Appendix G

Related Results

G.1 Girsanov Transformation

The following result is frequently referred to as the Girsanov transformation or Girsanov theorem. The prototype of this formula for one-dimensional Brownian motion was developed by Cameron and Martin [6, 7, 8]. Subsequently, Girsanov [18] generalized their transformation to a multi-dimensional Brownian motion and he used the modern notation and terminology of Itô's theory of stochastic integration. For a proof of this result, see Chung and Williams [11], Section 9.4, Karatzas and Shreve [27], Section 3.5, or Revuz and Yor [35], Chapter VIII.

Suppose that W is an n-dimensional Brownian motion (for some $n \geq 1$) defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfying the usual conditions such that $\{W_t, \mathcal{F}_t, t \in [0,T]\}$ is a continuous n-dimensional martingale. Assume without loss of generality that $\mathcal{F} = \mathcal{F}_T$. Let $\kappa = \{\kappa_t, t \in [0,T]\}$ be an n-dimensional, adapted process such that $\kappa : [0,T] \times \Omega \to \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\kappa(t,\omega) = \kappa_t(\omega)$ for all $t \in [0,T]$ and $\omega \in \Omega$, and suppose that $\int_0^T |\kappa_t|^2 \, dt < \infty$ P-a.s. Then the stochastic integral process

$$\int_0^t \kappa_s \cdot dW_s = \sum_{i=1}^n \int_0^t \kappa_s^i \, dW_s^i$$

is well defined as a continuous local martingale. The integral $\int_0^t |\kappa_s|^2 \, ds$, is well-defined for all $t \in [0,T]$, P-a.s. As explained in the remark in Section F, by defining the integrals to be zero on an exceptional P-null set, we may assume that they define a continuous adapted process. Define

$$\Lambda_t = \exp\left(\int_0^t \kappa_s \cdot dW_s - \frac{1}{2} \int_0^t |\kappa_s|^2 \, ds \right), \quad t \in [0,T].$$

Then, one can show using Itô's formula that $\Lambda = \{\Lambda_t, t \in [0,T]\}$ is a continuous local martingale satisfying P-a.s.,

$$\Lambda_t = 1 + \int_0^t \Lambda_s \, \kappa_s \cdot dW_s, \quad t \in [0,T].$$

Define

$$\tilde{W}_t = W_t - \int_0^t \kappa_s \, ds, \quad t \in [0,T].$$

For the following theorem, recall that we are assuming that $\mathcal{F} = \mathcal{F}_T$.

Theorem G.1.1 *(Girsanov's theorem)*
Suppose that $\Lambda = \{\Lambda_t, t \in [0,T]\}$ is a martingale. Define a new probability measure Q on $(\Omega, \mathcal{F})$ by

$$Q(A) = E^P[1_A \Lambda_T] \quad \textit{for all} \quad A \in \mathcal{F}.$$

Then $\tilde{W}$ is a standard n-dimensional Brownian motion under Q.

Remark. In fact, Λ will be a martingale if

$$E^P[\Lambda_T] = 1.$$

A sufficient condition for this is what is known as *Novikov's criterion:*

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T |\kappa_t|^2 \, dt\right)\right] < \infty.$$

For a proof of this sufficiency, see Karatzas and Shreve [27], Corollary 3.5.13, or Revuz and Yor [35], Proposition VIII.1.15.

G.2 Martingale Representation Theorem

Suppose that $W = \{W_t, t \in [0,T]\}$ defined on the complete probability space $(\Omega, \mathcal{F}, P)$ is a standard n-dimensional Brownian motion (for some $n \geq 1$) and let $\{\mathcal{F}_t, t \in [0,T]\}$ be its standard filtration. Without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. The following result will be used several times in the chapters on continuous models. It is important for this result that the filtration is the standard one generated by the Brownian motion. For a proof of this theorem, see for example, Revuz and Yor [35], Theorem V.3.4.

Theorem G.2.1 *(Martingale Representation Theorem)*
Suppose that $\{M_t, \mathcal{F}_t, t \in [0,T]\}$ is a right continuous local martingale. There is an adapted, n-dimensional process $\eta = \{\eta_t, t \in [0,T]\}$ satisfying

(i) $\eta : [0,T] \times \Omega \to \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\eta(t,\omega) = \eta_t(\omega)$ for $t \in [0,T]$ and $\omega \in \Omega$,

(ii) $\int_0^T |\eta_s|^2 \, ds < \infty$ P-a.s.,

such that P-a.s.,

$$M_t = M_0 + \int_0^t \eta_s \cdot dW_s \quad \textit{for} \quad t \in [0,T].$$