MATH. 20F MIDTERM 2

You have **50 minutes** for this exam. Please write legibly and show all working. **No calculators are allowed.** Write your name and ID number.

Name:

ID Number:

(1i) (6 points) Suppose that B is a matrix to which the following sequence of ERO's is performed:

(a) Exchange rows 1 and 3;
(b) Add 3 times of Row 2 to Row 3;
(c) Multiply row 1 by 2,

yielding the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 9 & 3 & 1 \end{pmatrix}.$$

Find $\det B$.

ERO (a) multiplies det by -1; ERO (b) does not change det, and ERO (c) multiplies det by 2. Hence, $\det C) = -2 \cdot \det B$. On the other hand, $\det B = -1$, so $\det C = 1/2$.

(ii) (4 points) If an $n \times n$ matrix A satisfies $A^T \cdot A = I_n$, what can you say about $\det A$?

We have:

$$1 = \det I_n = \det(A^T A) = \det(A^T) \cdot \det A = (\det A)^2.$$

So

$$\det A = \pm 1.$$

(2) Let $V = M_{3\times 3}$ be the vector space of 3×3 matrices. Let

$$W_1 = \{A \in V : A \text{ is invertible}\} \quad \text{and} \quad W_2 = \{A \in V : A^T = A\}$$

(i) (10 points) Decide if W_1 and W_2 are subspaces of V. Justify your answer.

W_1 is not a subspace since it does not contain the zero matrix. For W_2, note that the zero matrix is is W_2. Moreover, suppose that A and B are in W_2. Then

$$(A+B)^T = A^T + B^T = A + B$$

and

$$(\lambda A)^T = \lambda \cdot A^T = \lambda A.$$

So W_2 is closed under addition and scalar multiplication. In particular, W_2 is a subspace.

(ii) (10 points) Let $T = V \longrightarrow V$ be the function defined by

$$T(A) = A^T - A.$$

Show that T is a linear transformation and find the dimension of its kernel.

We have:

$$T(A+B) = (A+B)^T - (A+B) = (A^T - A) + (B^T - B) = T(A) + T(B)$$

and

$$T(\lambda \cdot A) = (\lambda A)^T - \lambda A = \lambda \cdot (A^T - A) = \lambda \cdot T(A).$$

Hence T is linear.

The kernel of T is the subspace W_2 from (i). A generic matrix in W_2 has the form

$$\begin{pmatrix} a & x & y \\ x & b & z \\ y & z & c \end{pmatrix}$$

This shows that a basis of W_2 is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

So $\dim W_2 = 6$.

(3) (10 points) Find bases for the null space, column space and row space of the following matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Performing ERO's, we see that

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

So a generic element in the nullspace has the form

$$(0, \alpha, 0, 0)^T = \alpha \cdot (0, 1, 0, 0)^T.$$

So a basis for the null space of A is $\{(0, 1, 0, 0)^T)\}$.

A basis for the row space is

$$\{(1, 0, 0, 1)^T, (0, 0, 1, -2)^T, (0, 0, 0, -1)^T\}$$

A basis for the column space is

$$\{(1, 2, 0)^T, (0, 1, -1)^T, (1, 0, 1)^T)\}.$$

(4i) (5 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2, and consider the basis

$$\mathcal{B} = \{1, 2t, 4t^2\}$$

of $\mathbb{P}_2$. Find the coordinate vector $[p]_{\mathcal{B}}$ of $p(t) = 1 + t + 2t^2$ with respect to $\mathcal{B}$.

Since

$$p(t) = 1 + \frac{1}{2} \cdot 2t + \frac{1}{2} \cdot 2t^2,$$

we have

$$[p]_{\mathcal{B}} = (1, 1/2, 1/2)^T.$$

(ii) (5 points) If $\mathcal{C} = \{t^2, 1, t+1\}$ is another basis of $\mathbb{P}_2$, find the matrix P such that

$$[p]_{\mathcal{B}} = P \cdot [p]_{\mathcal{C}}.$$

The matrix P has columns equal to $[t^2]_{\mathcal{B}}$, $[1]_{\mathcal{B}}$ and $[t+1]_{\mathcal{B}}$. We have

$$[t^2]_{\mathcal{B}} = (0, 0, 1/4)^T,$$
$$[1]_{\mathcal{B}} = (1, 0, 0,)^T$$
$$[t+1]_{\mathcal{B}} = (1, 1/2, 0)^T$$

Hence

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1/2 \\ 1/4 & 0 & 0 \end{pmatrix}$$