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Fluctuation process

\[
\hat{V}^r(t) = \sqrt{r} (\hat{X}^r(t) - \bar{X}(t)) \\
= \hat{V}^r(0) + \sum_k \hat{N}_k^r (\hat{\Phi}_k^r(t)) (\nu_k^r - \nu_k) + \sum_k \sqrt{r} \left( \int_0^t (f_k^r(\hat{X}^r(s)) - f_k(\bar{X}^r(s))) ds \right) (\nu_k^r - \nu_k) \\
+ \sum_k \left( \int_0^t \nabla f_k(\rho^r(s)) \cdot \hat{V}^r(s) ds \right) (\nu_k^r - \nu_k)
\]

where

\[
\hat{\Phi}_k^r(t) = \int_0^t f_k^r(\bar{X}^r(s)) ds
\]

Here, \(\rho^r(s)\) is between \(\bar{X}^r(s)\) and \(X(s)\).

We know

\[
(V(0), \sum_k W_k(\Phi(\cdot))(\nu_k^r - \nu_k), \bar{X}(\cdot)) \Rightarrow (V(0), \sum_k W_k(\Phi(\cdot))(\nu_k^r - \nu_k), \bar{X}(\cdot))
\]
as \(r \to \infty\).

Since \(\{\bar{X}^r\}\) converges to \(\bar{X}\), which is continuous, we have \(\{\bar{X}^r\}\) is \(C\)-tight. In particular, for each \(T > 0\), and \(\epsilon > 0\), there exist \(K_1, r_1\), such that

\[
P(\|\hat{X}^r(\cdot)\|_T > K_1) < \epsilon \quad \forall r \geq r_1.
\]

Recall that in the expression of \(\hat{V}^r\), \(f_k^r\) and \(f_k\) are given by the following:

\[
f_k^r(c) = \prod_{i=1}^n c_i \left( c_i - \frac{1}{r} \right) \cdots \left( c_i - \frac{\nu_k}{r} + \frac{1}{r} \right)
\]

\[
f_k(c) = \prod_{i=1}^n c_i^{\nu_{ik}}.
\]

It follows that there exist \(K_2, r_2\) such that

\[
P\left( \|f_k^r(\bar{X}^r(\cdot)) - f_k(\bar{X}(\cdot))\|_T > \frac{1}{r} K_2 \right) < \epsilon, \quad \forall r \geq r_2.
\]

Hence,

\[
\hat{\Psi}^r = \sum_k \sqrt{r} \int_0^t (f_k^r(\hat{X}^r(s)) - f_k(\bar{X}^r(s))) ds (\nu_k^r - \nu_k) \Rightarrow 0 \quad \text{as } r \to \infty
\]
or equivalently, for each \(T > 0\), \(\epsilon > 0\), there is \(r_3\) such that

\[
P(\|\hat{\Psi}^r(\cdot)\|_T > \epsilon) < \epsilon, \quad \forall r \geq r_3.
\]

It follows from (*) that there is \(K_4\) and \(r_4\) such that

\[
P(\|\nabla f_k(\cdot)\|_T > K_4) < \epsilon, \quad \forall r \geq r_4, \quad \forall k.
\]

Our aim now is to show that \(\{\hat{V}^r\}\) is \(C\)-tight.
Recall Gronwall’s inequality: suppose that $f, g$ are real valued functions that are Lebesgue integrable on $[0, T]$ and such that for some $C > 0$,
\[ f(t) \leq g(t) + C \int_0^t f(s) ds, \forall t \in [0, T]. \]

Then,
\[ f(t) \leq g(t) + C \int_0^t e^{C(t-s)} g(s) ds, \forall t \in [0, T]. \]

Given $T > 0$ and $\epsilon > 0$ there exist $K_5, r_5$ such that
\[
P(\|\hat{V}^r(0) + \sum_k \hat{N}^r_k(\Phi^r_k(\cdot))(\nu^r_k - \nu_k) + \hat{\Psi}^r(\cdot)\|_T \leq K_5, \quad \|\nabla f_k(\rho^r(\cdot))\|_T \leq \frac{K_5}{(|\nu^r_k - \nu_k| + 1)K}, \quad k = 1, \ldots, K) \geq 1 - \epsilon
\]
when $r \geq r_5$. Call the set of which the probability is taken on the left side above the “good set”. On the good set, we have
\[
|\hat{V}^r(t)| \leq K_5 + \left( \sum_{k=1}^K \int_0^t \frac{K_5}{(|\nu^r_k - \nu_k| + 1)K} |\hat{V}^r(s)| ds |\nu^r_k - \nu_k| \right)
\leq K_5 + \int_0^t K_5 |\hat{V}^r(s)| ds \quad \forall t \in [0, T].
\]

By Gronwall’s inequality, on the same set
\[
|\hat{V}^r(t)| \leq K_5 + K_5 \int_0^t e^{K_5(t-s)} K_5 ds \leq K_6(T) \quad \forall t \in [0, T].
\]

Hence
\[
P(\|\hat{V}^r(\cdot)\|_T \leq K_6(T)) \geq 1 - \epsilon \quad \forall r \geq r_5.
\]

Thus, we have the necessary oscillation control of the last term $\ast\ast$ for $C-$tightness. It follows that $\{\hat{V}^r\}$ is $C-$tight.

