GI/GI/1 fluid model

\[ \bar{Q}(t) = \bar{Q}(0) + (\lambda - \mu) t + \mu \bar{I}(t). \]

\( \bar{I}(0) \) is cts., non-decr. \( \lambda \) can incr. only when \( \bar{Q} \) is 0.
Fluid model is stable if \( \exists N > 0 \) s.t. for any fluid model sol'n \( \bar{Q}, \bar{Q}(t) = 0, \forall t \geq N/|\bar{Q}(0)|. \) (Scale by \( |\bar{Q}(0)|).)

Dai's Thm (1995) - see Bramson notes.

Impose mild conditions.
Assume interarrival times \( \{u_i\}_{i=1}^\infty \) have unbounded support, i.e. \( \forall K > 0 \) \( P(u_i > K) > 0. \) [Can be arbitrarily large.]
Consider \( \bar{x}^n(t) = (U^n(t), V^n(t)), (S_i^n)_{i=1}^\infty. \)

[Keep track of residual interarrival time \( U^n(t) \), residual service time for job in service \( V^n(t) \), \( \{S_i^n\}_{i=1}^\infty \) the "age" of \( i \)th job in the queue. The first job is in the head of the line.]
We make the sequence infinite. If no \( i \)th job, make \( S_i = -1. \)
Want to know how old the jobs are if we've more than 1 queue.

Recover queue length: \( Q^n(t) = \sum_{i=1}^{\infty} \{ S_i^n > 0 \} \).

Want to have \( \{ S_i^n \} \) in some deterministic space. So make its length infinite. \( \mathbb{R}_+ \times \mathbb{R}_+ \times (\mathbb{R}_+ \cup \{ -1 \})^N \) Markov state descriptor for GI/GI/1 queue.

Suppose fluid model is stable, then \( x^n \) is positive Harris recurrent for any \( n \). [Harris is from USC.]

Now have continuous state! \( \sup_{x \in A} \mathbb{E}_x [T^n_A(\delta)] < \infty \), where

\[
T^n_A(\delta) = \inf \{ t > 0 : x^n(t) \notin A \}, \quad \delta > 0 , \text{ continuous analog of "first return time"}, \quad A \text{ is some "petite" set.}
\]

Defn. A petite set \( A \) is a non-empty, measurable set for which \( \exists \) a prob. measure \( \mu \) on \( (0, +\infty) \) & a non-trivial pos. measure \( \nu \), s.t. \( \nu(B) \leq \int_0^{+\infty} \mu_t(x, B)\mu(dx) \) \( \forall x \in A \& B \) meas. sets. \( \square \)

\[
[p^n_t(x, B) = P_x(x^n(t) \in B)]
\]
$p^*_t(x, \delta)$ is bounded below in some sense by some reference measure $\nu$. Continuous analog of communicating property.

The converse of Dai's Thm is generally not true. Dai's Thm is useful in proving stability.

GI/GI/1 diffusion approx.

Consider $\lambda = \mu$ (balanced).

(Can also consider $n(\lambda^* - \mu^*) \to 0$ as $n \to \infty$.)

\[ Q^n(t) = Q^n(0) + A(t) - S(T^n(t)). \]

\[ \hat{Q}^n(t) = \frac{Q^n(n^2t)}{n} = \hat{Q}^n(0) + \frac{A(n^2t) - \lambda n^2t}{n} \]

\[ - \frac{S(T^n(n^2t) - \mu T^n(n^2t))}{n} + \frac{\lambda n^2t - \mu n^2t}{n} + \frac{\mu n^2t - \mu T^n(n^2t)}{n} \]

\[ = \hat{Q}^n(0) + \hat{A}^n(t) - \hat{S}^n(T^n(t)) + \mu \hat{I}^n(t). \]

\[ \hat{A}^n(t) = \frac{A(n^2t) - \lambda n^2t}{n}, \quad \hat{S}^n(t) = \frac{S(n^2t) - \mu n^2t}{n}, \quad \hat{T}^n(t) \]

\[ = \frac{T^n(n^2t)}{\lambda n^2}. \]

\[ T^n(n^2t) = \int_0^{n^2t} \mathbf{1}_{\{Q^n(s) > 0\}} \, ds \times \frac{n^2t - T^n(n^2t)}{n} \]

\[ = \frac{1}{n} \int_0^{n^2t} \mathbf{1}_{\{Q^n(s) = 0\}} \, ds = \hat{I}^n(t) = \frac{I^n(n^2t)}{n} \quad \text{non-decr. + incr. only} \]
when \( \hat{Q}^n = 0 \).]

