Comment to last time

- E+U of solution of SP for generalized M-matrix
  Harris and Reiman (1981)
- Fix pt theorem, contraction argument - Banach.

How to use the Thm proved last time.
- Semi-martingale Reflecting Brownian motion (SRBM)
  papers about it on webpage.
  - survey (R. Williams)
  - Brownian particles on the real line with
    rank dependent drift (Par + Pitman)
    \[ \cdots \] gap process (RBM)

\[ \text{SRBM} \]

Fix dim \( d \) (positive integer)

Let \( S = \{ x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0 \} \)

Let \( \theta \in \mathbb{R}^d \), \( T \) is \( d \times d \) non-degenerate covariance matrix
  (sym, strictly ps def.)

\( R \), \( d \times d \) matrix

An SRBM with data \( (S, \theta, T, R) \) is a cts process, adapted \( d \)-dim process \( \xi \), defined on a filtered space
  \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}) \) together with a family of prob measures
  \( \{ P_x, x \in S \} \) defined there, s.t. for each \( x \in S \),
  under \( P_x \) we have:
\[ Z(t) \in S, \]
\[ Z(t) = X(t) + R Y(t) \quad t \geq 0 \quad (P_x-a.s.) \]

where

1. \( X \) is a d-dim BM with a constant drift \( \theta \) and covariance matrix \( \Sigma \)
2. \( X(t) - \theta t, (\Omega, \mathcal{F}_t, P_x) \) is a martingale under \( P_x \), \( X(0) = X \) (\( P_x \)-a.s.)
3. \( Y(t) \) is an adapted d-dim process s.t.
   a. \( Y(0) = 0 \)
   b. \( \text{non-decreasing (in each component)} \)
   c. \( \int_{(0,\infty)} Z_i(s) \, dY_i(s) = 0 \quad \forall i = 1, \ldots, d \)

\[ \text{Behavior of such an SKBBM under } P_x \]

\[ R = \begin{pmatrix} R^{(1)} & R^{(2)} & R^{(3)} \\ \uparrow & \uparrow & \uparrow \\ \text{col } 1 & \text{col } 2 & \text{col } 3 \end{pmatrix} \]

- \( R \) is a \( 3 \times 3 \) matrix, \( R^{(i)} \) is a vector on \( \mathbb{R}^3, z_i = 0 \)
- Away from boundary behaviour like \( X(t) \) (BM)
- \( R Y(t) \) kick in to hold it in the state space

\[ \Delta \Omega = C \left( [0,\infty), [0]^d \right) \]
\[ \Delta F = C \left( t \cap \{X(s), 0 \leq s \leq \infty, x \in \Omega \} \right) \]
\[ \Delta F_x = C \left( t \cap \{X(s), 0 \leq s \leq t, x \in \Omega \} \right) \]
\[ P_x \quad \text{Ca nonical path space} \]
Then suppose $R$ is a generalized $M$-matrix $(\Phi, \Psi)$ specify the solution of the Skorokhod problem associated with $R$.

Let $X$, $(P_x, x \in S)$ be a filtered space $(\Omega, F, F_t, \mathbb{P})$ s.t. under $P_x$, $X$ is a BM with drift $\Theta$, covariance matrix $T$.

$x(t) = x$, $P_x - a.s.$ $X(t) = \Theta t, \forall t, t \geq 0$ is a $P_x$-martingale.

Let $Y = \Phi(x)$ $Z = \Psi(x)$, then $Z$ with $(P_x, x \in S)$ defines an SRBM with data $(S, \Theta, T, R)$. If it is unique in law (i.e. $Z, Z'$ are SRBM, with $(P_x, x \in S)$, $P_x, x \in S$ as prob measures.

By uniqueness of the solution of SP, $Z = \Psi(x)$ $\&$ $Z' = \Psi(x')$

Since $X$ under $P_x$ $\overset{d}{=} X'$ under $P_x'$, then $Z$ under $P_x$ $\overset{d}{=} Z'$ under $P_x'$ for $x \in S$.

In fact $Z$ under $(P_x, x \in S)$ is strong Markov.

If $T$ is a $P_x-a.s.$ finite stopping time, then \[ P_x(\tilde{Z}_{T^+} \in A | F_T) = P_{\tilde{Z}_{T^+}}(\tilde{Z} \in A) = R_T(z, z') \]

$P^x(A) = P_x(\tilde{Z} \in A)$

$\Phi = \Pi d_{|1, d} - 1d_{|1, d}$, $\Psi = d_{|1, d} - 1d_{|1, d}$

Given $x \in \Pi_d$, $z = \Phi(x)$, $y = \Psi(x)$ is solution of the SP for $x$.

\begin{enumerate}
  \item $z = x + R y$
  \item $z(t) \geq 0 \forall t \geq 0$
  \item $y(t) \geq 0 \forall t$ $y$ is non-decreasing
  \item $\int_{(0, \infty)} z(t) d\xi(t) = 0 \forall \xi$
\end{enumerate}
For single class open queuing network

\[ \begin{align*}
\text{III} & \rightarrow 0 \rightarrow \text{III} \rightarrow 0 \\
\text{III} & \rightarrow 0 \rightarrow (\text{II}) \rightarrow 0
\end{align*} \]

Associated with diffusion approx is an SRBM
with \( R = (I - P') \) diag (\( \mu \))
\( \mu_i = \text{average service rate at queue} \)
\( p_{ji} = \text{prob after completing service at server } j \)
goes to server \( i \)

Open network \( \leftrightarrow \) \( P \) has spectral radius < 1
\( \leftrightarrow \) \( P \) is a transition matrix for a transient Markov chain

This \( R \) is an M-matrix

Multi-class queuing network diffusion approx
however can lead to \( R \) matrix that are
not generalized M-matrix

**Def.** A real matrix \( R \) is called an S-matrix
is \( \exists \ E \in \mathbb{R}^{d \times d} \) s.t. \( R > 0 \) (S for Stiemke)
\( R \) is completely \(-S\) if every principle
submatrix of \( R \) is an \(-S\)-matrix

Principle submatrices are obtained by eliminating
all rows + cols of \( R \) with a given set of indices
Geometric Interpretation

$R^{(i)}$ is on face where $z_i = 0$
$R$ is completely $-$ if at each boundary point there is a pos. linear combination of the vectors of reflections that one can use there that points into the orthant.

Algebraically

Suppose $2$, $z_i = 0 \forall i \in \mathbb{A}$, $z_i > 0$,
$\forall i \in \mathbb{A}; \exists v_i$ for $i \in \mathbb{A}$ s.t. $v_i > 0 \forall i \in \mathbb{A}$
$(\sum_{i \in \mathbb{A}} v_i R^{(i)}) x > 0 \forall j \in \mathbb{A}$

Thus suppose $R$ is completely $-$ for each $x \in \mathbb{R}^d$, there exists $(y, z) \in \mathbb{R}^{d+1}_+$ that solves $SP$ for $x$.

Note: $(y, z)$ in general is not unique.

Counter example to uniqueness

(Mauldenbaum, van der Heyde)

$R = \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}$

$z^{(1)}$ at time 1 is at $(1)$
$y^{(1)}$ is well defined.

Zig zag is crucial construction for uniqueness.