MATH. 20D, SAMPLE FINAL

You have **3 hours** for this exam. Please write legibly and show all working. **No calculators are allowed.** Write your name, ID number and your TA's name below. The total number of points for this exam is 180.

Name:

ID Number:

TA's name:

(1) (40 points) Solve the following initial value problems:

(a) $(4y - x) \cdot y' = 9x^2 + y - 1$, $y(1) = 0$.

(b) $(2y - 5) \cdot y' = 3x^2 - e^x$, $y(0) = 1$.

(c) $xy' + 2y = \sin t$, $y(\pi/2) = 1$.

(d) $y'' - 6y' + 9y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$.

(2a) (15 points) Consider the differential equation

$$y'' - 2xy' + \lambda y = 0.$$

Find the first 4 terms in the Taylor expansion of 2 linearly independent solutions.

(b) (10 points) Find 2 linearly independent power series solutions.

(c) (10 points) When λ is a non-negative even integer, show that there is a polynomial solution.

(3a) (10 points) Suppose that y_1 and y_2 are two solutions of

$$ty'' - 2y' + (3 + t)y = 0,$$

satisfying the initial conditions

$$y_1(1) = 1 \quad \text{and} \quad y_1'(1) = 0$$

and

$$y_2(1) = 3 \quad \text{and} \quad y_2'(1) = 2.$$

Show that y_1 and y_2 form a set of fundamental solutions (i.e. that they are linearly independent).

(b) (15 points) Find the Wronskian of y_1 and y_2.

(4a) (15 points) Express the Laplace transform of the following functions in terms of the Laplace transform of f:

(i) $g(t) = f'(t)$.

(ii) $g(t) = f(ct)$ where c is a positive constant.

(iii) $g(t) = e^{\lambda t} \cdot f(t)$.

(b) (20 points) Use the method of Laplace transform to find the solution of

$$y'' + 3y' + 2y = f(t) \quad y(0) = 0, \quad y'(0) = 0 \quad (t \geq 0)$$

where

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 10; \\ 0, & \text{if } t \geq 0. \end{cases}$$

(5) Consider the following system of differnetial equation $x'(t) = Ax(t)$ with

$$A = \begin{pmatrix} 2 & \alpha \\ 1 & 1 \end{pmatrix}.$$

(a) (15 points) Solve the system of equations when $\alpha = 2$, given the initial condition $x(0) = (1, 2)^t$.

(b) (15 points) Do the same for $\alpha = -1/4$.

(c) (15 points) For general α, sketch some trajectories in the phase plane which shows what happens in the long run, taking care to distinguish different cases for different values of α.