Appendix F

Itô Process

Suppose that W is an n-dimensional Brownian motion (for some $n \geq 1$) defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfying the usual conditions such that $\{W_t, \mathcal{F}_t, t \in [0,T]\}$ is a continuous n-dimensional martingale. Assume without loss of generality that $\mathcal{F} = \mathcal{F}_T$.

An *Itô process* driven by the Brownian motion W is a d-dimensional (for some $d \geq 1$), continuous, adapted process $X = \{X_t, t \in [0,T]\}$ satisfying P-a.s.,

$$X_t = X_0 + \int_0^t R_s \, ds + \int_0^t Z_s \cdot dW_s \quad \text{for all } t \in [0,T], \tag{F.1}$$

where

$$\left(\int_0^t Z_s \cdot dW_s \right)^i = \sum_{j=1}^n \int_0^t Z_s^{ij} \, dW_s^j \quad \text{for } i = 1, \ldots, d, \tag{F.2}$$

and

(i) $X_0 \in \mathcal{F}_0$,

(ii) $R = \{R_t, t \in [0,T]\}$ is a d-dimensional, adapted process such that $R : [0,T] \times \Omega \to \mathbb{R}^d$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $R(t,\omega) = R_t(\omega)$ for each $t \in [0,T]$ and $\omega \in \Omega$, and $\int_0^T |R_s| \, ds < \infty$ P-a.s. where $|R_s| = \left(\sum_{i=1}^d (R_s^i)^2 \right)^{\frac{1}{2}}$ for $s \in [0,T]$,

(iii) $Z = \{Z_t, t \in [0,T]\}$ is an adapted process taking values in the set of $d \times n$ matrices with real entries (denoted by $\mathbb{R}^{d \times n}$), $Z : [0,T] \times \Omega \to \mathbb{R}^{d \times n}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $Z(t,\omega) = Z_t(\omega)$ for each $t \in [0,T]$ and $\omega \in \Omega$, and $\int_0^T |Z_s|^2 \, ds < \infty$ P-a.s. where $|Z_s| = \left(\sum_{i=1}^d \sum_{j=1}^n (Z_s^{ij})^2 \right)^{\frac{1}{2}}$ for $s \in [0,T]$.

Remark. Given the above assumptions on R, P-a.s., the integral $\int_0^t R_s \, ds$ is well defined pathwise and componentwise for all $t \in [0,T]$. For convenience, on the exceptional null set, we define these integrals for $t \in [0,T]$ to all be zero. It is known (cf. Chung and Williams [9], Theorem 3.7, Lemma 3.5(ii)), that there is a d-dimensional predictable process U such that

$$(\lambda \times P)(\{(t,\omega) \in [0,T] \times \Omega : R_t(\omega) \neq U_t(\omega)\}) = 0. \tag{F.3}$$

(A d-dimensional process $U = \{U_t, t \in [0,T]\}$ is *predictable* if $U : [0,T] \times \Omega \to \mathbb{R}^d$ given by $U(t,\omega) = U_t(\omega)$ for $t \in [0,T]$ and $\omega \in \Omega$, is measurable with the predictable σ-algebra on $[0,T] \times \Omega$ and the Borel σ-algebra on $\mathbb{R}^d$.) It follows (F.3) that the continuous, adapted process $\{\int_0^t U_s\, ds, t \in [0,T]\}$ is indistinguishable from $\{\int_0^t R_s\, ds, t \in [0,T]\}$, and so (since the filtration satisfies the usual conditions) the latter is also a continuous, adapted process. The integrals in (F.2) with respect to W are stochastic integrals as defined in Section E. By the assumptions on Z, these define continuous local martingales.

Sometimes for brevity we shall write the stochastic differential equation satisfied by an Itô process X in differential form:

$$dX_t = R_t\, dt + Z_t \cdot dW_t, \tag{F.4}$$

where the rigorous interpretation of this is as (F.1).

For fixed $i, j \in \{1, \ldots, d\}$, the *mutual variation process* $\langle X^i, X^j \rangle$ associated with the i^{th} and j^{th} components of the Itô process X is given P-a.s., by

$$\langle X^i, X^j \rangle_t = \int_0^t (ZZ')_s^{ij}\, ds = \sum_{k=1}^n \int_0^t Z_s^{ik} Z_s^{jk}\, ds, \quad t \in [0,T], \tag{F.5}$$

where $'$ denotes transpose. When $i = j$, this yields the *quadratic variation process* associated with the component X^i. This process is sometimes also denoted by $[X^i]$ and P-a.s. is given by

$$[X^i]_t = \sum_{k=1}^n \int_0^t (Z_s^{ik})^2\, ds, \quad t \in [0,T]. \tag{F.6}$$

F.1 Itô Formula

The following result is known as the *Itô formula.* For a proof of results that imply this, see Chung and Williams [11], Theorem 5.10. Let X be a d-dimensional Itô process as described in Section F above. Suppose that D is a domain (i.e., an open connected set) in $\mathbb{R}^d$ such that

$$P(X_t \in D \text{ for all } t \in [0,T]) = 1.$$

Let $f : [0,T] \times D \to \mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}$ exist and are continuous on $[0,T] \times D$. Then P-a.s. for all $t \in$

$[0, T]$,

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)\, R_s^i\, ds \\ &\quad + \sum_{i=1}^{d} \sum_{j=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)\, Z_s^{ij}\, dW_s^j \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s)(ZZ')_s^{ij}\, ds. \end{aligned}$$

Here the integrals with respect to dW are stochastic integrals and those with respect to ds are defined a.s. as Lebesgue integrals. Using the notation (F.4) and (F.5), it is often convenient to write the above as

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)\, dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s)\, d\langle X^i, X^j \rangle_s, \end{aligned}$$

or in differential form

$$\begin{aligned} df(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t)\, dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(t, X_t)\, dX_t^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)\, d\langle X^i, X^j \rangle_t. \end{aligned}$$

In the simple case when $d = 1$ and $f : \mathbb{R} \to D$ is twice continuously differentiable, the Itô formula reduces to

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\, R_s\, ds + \int_0^t f'(X_s)\, Z_s \cdot dW_s \\ &\quad + \frac{1}{2} \int_0^t f''(X_s)\, |Z_s|^2\, ds. \end{aligned}$$