MATH 286: Lecture 1

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1 Introduction

1.1 Ways to think about SDEs

Roughly, we can think of a stochastic differential equation (SDE) as an ordinary differential equation (ODE) with an added random perturbation in the dynamics. For example, if $X = \{X_t, t \geq 0\}$ is a real valued process describing the state of a system at each time t, a stochastic differential equation governs the time evolution of this process if X satisfies the following equation in some sense:

$$\frac{dX_t}{dt} = \mu(t, X_t) + \sigma(t, X_t)W_t, \quad t \in (0, \infty), \tag{1}$$

where $W = \{W_t, t \geq 0\}$ is so-called "white noise". In the equation (1), $\frac{dX_t}{dt}$ and W_t are yet to be defined rigorously. Now, we would "like" $\{W_t, t \geq 0\}$ to be a continuous time analog of a certain "sequence" of $i.i.d$ mean zero random variables having the following properties:

1. W_t is independent of W_s whenever $t \neq s$,
2. $E\{W(t)\} = 0$ for each $t \geq 0$,
3. stationarity: $(W_{t_1+t}, \ldots, W_{t_n+t})$ has the same distribution as $(W_{t_1}, \ldots, W_{t_n})$ for any t, $t_1 < t_2 < \ldots < t_n$ in $[0, \infty)$,
4. W_t is a generalized Gaussian random variable with variance $+\infty$, for each $t \geq 0$.

Unfortunately, there is no real valued stochastic process with these properties. If there were such a process, by extrapolating from the spectral theory for Gaussian processes, one might expect that, for each $t \geq 0$, the covariance function $\theta(t) = E\{W_0 W_t\} = \int_{\mathbb{R}} e^{i\lambda t} f(\lambda) d\lambda$, where f is the so-called "spectral density". The second and fourth properties above would imply that, for each $t > 0$, $\theta(t) = 0$, and $\theta(0) = +\infty$, which in turn would suggest that the spectral density f is constant. This constant density function is why W is called white noise (all of the frequencies are equally represented).

In order to circumvent the aforementioned difficulty, we will resort to an integral version of the equation (1):

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{2}$$

where $B = \{B_t, t \geq 0\}$ is a Brownian motion. Formally, we are thinking that

$$dB_t = W_t dt.$$

We know that this is only true in some rough intuitive sense because a Brownian motion is almost surely nowhere differentiable, i.e., for almost every realization of a Brownian motion path, $t \to B(t)$ is not differentiable for all t.

Another way to view SDEs, which is employed often by physicists, is to consider the distribution of a solution of an SDE at each time t. For example, under suitable conditions, a solution of (2) is a diffusion process (continuous strong Markov process), and then one may be interested in studying the process through functions of the form:

$$u(s, x) = E[f(X_t)|X_s = x], \text{ for } 0 < s < t < \infty, \ x \in \mathbb{R}.$$

Under certain "nice" conditions on the process X and the function f, the function $u(s, x)$ satisfies

$$0 = \frac{\partial u(s,x)}{\partial s} + \mu(s,x)\frac{\partial u(s,x)}{\partial x} + \frac{1}{2}\sigma^2(s,x)\frac{\partial^2 u(s,x)}{\partial x^2} \tag{3}$$

for each $0 < s < t$ and $x \in \mathbb{R}$. The parabolic partial differential equation (3) is referred to as a backward Kolomogorov equation (by mathematicians). There are related adjoint equations called variously Kolmogorov forward equations,

For us, an essential tool for analyzing SDEs will be Itô's formula. In particular, this will allow us to connect solutions of equations like (2) with partial differential equations like (3).

Yet another way to view SDEs is via their infinitesimal characteristics. (This is common in developing approximations to discrete models, e.g., those arising in population genetics.) Under mild conditions, the infinitesimal drift and variance associated with the solution of an SDE are given by

$$\mu(t,x) = \lim_{h\to 0} E\left[X(t+h) - X(t)|X(t) = x\right] \tag{4}$$

$$\sigma^2(t,x) = \lim_{h\to 0} E\left[(X(t+h) - X(t))^2|X(t) = x\right]. \tag{5}$$

The quantity $\sigma(t, x)$ is often refered to as the infinitesimal volatility or dispersion.

1.2 Applications

Example 1: Noisy Bonhoeffer-van der Pol oscillator. A noisy 2-dimensional simplification of the 4-dimensional Hodgkin-Huxley system for modeling the

firing of a single neuron. Questions of interest include qualitative behavior and the effect of increasing noise. See scanned transparencies.
Example 2: Black-Scholes model from mathematical finance. When X_t represents the value of a bond at time t, Y_t represents the value of a stock at time t, and $0 < r < \mu$, $\sigma > 0$, the Black-Scholes model proposes that X, Y satisfy

$$dX_t = rX_t dt$$

and

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t,$$

for $t \geq 0$. A solution of the second equation is called a geometric Brownian motion. These types of SDEs are often used in finance models related to problems of portfolio optimization and option pricing.