Consider $\{X^n\}_{n=1}^{\infty}$ a sequence of $d$-dim stochastic processes.

Let $\mathcal{F}^n$ be the law of $X^n$ on $(\mathbb{D}^n, \mathcal{D}^n)$.

$\{\mathcal{F}^n\}_{n=1}^{\infty}$ is tight iff $\{X^n\}_{n=1}^{\infty}$ is tight.

The sequence $\{X^n\}_{n=1}^{\infty}$ is tight iff for each $T>0$ and $\varepsilon>0$

(i) $\lim_{K \to \infty} \lim_{n \to \infty} P(\|X^n\|_T > K) = 0$

(ii) $\lim_{n \to \infty} \lim_{\delta \to 0} P(\mathcal{W}(X^n, \delta, T) > \varepsilon) = 0$

Visual modulus of continuity

$\mathcal{W}(X, \delta, T) = \sup_{u \leq \delta \leq T} \sup_{U \in \mathcal{G}} |X(u) - X(v)|$, $X \in \mathbb{D}^d$

Definition: $\{X^n\}_{n=1}^{\infty}$ is $C$-tight if

(i) It is tight

(ii) Any weak limit of $\{X^n\}$ has continuous paths almost surely.

The sequence $\{X^n\}_{n=1}^{\infty}$ is $C$-tight iff for each $T>0$ and $\varepsilon>0$

(i) $\lim_{K \to \infty} \lim_{n \to \infty} P(\|X^n\|_T > K) = 0$

(ii) $\lim_{\delta \to 0} \lim_{n \to \infty} P(\mathcal{W}(X^n, \delta, T) > \varepsilon) = 0$
A weak limit of \( \{X_n\}_{n=1}^\infty \) is a stochastic process \( X \) such that \( X^{(n)} \Rightarrow X \) as \( n \to \infty \) for some subsequence \( \{n_k\} \) of \( \{n\} \).

**Random Time Change Lemma**

Let \( X \in \mathbb{D}^d \). \( \psi \in D_0 = \{\psi : (0,\infty) \to [0,\infty), \psi \text{ is non-decreasing, } \psi \text{ is r.c.l.l.}\} \)

**Exercise:** Check \( X \circ \psi \in \mathbb{D}^d \)

**Facts:**

1. For \( t \in [0,\infty) \), \( X \to X(t) \), for \( X \in \mathbb{D}^d \) is continuous at \( t \) if \( X \) is continuous at \( t \).

**Proof:** Suppose \( X \in \mathbb{D}^d \) and \( X \) is continuous at \( t \). Suppose \( X_n \to X \) in \( \mathbb{D}^d \). Let \( T > t \). A sequence of time changes \( \{Y_n\}_{n=1}^{\infty} = \Gamma \).

1. \( \|Y_n(t) - \psi(t)\| \to 0 \) as \( n \to \infty \) where \( \psi(t) = \psi(t) \) for \( t \geq 0 \).
2. \( \|X_n(t) - X(Y_n(t))\| \to 0 \) as \( n \to \infty \).

\[
|X_n(t) - X(t)| \leq |X_n(t) - X(Y_n(t))| + |X(Y_n(t)) - X(t)|
\]

- \( 0 \) as \( n \to \infty \) by (1)
- \( 0 \) as \( n \to \infty \) by (2)

\( X \) is continuous at \( t \) by (1)
Since $X$ is discontinuous at at most countably many points, 
$x(t_n) \Rightarrow x(t)$ as $n \to \infty$ for all but countably many $t$.

Furthermore, if $X$ is continuous at $t_1, \ldots, t_m$, then
\[(x(t_1), \ldots, x(t_m)) \Rightarrow (x(t_1), \ldots, x(t_m)) \text{ as } n \to \infty\]

(2). For $X$ a 0-dimensional process,
\[J_t = \{ W : x(t) \neq x_t(w) \} \text{ for } t \geq 0\]
\[P(J_t) = 0 \text{ for all but countably many } t.\]

(3). Suppose $X^n \Rightarrow X$ as $n \to \infty$.

