Appendix E

Stochastic Integrals (with respect to Brownian motion)

Suppose that W is a standard one-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t\}$ is a filtration such that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions and $\{W_t, \mathcal{F}_t, t \in [0,T]\}$ is a continuous martingale. (One could use the standard filtration associated with the Brownian motion for $\{\mathcal{F}_t\}$, however, we do not assume that a priori.) Without loss of generality, we assume that $\mathcal{F} = \mathcal{F}_T$.

Let $\mathcal{L}_s$ denote the set of one-dimensional processes $Y = \{Y_t, t \in [0,T]\}$ such that Y can be written in the simple form:

$$Y_t(\omega) = H_0(\omega)1_{\{0\}}(t) + \sum_{i=1}^{n-1} H_i(\omega)1_{(t_i,t_{i+1}]}(t) \quad \text{for all } t \in [0,T],\ \omega \in \Omega,$$

for some $0 = t_0 = t_1 < t_2 < \cdots < t_n = T < \infty$, $H_i : \Omega \to \mathbb{R}$ that is bounded and $\mathcal{F}_{t_i}$-measurable for $i = 0, 1, \ldots, n-1$. Note that for each fixed ω, as a function of t, Y is a step function. Also, the value of Y_0 depends only on the information available at time zero, and for each $i = 1, \ldots, n-1$, the value of the random variable Y_t for t in the interval $(t_i, t_{i+1}]$ only depends on the information available up to the time t_i. For these reasons Y is called a *simple predictable* function. (The σ-algebra induced on $[0,T] \times \Omega$ by the simple predictable processes is called the *predictable* σ-algebra.)

Define for each $\omega \in \Omega$ and $t \in [0,T]$,

$$\left(\int_0^t Y_s \, dW_s\right)(\omega) = \sum_{i=1}^{n-1} H_i(\omega)(W_{t\wedge t_{i+1}} - W_{t\wedge t_i})(\omega).$$

This family of stochastic integrals (one integral for each time t) is linear in $Y \in \mathcal{L}_s$ and has the following properties:

(i) $\left\{\int_0^t Y_s\, dW_s, \mathcal{F}_t, t \in [0,T]\right\}$ is a continuous L^2-martingale,

(ii) $E\left[\left(\int_0^t Y_s\, dW_s\right)^2\right] = E\left[\int_0^t Y_s^2\, ds\right]$ for all $t \in [0,T]$,

(iii) $E\left[\sup_{t\in[0,T]} \left(\int_0^t Y_s\, dW_s\right)^2\right] \le 4E\left[\int_0^T Y_s^2\, ds\right]$.

The first two properties follow by simple but tedious verification. Property (ii) is referred to as the L^2-isometry. The third property is a consequence of (i) and (ii) and Doob's inequality for continuous L^2-martingales.

Let $\mathcal{L}$ denote the set of all one-dimensional processes $Y = \{Y_t, t \in [0,T]\}$ such that Y is adapted, $Y : [0,T] \times \Omega \to \mathbb{R}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $Y(t,\omega) = Y_t(\omega)$ for all $t \in [0,T]$ and $\omega \in \Omega$, and

$$E\left[\int_0^T Y_s^2\, ds\right] < \infty.$$

Then $\mathcal{L} \subset L^2([0,T] \times \Omega, \mathcal{B}_T \times \mathcal{F}_T, \lambda \times P)$, where λ denotes Lebesgue measure on $[0,T]$. The L^2-norm of an element $Y \in \mathcal{L}$ is given by

$$\|Y\|_{\mathcal{L}} = \left(E\left[\int_0^T Y_s^2\, ds\right]\right)^{\frac{1}{2}}.$$

