Topics in Probability (Math 289A): Lecture 13

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Skorokhod Problem Given \( x \in \mathbb{D}_+^d \), want to find \((y, z) \in \mathbb{D}_+^d \times \mathbb{D}_+^d \) such that \( z(t) = x(t) + Ry(t) \in \mathbb{R}_+^d \ \forall t \geq 0 \) and \( y(0) = 0 \), \( y \) is non-decreasing, \( \int_{[0,\infty)} z(t)dy(t) = 0 \).

If \( R = (v_1 | \cdots | v_d) \) is completely-\( S \), then there is a solution of the Skorokhod problem for each \( x \in \mathbb{D}_+^d \).

Solutions may not be unique even if \( R \) is a \( P \)-matrix. This non-uniqueness is a problem for “strong” construction of reflecting Brownian motion.

Weak Formulation for Semimartingale Reflecting Brownian Motion (SRBM)

An SRBM associated with \( R \) and starting from \( x_0 \in \mathbb{R}_+^d \) is a \( d \)-diml continuous adapted process \( Z \) defined on some filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) such that \( Z(t) = x_0 + B(t) + Ry(t) \in \mathbb{R}_+^d \ \forall t \geq 0 \). where

(i) \( B \) is a standard \( d \)-diml Brownian motion that is an \( \{\mathcal{F}_t\} \)-martingale;

(ii) \( Y(0) = 0 \), \( Y \) is continuous, non-decreasing and \( \{\mathcal{F}_t\} \) adapted, \( \int_{[0,t]} Z(s)dy(s) = 0 \).

Theorem (Reiman-Williams, 87; Taylor-Williams, 93) A necessary condition to have an SRBM (in weak sense) for each \( x_0 \in \mathbb{R}_+^d \) is that \( R \) is completely-\( S \). If \( R \) is completely-\( S \), then for each \( x_0 \in \mathbb{R}_+^d \), there exists an SRBM (in weak sense) and for such an SRBM with associated \((B, Y), (Z, B, Y)\) is unique in law.

( A cool theorem proved by Reiman-Williams: \( \int_{[0,t]} 1_{F_i \cap \mathcal{F}_j} (Z(s))dY_i(s) = 0 \) a.s., \( i \neq j, \forall t > 0 \), here \( F_i = \{x \in \mathbb{R}_+^d : x_i = 0\} \).)

Example of an SDE which has a weak solution but no strong solution: \( dX_t = sgn(X_t)dB_t \)

here \( sgn(x) = \begin{cases} 
1 & \text{if } x > 0, \\
-1 & \text{if } x \leq 0.
\end{cases} \)

Oscillation Inequalities (Section 3.5 in Skorokhod problem notes) Suppose \( R \) is completely-\( S \). Suppose that \( \delta \geq 0, 0 \leq t_1 < t_2 < \infty \) and \( z, x, y \in \mathbb{D}([t_1, t_2], \mathbb{R}^d) \) such that

(i) \( z(t) = x(t) + Ry(t), t \in [t_1, t_2] \),

(ii) \( z(t) \in \mathbb{R}_+^d, t \in [t_1, t_2] \),

(iii) \( y(t) \geq 0, y \) is non-decreasing, \( \int_{(t_1, t_2]} 1_{(\delta, \infty)}(z_i(s))dy_i(s) = 0 \).

Then there is a constant \( C > 0 \) depending only on \( R \) such that

\[
\text{Osc}(y, [t_1, t_2]) \leq C(\text{Osc}(x, [t_1, t_2]) + \delta)
\]
Osc(z,[t_1,t_2]) \leq C(Osc(x,[t_1,t_2]) + \delta)

where, as before.

\[
Osc(f,[t_1,t_2]) = \sup\{|f(t) - f(s)| : t_1 \leq s < t \leq t_2\}
\]

Can see proof in notes.

**Idea for Proof of Existence of SRBM**: Suppose set \(X(t) = x_0 + B(t)\), and \((Z^\delta,Y^\delta)\) is obtained by jumping back into the orthant a distance \(\delta\) whenever a boundary face is reached. From construction, \((Z^\delta,Y^\delta,X)\) are adapted to filtration generated by \(X\): \(X\) is a martingale wrt filtration generated by \((Z^\delta,Y^\delta,X)\). Let \(\delta_n \to 0\) as \(n \to \infty\), we can use oscillation inequality to prove that \(\{(Z^{\delta_n},X,Y^{\delta_n})\}_{n=1}^{\infty}\) is \(C\)-tight:

\[
Y^{\delta_n}(T) \leq Osc(Y^{\delta_n},[0,T]) \leq C(Osc(X,[0,T]) + \delta)
\]

Suppose \((\tilde{Z},\tilde{X},\tilde{Y})\) is some weak limit point \((Z^\delta \to \tilde{Z},X \to \tilde{X},Y^\delta \to \tilde{Y},\tilde{Y}(0) = 0)\), then \((\tilde{Z},\tilde{X},\tilde{Y})\) is an SRBM.

**Idea for Proof of Uniqueness of SRBM** (Taylor-Williams):

\(Z(t) = x_0 + B(t) + RY(t)\). Do proof by induction on dimension. True for \(d = 1\). Suppose true for all completely-\(S\) \(R\) matrices in dimension \(< d\). Want to show true for dimension \(d\) (reduce to the case where we start from the origin). Use Brownian scaling and ergodicity of process hitting spherical caps of radius \(2^{-n}\) as it comes out of the origin.

What if \(R\) depends on position?

Weak Uniqueness is still very open other than when have strong spectral radius type condition. (Shashiashvili, Dupuis-Ishii–case 2)


\(R\) is completely-\(S \iff R^T\) is completely-\(S\).