

MATH. 20D, MIDTERM 3

You have 45 minutes for this exam. Please write legibly and show all working. No calculators are allowed. Write your name, ID number and your TA's name below.

Name: Solutions

ID Number:

TA's name:

(1) Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) (10 points) Calculate the determinant of A and find its inverse.

(b) (20 points) Find the eigenvalues and eigenvectors of A .

(c) (25 points) Consider the differential equation:

$$x'(t) = A \cdot x(t).$$

Find the general solution.

$$(a) \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} = (1)(2)(1) + 0 + 0 - 0 - 0 - 0 = \boxed{2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R2/2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R1-R2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1/2 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\begin{array}{l} R1+R3 \\ R2-2R3 \end{array} \xrightarrow{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & 1 \\ 0 & 1 & 0 & 0 & 1/2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \Rightarrow \boxed{A^{-1} = \begin{pmatrix} 1 & -1/2 & 1 \\ 0 & 1/2 & -2 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$(b) \text{ eigenvalues: } \det(A - rI) = \begin{vmatrix} (1-r) & 1 & 1 \\ 0 & (2-r) & 4 \\ 0 & 0 & (1-r) \end{vmatrix} = (1-r)^2(2-r)$$

so $\boxed{r=2}$ and $\boxed{r=1}$ (with multiplicity 2)

$$\boxed{r=2} \quad (A-2I)x=0$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{So } \begin{cases} -x_1 + x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow X^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

$$\boxed{r=1} \quad (A-I)x=0$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{So } \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow X^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t$$

Another solution:

$$X^{(3)} = te^t \vec{v} + e^t \vec{\eta}$$

$$X^{(3)'} = e^t \vec{v} + te^t \vec{v} + e^t \vec{\eta}$$

$$X' = AX \Rightarrow e^t \vec{v} + te^t \vec{v} + e^t \vec{\eta} = Ate^t \vec{v} + Ae^t \vec{\eta}$$

$$\Rightarrow A\vec{v} = \vec{v} \quad \text{and} \quad A\vec{\eta} = \vec{v} + \vec{\eta}$$

$$\boxed{\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$(A-I)\vec{\eta} = \vec{v}$$

$$\text{solve } \begin{pmatrix} 0 & 1 & 1 & | & 1 \\ 0 & 0 & 4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & | & 1 \\ 0 & 0 & 3 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & -1/3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & | & 4/3 \\ 0 & 0 & 1 & | & -1/3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow X^{(3)} = te^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^t \left(k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4/3 \\ -1/3 \end{pmatrix} \right) \Rightarrow \vec{\eta} = \begin{pmatrix} k \\ 4/3 \\ -1/3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4/3 \\ -1/3 \end{pmatrix}$$

$$\Rightarrow X = c_1 X^{(1)} + c_2 X^{(2)} + c_3 X^{(3)} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_3 \left[te^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 4/3 \\ -1/3 \end{pmatrix} \right]$$

(2) Consider the differential equation

$$x'(t) = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix} x(t).$$

(a) (5 points) Find the eigenvalues of the coefficient matrix.

(b) (20 points) Classify the type of equilibrium point at the origin of the phase plane (i.e. whether it is stable, unstable or a saddle point), taking care to distinguish different cases based on the range of values of α . You should justify your answers.

(c) (10 points) For each range of values of α you described in (b), sketch a few trajectories in the phase plane.

$$(a) \det(A - \lambda r) = \begin{vmatrix} 1-r & \alpha \\ 1 & 1-r \end{vmatrix} = (1-r)^2 - \alpha = r^2 - 2r + (1-\alpha) = 0$$

$$\Rightarrow r = \frac{2 \pm \sqrt{4 - 4(1-\alpha)}}{2} = \frac{2 \pm \sqrt{4\alpha}}{2} = \boxed{1 \pm \sqrt{\alpha}}$$

(b) $\boxed{\alpha < 0}$ complex eigenvalues \Rightarrow spiral
positive real part \Rightarrow unstable

$\boxed{0 < \alpha < 1}$ both eigenvalues positive \Rightarrow node
since positive \Rightarrow unstable

$\boxed{\alpha > 1}$ eigenvalues real with opposite signs
 \Rightarrow saddle point (always unstable)