Consider a weak limit point $V$ of $\{\hat{V}^r\}$. Without loss of generality, can assume the convergence of a subsequence of $\{\hat{V}^r\}$ to $V$ is also joint with the same subsequence of $\{(\hat{V}^r(0), \sum_k \hat{N}^r_k(\Phi^r_k(\cdot))(\nu^r_k - \nu_k), \hat{\Psi}^r(\cdot))\}$. The limit $V$ will satisfy
\[
V(t) = V(0) + \sum_k W_k \left( \int_0^t f_k(\bar{X}(s)) ds \right) (\nu^r_k - \nu_k) + \sum_k \left( \int_0^t \nabla f_k(\bar{X}(s)) \cdot V(s) ds \right) (\nu^r_k - \nu_k),
\]
\[ \forall t \geq 0. \]

$V$ is equivalent in law to a solution of
\[
\bar{V}(t) = V(0) + \sum_k \left( \int_0^t f_k(\bar{X}(s)) d\bar{W}_k(s) \right) (\nu^r_k - \nu_k) + \sum_k \left( \int_0^t \nabla f_k(\bar{X}(s)) \cdot \bar{V}(s) ds \right) (\nu^r_k - \nu_k).
\]

The S.D.E for $\bar{V}$ has a unique solution in law. Hence, $\{\hat{V}^r\}$ in fact converges to $V$ (or $\bar{V}$).

We have shown that $\sqrt{r}(\hat{X}^r(\cdot) - \bar{X}(\cdot)) \Rightarrow V$ as $r \rightarrow \infty$. We can interpret this as
\[
\hat{X}^r(\cdot) - \bar{X}(\cdot) \approx \frac{1}{\sqrt{r}} V.
\]
This is called the functional central limit theorem approximation or the Van Kampen approximation. A special case is when $X(t) \equiv x_0$, $\forall t \geq 0$ where $x_0$ is an equilibrium point for fluid model.

Another Diffusion Approximation
We have

$$
\dot{X}^r(t) = \dot{X}^r(0) + \frac{1}{r} \sum_k \tilde{N}_k \left( r \int_0^t f_k(\dot{X}^r(s))ds \right) (\nu'_k - \nu_k) + \sum_k \left( \int_0^t f_k(\dot{X}^r(s))ds \right) (\nu'_k - \nu_k)
$$

We use $\{\dot{X}^r\}$ to approximate $\{X^r\}$, where

$$
\dot{X}^r(t) = \dot{X}^r(0) + \frac{1}{r} \sum_k \tilde{W}_k \left( r \int_0^t f_k(\dot{X}^r(s))ds \right) (\nu'_k - \nu_k) + \sum_k \left( \int_0^t f_k(\dot{X}^r(s))ds \right) (\nu'_k - \nu_k)
$$

Now $\tilde{W}_k(\cdot) = \frac{W_k(r \cdot)}{\sqrt{r}}$ is a Brownian motion and we can rewrite $\dot{X}^r$ as:

$$
\dot{X}^r(t) = \dot{X}^r(0) + \frac{1}{\sqrt{r}} \sum_k \tilde{W}_k \left( \int_0^t f_k(\dot{X}^r(s))ds \right) (\nu'_k - \nu_k) + \sum_k \left( \int_0^t f_k(\dot{X}^r(s))ds \right) (\nu'_k - \nu_k)
$$

From Kurtz’s paper, if

$$
T_m = \inf\{t \geq 0 : |\dot{X}^r(t)| \vee |\dot{X}^r(t)| \geq m\}
$$

then we have:

$$
\sup_{t \in [0, T \wedge T_m]} |\dot{X}^r(t) - \dot{X}^r(t)| \leq \frac{\Gamma r \log r}{T} \quad a.s
$$

The above approximation is often called the Langevin approximation or the diffusion approximation.

Queueing Models
Consider a single server GI/GI/1 queue (here GI indicates a sequence of i.i.d. random variables).

$$
\text{GI} \xrightarrow{\text{FIFO}} \text{queue} \xrightarrow{\text{GI}} \text{1}
$$

We let

$$
Q(t) = \text{queue length at time } t
$$

$$
= Q(0) + \underbrace{A(t)}_{\# \text{ of arrivals up to time } t} - \underbrace{D(t)}_{\# \text{ of departures up to time } t}
$$

Here, we rewrite $D(t)$ as $S(T(t))$ where $S$ is the renewal process that goes with the i.i.d service times, and $S(t)$ is the number of service completions that would occur by time $t$ if the server never idles. We let $T(t)$ be the busy time of the server up to time $t$, and it can be expressed as: $T(t) = \int_0^t 1\{Q(s) > 0\}ds$

From the fluid model, we have:

$$
\dot{Q}^r(t) = \frac{Q'(rt)}{r} = \frac{Q'(0)}{r} + \frac{A(rt)}{r} - \frac{S(T(rt))}{r}
$$

and $\dot{Q}^r \Rightarrow \bar{Q}$. We also write

$$
\frac{S(T(rt))}{r} = \frac{S(T(rt))}{r}, \quad \text{and} \quad \frac{T(t)}{r} = \frac{T(rt)}{r}
$$
\( \{T^r\} \) is tight. Let \( \tilde{T} \) be a weak limit point of this sequence. Then we can write

\[
\bar{Q}(t) = \bar{Q}(0) + \lambda t - \mu \tilde{T}(t) = \bar{Q}(0) + (\lambda - \mu)t + \mu \bar{I}(t)
\]

where, \( I(t) \) is the idle time and \( \bar{I}(t) = t - T(t) \), so that \( \bar{I}^e(t) = \frac{t^e}{r} - \frac{T(x)}{r} \) and \( \bar{I}(t) = t - \tilde{T}(t) \). Notice \( \bar{I}(\cdot) \) increases only when \( \bar{Q}(\cdot) \) is zero.