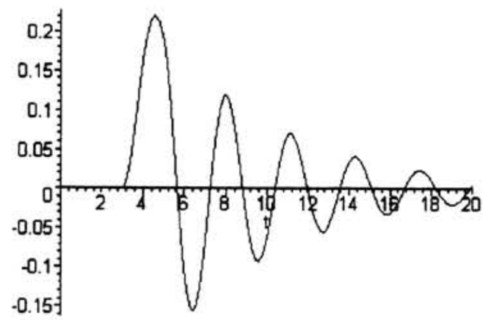
$$y(t) = h(t - 4 + k)\, u_{4-k}(t) - h(t - 4 - k)\, u_{4+k}(t)\,.$$

(c).

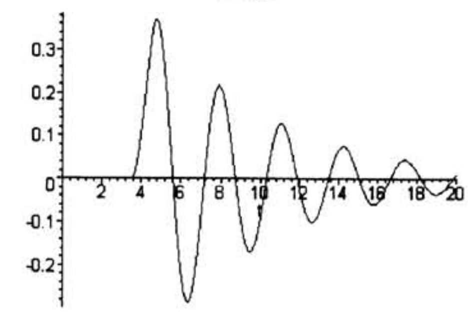
k = 2



k = 1



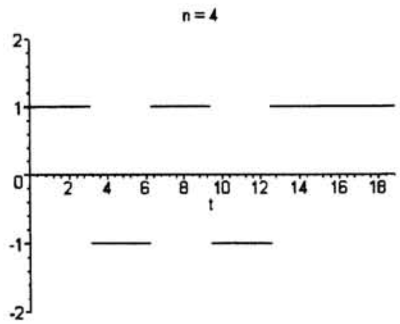
k = 1/2



As the parameter k decreases, the solution remains *null* for a longer period of time.

Since the *magnitude* of the impulsive force *increases*, the initial *overshoot* of the response also increases. The *duration* of the impulse decreases. All solutions eventually decay to $y = 0$.

19(a).



(c). From Part (b),

$$u(t) = 1 - cos\, t + 2\sum_{k=1}^{n}(-1)^k[1 - cos(t - k\pi)]u_{k\pi}(t)\,.$$