NUMBER FIELDS HW 1

(1) Which of the following are algebraic integers?

$$(\sqrt{3}+\sqrt{5})/2, \quad (\sqrt{3}+\sqrt{7})/2, \quad (1+\sqrt[3]{10}+\sqrt[3]{100})/3$$

(2) If a_i are algebraic integers and $\alpha \in \mathbb{C}$ satisfies:

$$\alpha^d + a_{d-1}\alpha^{d-1} + ... + a_0 = 0,$$

show that α is an algebraic integer.

(3) Let V is a finite diemsnional vector space over a field F, and let

$$\langle -, - \rangle : V \times V \longrightarrow F$$

be a nondegenerate bilinear form. Suppose that $\{v_i\}$ is a basis of V. Show that there is a unique basis $\{v_i^*\}$ of V so that $\langle v_i, v_j^* \rangle = \delta_{ij}$. The basis $\{v_i^*\}$ is called the **dual basis** to $\{v_i\}$ with respect to $\langle -, - \rangle$.

(4) Let $\mathcal{O}_f = \mathbb{Z} + \mathbb{Z} \cdot f\sqrt{D}$. Show that $\mathcal{O}_f$ is an order in $\mathbb{Q}(\sqrt{D})$. Compute $Disc(\mathcal{O}_f)$.

(5) If $\mathcal{O}$ is an order spanning a number field K, and $Disc(\mathcal{O})$ is squarefree. Must $\mathcal{O}$ be equal to the ring of integers $\mathcal{O}_K$?

(6) Prove Stickelberger's criterion: if K is a number field, then $Disc(K) = 0$ or 1 mod 4. (Hint: use the fact that if $\mathcal{O}_K = \oplus_i \mathbb{Z}\alpha_i$, then $Disc(K) = |\det(\sigma_i(\alpha_j))^2|$; write the determinant as $P - Q$, where P is the sum over even permutations and Q is the sum over odd permutations, and use $(P-Q)^2 = (P+Q)^2 - 4PQ$).

(7) Let d be a cubefree integer not divisible by 3 and let $K = \mathbb{Q}(\sqrt[3]{d})$.

(i) Show that $\mathbb{Z}[\sqrt[3]{d}]$ has discriminant $27d^2$.

(ii) If p is a prime factor of d and $\alpha = (a + b\sqrt[3]{d} + c\sqrt[3]{d^2})/p$ lies in $\mathcal{O}_K$ (with a, b, c rational), show by taking traces that $p|a$. Then show by squaring that $\alpha \in \mathbb{Z}[\sqrt[3]{d}]$.

(iii) Deduce that either $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{d}]$ or that $\mathcal{O}_K$ is spanned by $\mathbb{Z}[\sqrt[3]{d}]$ and some

$$\beta = (a + b\sqrt[3]{d} + \sqrt[3]{d^2})/3.$$

In the seocnd case, show by squaring that $a \equiv 1$ mod 3 and $b \equiv d$ mod 3, so that we may take β to be

$$\beta_0 = (1 + bd\sqrt[3]{d} + \sqrt[3]{d^2})/3.$$

(iv) Show that if $d \neq \pm 1$ mod 9, then $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{d}]$, and otherwise $\mathcal{O}_K$ is generated by $\mathbb{Z}[\sqrt[3]{d}]$ and β_0.

(8) Do (the very long) Exercises 29 and 42 from Chapter 2 of D. Marcus's "Number Fields". These exercises determine the ring of integers of a biquadratic extension $\mathbb{Q}(\sqrt{m}, \sqrt{n})$.

(9) Do Ex. 30 from Marcus. This exercise shows that the ring of inetegrs $\mathcal{O}_K$ is not always of the form $\mathbb{Z}[\alpha]$.