Topics in Probability (Math 289A): Lecture 13

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Skorokhod Problem Given $x \in \mathbb{D}_+^d$, want to find $(y, z) \in \mathbb{D}_+^d \times \mathbb{D}_+^d$ such that $z(t) = x(t) + Ry(t) \in \mathbb{R}_+^d \ \ \forall t \geqslant 0$ and $y(0) = 0, y$ is non-decreasing, $\int_{[0,\infty)} z(t)dy(t) = 0$.
If $R = (v_1| \cdots |v_d)$ is completely-$\boldsymbol{S}$, then there is a solution of the Skorokhod problem for each $x \in \mathbb{D}_+^d$.
Solutions may not be unique even if R is a $\boldsymbol{P}$-matrix. This non-uniqueness is a problem for "strong" construction of reflecting Brownian motion.

Weak Formulation for Semimartingale Reflecting Brownian Motion (SRBM)
An SRBM associated with R and starting from $x_0 \in \mathbb{R}_+^d$ is a d-diml continuous adapted process Z defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ such that $Z(t) = x_0 + B(t) + RY(t) \in \mathbb{R}_+^d, \forall t \geqslant 0$. where
(i) B is a standard d-diml Brownian motion that is an $\{\mathcal{F}_t\}$-martingale;
(ii) $Y(0) = 0, Y$ is continuous, non-decreasing and $\{\mathcal{F}_t\}$ adapted, $\int_{[0,t]} Z(s)dY(s) = 0$.

Theorem (Reiman-Williams, 87; Taylor-Williams, 93) A necessary condition to have an SRBM (in weak sense) for each $x_0 \in \mathbb{R}_+^d$ is that R is completely-$\boldsymbol{S}$. If R is completely-$\boldsymbol{S}$, then for each $x_0 \in \mathbb{R}_+^d$, there exists an SRBM (in weak sense) and for such an SRBM with associated (B, Y), (Z, B, Y) is unique in law.
(A cool theorem proved by Reiman-Williams: $\int_{[0,t]} 1_{F_i \cap F_j}(Z(s))dY_i(s) = 0$ a.s., $i \neq j, \forall t \geqslant 0$, here $F_i = \{x \in \mathbb{R}_+^d : x_i = 0\}$.)
Example of an SDE which has a weak solution but no strong solution: $dX_t = sgn(X_t)dB_t$

here $sgn(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leqslant 0. \end{cases}$

Oscillation Inequalities (Section 3.5 in Skorokhod problem notes) Suppose R is completely-$\boldsymbol{S}$. Suppose that $\delta \geqslant 0, 0 \leqslant t_1 < t_2 < \infty$ and $z, x, y \in \mathbb{D}([t_1, t_2], \mathbb{R}^d)$ such that
(i) $z(t) = x(t) + Ry(t), t \in [t_1, t_2]$,
(ii) $z(t) \in \mathbb{R}_+^d, t \in [t_1, t_2]$,
(iii) $y(t) \geqslant 0, y$ is non-decreasing, $\int_{(t_1,t_2]} 1_{(\delta,\infty)}(z_i(s))dy_i(s) = 0$.
Then there is a constant $C > 0$ depending only on R such that

$$Osc(y, [t_1, t_2]) \leqslant C(Osc(x, [t_1, t_2]) + \delta)$$

$$Osc(z, [t_1, t_2]) \leqslant C(Osc(x, [t_1, t_2]) + \delta)$$

where, as before.

$$Osc(f, [t_1, t_2]) = sup\{|f(t) - f(s)| : t_1 \leqslant s < t \leqslant t_2\}$$

Can see proof in notes.

Idea for Proof of Existence of SRBM: Suppose set $X(t) = x_0 + B(t)$, and (Z^δ, Y^δ) is obtained by jumping back into the orthant a distance δ whenever a boundary face is reached. From construction, (Z^δ, Y^δ, X) are adapted to filtration generated by X: X is a martingale wrt filtration generated by (Z^δ, Y^δ, X). Let $\delta_n \to 0$ as $n \to \infty$, we can use oscillation inequality to prove that $\{(Z^{\delta_n}, X, Y^{\delta_n})\}_{n=1}^\infty$ is C-tight:

$$Y^{\delta_n}(T) \leqslant Osc(Y^{\delta_n}, [0, T]) \leqslant C(Osc(X, [0, T]) + \delta)$$

Suppose $(\tilde{Z}, \tilde{X}, \tilde{Y})$ is some weak limit point $(Z^\delta \to \tilde{Z}, X \to \tilde{X}, Y^\delta \to \tilde{Y}, \tilde{Y}(0) = 0)$, then $(\tilde{Z}, \tilde{X}, \tilde{Y})$ is an SRBM.

Idea for Proof of Uniqueness of SRBM (Taylor-Williams):
$Z(t) = x_0 + B(t) + RY(t)$. Do proof by induction on dimension. True for $d = 1$. Suppose true for all completely-$\boldsymbol{S}$ R matrices in dimension $< d$. Want to show true for dimension d (reduce to the case where we start from the origin). Use Brownian scaling and ergodicity of process hitting spherical caps of radius 2^{-n} as it comes out of the origin.

What if R depends on position?
Weak Uniqueness is still very open other than when have strong spectral radius type condition. (Shashiashvili, Dupuis-Ishii–case 2)
Existence: via oscillation inequality (Kang-Williams, 2007, The Annals of Applied Probability, Volume 17, 741-779) (piecewise smooth boundries)

R is completely-$\boldsymbol{S} \iff R^T$ is completely-$\boldsymbol{S}$.