It is known (cf. Chung and Williams [11], Sections 3.3 and 2.5) that $\mathcal{L}_s$ is dense in $\mathcal{L}$ with respect to the L^2-norm $\|\cdot\|_{\mathcal{L}}$. This property, the isometry (ii), and its consequence (iii), are the keys to extending the definition of the stochastic integral process $\{\int_0^t Y_s\, dW_s,\ t \in [0,T]\}$ to all integrands $Y \in \mathcal{L}$. Briefly, this extension is accomplished as follows. Firstly, for any $Y \in \mathcal{L}$, there is a sequence $\{Y^{(m)}\}_{m=1}^{\infty}$ in $\mathcal{L}_s$ such that $\|Y - Y^{(m)}\|_{\mathcal{L}} \to 0$ as $m \to \infty$. It follows that such a sequence is Cauchy in $\mathcal{L}$ and hence by the linearity of the stochastic integral and the isometry property (ii) (with $t = T$), the corresponding sequence of stochastic integrals $\{\int_0^T Y_s^{(m)}\, dW_s\}_{n=1}^{\infty}$ is Cauchy in $L^2(\Omega, \mathcal{F}_T, P)$. In fact, using (iii) on differences of the stochastic integral processes and a standard Borel-Cantelli argument, one can show that there is a subsequence $\{m_k\}$ of $\{m\}$ and a continuous, adapted process, which we denote by $\{\int_0^t Y_s\, dW_s,\ t \in [0,T]\}$, such that P-a.s., as $k \to \infty$, $\int_0^t Y_s^{(m_k)}\, dW_s$ converges uniformly for $t \in [0,T]$ to $\int_0^t Y_s\, dW_s$. Up to indistinguishability, this limit process $\{\int_0^t Y_s\, dW_s,\ t \in [0,T]\}$ does not depend on the particular convergent subsequence, nor on the original sequence chosen to approximate Y. This process $\{\int_0^t Y_s\, dW_s,\ t \in [0,T]\}$ inherits the martingale and isometry properties, (i) and (ii), from the approximating sequence.

A final step in defining stochastic integrals is to localize the class of integrands with the attendant stochastic integrals yielding local martingales starting from zero. For this, let $\mathcal{L}_{loc}$ denote the set of one-dimensional processes

$Y = \{Y_t, t \in [0,T]\}$ such that Y is adapted, $Y : [0,T] \times \Omega \to \mathbb{R}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $Y(t,\omega) = Y_t(\omega)$ for all $t \in [0,T]$ and $\omega \in \Omega$, and

$$\int_0^T Y_s^2 \, ds < \infty \qquad P\text{-a.s.}$$

Given $Y \in \mathcal{L}_{loc}$, for each positive integer m, consider the stopping time

$$\tau_m = \inf \left\{ t \in [0,T] : \int_0^t Y_s^2 \, ds \geq m \right\}.$$

It is readily verified that for each m, the process $Y^{(m)}$ defined by

$$Y_s^{(m)}(\omega) = 1_{[0,\tau_m(\omega)]}(s) Y_s(\omega) \qquad \text{for } s \in [0,T], \ \omega \in \Omega,$$

is in $\mathcal{L}$. One can show that the stochastic integral processes $\{\int_0^t Y_s^{(m)} \, dW_s, \, t \in [0,T]\}$ are consistent (almost surely) as m increases. It then follows that there is a continuous local martingale that starts from zero (with localizing sequence $\{\tau_m\}$), which we denote by $\{\int_0^t Y_s \, dW_s, t \in [0,T]\}$, that P-a.s. satisfies

$$\int_0^t Y_s \, dW_s = \lim_{m \to \infty} \int_0^t Y_s^{(m)} \, dW_s \quad \text{for all} \quad t \in [0,T].$$

This limit process is unique up to indistinguishability.

Now suppose that W is an n-dimensional standard Brownian motion W defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t, t \in [0,T]\}$ is a filtration such that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions and $\{W_t, \mathcal{F}_t, t \in [0,T]\}$ is a continuous, n-dimensional martingale. (Again we could use the standard filtration associated with W for $\{\mathcal{F}_t\}$, but we do not assume that a priori.) Assume without loss of generality that $\mathcal{F} = \mathcal{F}_T$. Define $\mathcal{L}$ and $\mathcal{L}_{loc}$ in an analogous manner to that described above using this filtration. Suppose that $Y = (Y^1, \ldots, Y^n)$ is an n-dimensional process such that for each $i = 1, \ldots, n$, $Y^i \in \mathcal{L}_{loc}$ for each i. Then each of the one-dimensional stochastic integral processes $\{\int_0^t Y_s^i \, dW_s^i, t \in [0,T]\}$ can be defined as above and then we define

$$\int_0^t Y_s \cdot dW_s = \sum_{i=1}^n \int_0^t Y_s^i \, dW_s^i, \quad t \in [0,T].$$

It can be shown using the independence of the components of W and the isometry for stochastic integrals that if $Y_i \in \mathcal{L}$ for all i, then

$$E\left[\left(\int_0^t Y_s \cdot dW_s\right)^2\right] = E\left[\int_0^t |Y_s|^2 \, ds\right], \quad t \in [0,T],$$

where $|Y_s| = \left(\sum_{i=1}^n (Y_s^i)^2\right)^{\frac{1}{2}}$ for all $s \in [0,T]$.