MATH. 20F SAMPLE MIDTERM 1 SOLUTIONS

You have **50 minutes** for this exam. Please write legibly and show all working. **No calculators are allowed.** Write your name and ID number.

Name:

ID Number:

(1) Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & a & 0 \\ 2 & 1 & 3 \end{pmatrix}.$$

(i) (8 points) Find all a's such that the homogeneous system $Ax = 0$ has non-trivial solutions.

After performing Gaussian elimination on A, one obtains the row echelon form

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & a-5 \end{pmatrix}.$$

It follows that there is a free variable if and only if $a = 5$. Hence, $Ax = 0$ has infinitely many solutions if and only if $a = 5$.

(ii) (7 points) For those values of a from (i), find the general solution of $Ax = 0$.

Taking $a = 5$ in the row echelon form from (i), we see that x_3 is a free variable. Setting $x_3 = \alpha$, we see that

$$x_2 = \frac{1}{3} \cdot \alpha \quad \text{and} \quad x_1 = -\frac{5}{3} \cdot \alpha.$$

So the general solution is

$$x = \alpha \cdot (-\frac{5}{3}, \frac{1}{3}, 1)^t.$$

(2i) (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If x and y in $\mathbb{R}^n$ satisfies

$$T(x) = 0 = T(y),$$

show that $T(x + y) = 0 = T(\lambda x)$ for any scalar λ.

We have:

$$T(x + y) = T(x) + T(y) = 0 + 0 = 0$$

and

$$T(\lambda x) = \lambda \cdot T(x) = \lambda \cdot 0 = 0.$$

(ii) (5 points) If T is one-to-one, show that $n \leq m$.

Suppose that $T(x) = A \cdot x$ for some $m \times n$ matrix A. Then T is one-to-one if and only if the homogeneous equation $Ax = 0$ has $x = 0$ as unique solution. This means that there are no free variables, or equivalently that every variable is basic. Since there are at most m pivots (since A has m rows) and exactly n variables (since A has n columns), we must have $n \leq m$.

(3) (15 points) Find the inverse of the matrtix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix},$$

and express A as a product of elementary matrices.

Perform Gaussian elimination on $(A|I_3)$ to reduce A to the identity matirx I_3:

$$A = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R2 \leftrightarrow R3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R2 \to R2 - 2 \cdot R1]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R3 \to \frac{1}{2} R3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{array}\right)$$

$$\xrightarrow[R2 \to R2 + R3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{array}\right)$$

$$\xrightarrow[R1 \to R1 - R3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -2 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{array}\right)$$

So

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -2 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Further if the elementary matrices corresponding to the above ERO's are E_1, E_2,...,E_5 respectively, then

$$A^{-1} = E_5 \cdot E_4 \cdot E_3 \cdot E_2 \cdot E_1$$

So

$$A = (E_5 \cdot E_4 \cdot E_3 \cdot E_2 \cdot E_1)^{-1} = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} \cdot E_4^{-1} \cdot E_5^{-1}.$$

Now the inverse of an elementary matrix associated to an ERO is the elementary matrix associated to the reverse ERO. SO we have

$$E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_5^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(4) (10 points) Decide if the following statements are true or false, giving justifications to your answers.

(a) If A is an $n \times n$ matrix such that $A^2 = A \cdot A = 0$, then $A = 0$.

False. Consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $A^2 = 0$ but $A \neq 0$.

(b) Any collection of 3 vectors in $\mathbb{R}^3$ such that none is a multiple of another must be linearly independent.

False. Consider the vectors

$$e_1 = (1,0,0)^t, \quad e_2 = (0,1,0)^t, \quad v = e_1 + e_2 = (1,1,0)^t$$

They are linearly dependent, since $v = e_1 + e_2$, but nno one is a multiple of another.