APPLIED ALGEBRA:
PROBLEM SHEET 5

(1) We showed in class that the ring of symmetric functions

$$\Lambda = \mathbb{Z}[e_1, e_2, ...] = \mathbb{Z}[h_1, h_2, ...].$$

On the other hand, for each n, we have a surjective projection map

$$\Lambda \longrightarrow \Lambda_n = \mathbb{Z}[x_1, ..., x_n]^{S_n}$$

given by setting $x_{n+1} = x_{n+2} = ... = 0$.

(i) By studying this projection map, show that

$$\Lambda_n = \mathbb{Z}[e_1, ..., e_n] = \mathbb{Z}[h_1, ..., h_n],$$

where, by abuse of notation, we write $e_1, .., e_n$ for the image of $e_1, ..., e_n$ under this map.

(ii) Determine the images of e_i and h_i for $i \geq n+1$, by expressing them as polynomials in the variables $e_1, .., e_n$ or $h_1, .., h_n$ respectively.

(2) We defined a ring automorphism ω of Λ by setting

$$\omega(e_i) = h_i.$$

(i) Show that $\omega(h_i) = e_i$.

(ii) Show that $\omega(p_k) = (-1)^{k-1} \cdot p_k$, and deduce that for any partition λ, $\omega(p_\lambda) = (-1)^{|\lambda|-l(\lambda)} \cdot p_\lambda$.

(3i) Show that $e_n = \det A$, where A is the $n \times n$ matrix whose (i,j)-th entry is h_{1-i+j}. Here, we interpret $h_0 = 1$ and $h_i = 0$ if $i < 0$.

(ii) Show that $h_n = \det B$ where B is the $n \times n$ matrix whose (i,j)-th entry is e_{i-i+j}.

(iii) Show that $p_n = \det C$ where

$$C = \begin{pmatrix} e_1 & 1 & 0 & 0 & ..... \\ 2e_2 & e_1 & 1 & 0 & .... \\ 3e_3 & e_2 & e_1 & 1 & 0.... \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

(4) We showed in class that e_n and h_n can be expressed as $\mathbb{Q}$-linear combinations of the p_λ's. By exploiting the identities of generating functions shown in class:

$$P(-t) = \frac{d}{dt} \log E(t) \quad \text{and} \quad P(t) = \frac{d}{dt} \log H(t),$$

show that

$$e_n = \sum_{\lambda:|\lambda|=n} \frac{(-1)^n}{z_\lambda} \cdot p_\lambda$$

and

$$h_n = \sum_{\lambda:|\lambda|=n} \frac{1}{z_\lambda} \cdot p_\lambda$$

where

$$z_\lambda = \prod_i i^{m_i} \cdot (m_i)! \quad \text{if } \lambda = (1^{m_1}, 2^{m_2}, ...,).$$

(5) Show that the coefficient of $x_1 x_2 ... x_n$ in $s_\lambda(x_1, ..., x_n)$ is

$$\frac{n!}{\prod_i (\lambda_i + n - i)!} \cdot \prod_{i<j} (\lambda_i - \lambda_i - i + j).$$

(6) Show that

$$a_\lambda(q^{n-1}, q^{n-2}, ...q, 1) = a_0(q^{\lambda_1+n-1}, q^{\lambda_2+n-2}, ..., q^{\lambda_n}).$$

(7) Compute the value of $s_\lambda(1, ...1)$

(8) The principal specialization of a symmetric function in the variables $\{x_1, .., x_n\}$ is obtained by replacing x_i by q^i for all i. Show that the Schur function specialization $s_\lambda(q, q^2, .., q^n)$ is the generating function for semistandard λ-tableaux with entries of size at most n.

(9) Express $s_\lambda \cdot e_1$ as a linear combination of Schur polynomials.