\[
\hat{S} = \frac{S}{n^2}, \quad \hat{X}^n(t) = n \int_0^t 1 \left\{ Q^n(n^2\hat{S}) = 0 \right\} d\hat{S} = n \int_0^t 1 \left\{ \hat{X}(\hat{S}) = 0 \right\} d\hat{S}.
\]

Claim. \( \hat{X}^n(t) = \mu^{-1} \Phi \left( \hat{Q}^n(0) + \hat{A}^n(\cdot) - \hat{S}^n(\frac{\hat{T}^n(\cdot)}{n})) \right)(t) \forall t \geq 0, \)

where \( \Phi(x)(t) = \sup_{0 \leq s \leq t} x^- (s), \quad x^- (s) = \max(-x(s), 0) \). \( \Phi: \mathbb{D} \to \mathbb{D} \) (cts. mapping). \( \square \)

Suppose \( \hat{Q}^n (0) \Rightarrow \bar{Q}(0) \) & \( \hat{Q}^n (0) \) are indept. of \( \hat{A}^n, \hat{S}^n \).

\( \text{FCLT} \Rightarrow (\hat{A}^n, \hat{S}^n) \Rightarrow (B_a, B_s) \).

\( B_a \) is 1-dim BM (variance par. \( \lambda^3 \sigma_a^2 \)) and \( B_s \) is 1-dim BM (variance par. \( \mu^3 \sigma_s^2 \)). \( B_a, B_s \) indept.

\( \{ \hat{T}^n \}_{n=1}^\infty \) are C-tight and any weak limit pt. \( \bar{Q}(t) = \bar{Q}(0) + \lambda t - \mu \hat{T}(t) \). \( \bar{Q}^n(t) = \frac{\hat{Q}^n(t)}{n} \Rightarrow \bar{Q}(0) = 0. \)

\( \bar{Q}(t) = \lambda t - \mu \hat{T}(t) = (\lambda - \mu) t + \mu \hat{I}(t) = \mu \hat{I}(t) \).

For balanced fluid model, \( \bar{Q}(t) = \bar{Q}(0) \forall t \).

\( \Rightarrow \bar{I}(t) = 0 \forall t \Rightarrow \bar{T}(t) = t \forall t. \)

So conclude \( \bar{T}^n \Rightarrow \bar{T}(t) = t, \text{ all } t. \)

\[ \text{Summa} \]
Summary

\((\hat{Q}^n(0), \hat{A}^n, \hat{S}^n, \hat{T}^n) \Rightarrow (\hat{Q}(0), B_a, B_s, \hat{T})\) as 
\(n \to \infty\). For \(\hat{X}^n(t) = \hat{Q}^n(0) + \hat{A}^n(t) - \hat{S}^n(\hat{T}^n(t))\),
\(\hat{X}^n \Rightarrow \hat{X}\) where \(\hat{X}(t) = \hat{Q}^n(0) + B_a(t) - B_s(t)\). BM
var. par. \(\lambda^3 \sigma_a^2 + \mu^3 \sigma_s^2\).

\(\hat{Q}^n(t) = \hat{X}^n(t) + \Phi(\hat{X}^n)(t) = \psi(\hat{X}^n)(t)\).

\(\psi(x)(t) = x(t) \Phi + \Phi(x)(t)\).

\(\psi: D \to D\) is cts. By cts. mapping thm, \(\psi(\hat{X}^n) \Rightarrow \psi(\hat{X}) \Rightarrow \hat{Q}^n \Rightarrow \hat{Q}\), where \(\hat{Q}(t) = \hat{X}(t) + \Phi(\hat{X})(t)\). BM, \(\Phi(\hat{X})\) cts., non-decr. + incr. only when \(\hat{Q} = 0\).
\(\hat{Q}\) is 1-dim'l reflected BM.

\((\hat{Q}^n, \hat{I}^n) \Rightarrow (\hat{Q}, \hat{I}), \hat{I} = \Phi(\hat{X})\). Local time of \(\hat{Q}\).