By Skorokhod Representation Thm, there are processes \((\tilde{x}_n)_{n=1}^{\infty}, \tilde{x}\)
all defined on the same probability space, such that
\[X^n = \tilde{x}_n \quad \forall n, \quad X = \tilde{x}, \quad \tilde{x}^n \Rightarrow \tilde{x} \text{ a.s. as } n \to \infty \quad (**)\]

There is a countable set $B \subset (0, \infty)$ such that for each $t \in (0, \infty) \setminus B$,
\[P(X \text{ is continuous at } t) = 1.\]

Hence for any $t_1, t_2, \ldots, t_m \in (0, \infty) \setminus B$, for a.e. $W$, $\tilde{x}$ is continuous
at \( t_1, \ldots, t_m \).

So by (*) and (i), 
\[
(\tilde{X}^n(t_1), \ldots, \tilde{X}^n(t_m)) \Rightarrow (\tilde{X}(t_1), \ldots, \tilde{X}(t_m))
\]
a.s. as \( n \to \infty \)

Hence, 
\[
(X^n(t_1), \ldots, X^n(t_m)) \Rightarrow (X(t_1), \ldots, X(t_m)) \quad \text{as } n \to \infty.
\]

**Summary:** \( X^n \Rightarrow X \) as \( n \to \infty \) implies convergence of the finite dimensional distributions of \( X^n \) to those of \( X \) at all but countably many times.

\[\text{\(\text{\(\text{\([\tilde{X}(t_1), \ldots, X(t_m)\]}\) for all } t_1, \ldots, t_m \text{ subsets of } (0, \infty)\)}\]
are the finite dimensional distribution (f.d.d) of \( X \).

(4). If \( \{X^n\}_{n=1}^{\infty} \) is tight and there is a dense set of times 
\( T \subseteq [0, \infty) \) such that 
\[
(X^n(t_1), \ldots, X^n(t_m)) \Rightarrow (X(t_1), \ldots, X(t_m))
\]
a.s. as \( n \to \infty \) for all \( t_1, \ldots, t_m \in T \), then \( X^n \Rightarrow X \) as \( n \to \infty \).

**Donsker's Invariance Principle**

Recall if \( \{V_i\}_{i=1}^{\infty} \) is an iid sequence of r.v.'s with finite mean
$\mu$ and variance $\sigma^2 > 0$

Central Limit Thm

$$\frac{\sum_{i=1}^{n} (V_i - \mu)}{\sigma \sqrt{n}} \xrightarrow{n \to \infty} N(0,1),$$

random r.v. mean $\mu$, variance $\sigma^2$.

$$\frac{n \sum_{i=1}^{n} V_i}{n} \to \mu \text{ in probability as } n \to \infty. \text{ Weak law of large Num.}$$

Random Walk

$$\sum_{i=1}^{n} (V_i - \mu)$$

$$\hat{V}_n(t) = \frac{\sum_{i=1}^{\lfloor nt \rfloor} (V_i - \mu)}{\sigma \sqrt{n}} \quad \lfloor . \rfloor = \text{integer part}$$

Donsker's Thm (Functional Central Limit Thm)

$$\{\hat{V}_n(t)\}_{n=1}^{\infty} \text{ converges in distribution to } W, \text{ a standard one-dimensional Brownian Motion.} \quad (\text{cts paths, independent increments, } W(t) - W(s) \sim \text{Normal}(0, t-s) \quad \text{WH}=0)$$

Central

Renewal Process Central Limit Thm

Let $\{W_i\}_{i=1}^{\infty}$ be i.i.d. (strictly) positive r.v's with mean $\lambda \in (0, \infty)$, and variance $\sigma^2 \in (0, \infty)$
\[ N(t) = \sup \{ k : u_1 + u_2 + \ldots + u_k \leq t \} \]

Renewal processes.

(Poisson process - exponential \( u_i \)'s)

\[ \hat{N}^n(t) = \frac{N(nt) - \lambda nt}{\sqrt{t/n}} \quad t > 0 \]

Then \( \hat{N}^n \to W \) as \( n \to \infty \) (Wiener - W)