MATH. 20F SAMPLE MIDTERM 2

You have **50 minutes** for this exam. Please write legibly and show all working. **No calculators are allowed.** Write your name and ID number.

Name:

ID Number:

(1i) (5 points) Find the determinant of

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 11 & 3 & 7 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Calculate the determinant by using cofactor expansion along the 2nd column and then the 3rd column:

$$\det A = -3 \cdot \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} = -3 \cdot 1 \cdot \det \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = (-3) \cdot 1 \cdot (-2) = 6$$

(ii) (5 points) If A is an $n \times n$ matrix, how are $\det(3A)$ and $\det(A)$ related?

The matrix $3A$ is obtained from A by multiplying each row by 3. Each such ERO multiplies the determinant by 3. Hence

$$\det(3A) = 3^n \cdot \det(A).$$

(2i) (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2, and let

$$W = \{p \in \mathbb{P}_2 : p(0) + p'(1) = 0\}.$$

Show that W is a subspace of $\mathbb{P}_2$ and find a basis for W.

It is clear that the zero polynomial is in W. If p and q are in W, then

$$(p+q)(0) + (p+q)'(1) = p(0) + q(0) + p'(1) + q'(1) = 0,$$

since we are assuming that $p(0)+p'(1) = 0 = q(0)+q'(1)$. Thus, W is closed under addition. Further, if λ is a scalar, then

$$(\lambda p)(0) + (\lambda p)'(1) = \lambda \cdot (p(0) + p'(1)) = 0.$$

So W is closed under scalar multiplication. We have shown that W is a subspace.

Suppose that $p(t) = a_0 + a_1 t + a_2 t^2$, then P lies in W if and only if

$$0 = p(0) + p'(1) = a_0 + a_1 + 2a_2.$$

Thus $p(t)$ lies in W if and only if it has the form

$$p(t) = -(a_1 + 2a_2) + a_1 t + a_2 t^2 = a_1 \cdot (t-1) + a_2 \cdot (t^2 - 2).$$

Thus W is spanned by $t-1$ and t^2-2. It is clear that these two polynomials are linearly independent, so they form a basis of W.

(ii) (10 points) Consider the map $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(p) = p''(t) + p(0).$$

Show that T is linear and determine its range.

We have:

$$T(p+q) = (p+q)''(t) + (p+q)(0) = p''(t) + q''(t) + p(0) + q(0) = T(p) + T(q)$$

and

$$T(\lambda \cdot p) = (\lambda p)''(t) + (\lambda p)(0) = \lambda \cdot (p''(t) + p(0)) = \lambda \cdot T(p).$$

Hence T is linear.

If $p(t) = a_0 + a_1 t + a_2 t^2$, then

$$T(p) = 2a_2 + a_0.$$

Hence the range of T is the subspace of constant polynomials, i.e. of polynomial of degree 0.

(3) Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$. Consider the following two bases of $\mathbb{R}^2$:

$$\mathcal{B} = \{e_1 + e_2, 2e_2\} \quad \text{and} \quad \mathcal{C} = \{e_1 - e_2, e_1\}.$$

(i) (5 points) Find the change of cooridnate matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

Since

$$e_1 + e_2 = -(e_1 - e_2) + 2e_1$$

and

$$2e_2 = -2 \cdot (e_1 - e_2) + 2e_1$$

the desired matrix is

$$P = \begin{pmatrix} -1 & -2 \\ 2 & 2 \end{pmatrix}$$

(ii) (5 points) If a vector w is such that $[w]_{\mathcal{B}} = (1, 3)^T$, what is $[w]_{\mathcal{C}}$?

$$[w]_{\mathcal{C}} = P \cdot [w]_{\mathcal{B}} = (-7, 8)^T.$$

(4) (10 points) Decide if the following statements are true or false. Justify your answers.

(a) If A is a 6×7 matrix whose row space has dimension 2, then the null space of A has dimension 5.

True, by the rank-nullity theorem.

(b) If A and B are related by a sequence of ERO's, then the column space of A and B are equal.

False. Consider the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$