Topic: Limit Theorems for high dimensional stochastic processes with applications to biological and engineered networks

1. First order approximation (functional law of large numbers): dynamical system; odes

2. Second order approximation (functional central limit theorem): diffusion process (continuous strong Markov process)

3. Stein’s method (stationary distribution)

4. Applications

References

1. Billingsley, Convergence of probability measures

2. Ethier + Kurtz, Markov Processes Characterization and Convergence

3. Jacod + Shiryaev, Limit Theorem for Stochastic Processes
Basic Setting \((\Omega, \mathcal{F}, P)\)

Stochastic Process - Ways to think about them

(i) "Fix \(t\)": \(\{X_t, t \in [0, \infty)\}\) where each \(X_t\) is a random variable, i.e., \(X_t : \Omega \rightarrow \mathbb{R}^d\) (\(d\)-dim Euclidean space) is measurable.

(ii) "Fix \(\omega\)": \(X.(\omega) : [0, \infty) \rightarrow \mathbb{R}^d\) sample path of \(X\). For example,

![Sample path of a stochastic process](image)

(All stochastic processes we consider will have sample paths that are at least right continuous with finite left limits) Then \(X : \Omega \rightarrow \mathbb{D}([0, \infty), \mathbb{R}^d)\). \(X(\omega) = X.(\omega)\) in notation. We use \(C([0, \infty), \mathbb{R}^d)\) to represent the subset of \(\mathbb{D}([0, \infty), \mathbb{R}^d)\) consisting of continuous functions.
Example: Poisson Process

(iii) $X : [0, \infty) \times \Omega \to \mathbb{R}^d$ and $X(t, \omega) = X_t(\omega)$.

**Convergence in function space**

Let $C^d = C([0, \infty), \mathbb{R}^d)$. A sequence $\{x_n\}_{n=1}^\infty$ in $C^d$ converges to $x$ in $C^d$ if for each fixed $T > 0$, $x_n \to x$ as $n \to \infty$ uniformly on $[0, T]$. Define $\|x\|_T = \sup_{t \in [0, T]} |x(t)|$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$ (w.l.o.g assuming $L^2$ norm).

So $x_n \to x$ uniformly on $[0, T] \iff \|x_n - x\|_T \to 0$ as $n \to \infty$. Convergence in $C^d$ means uniform convergence on compact time intervals. Metric on $C^d$ consistent with this is $d(x, y) = \sum_{m=1}^\infty 2^{-m}(\|x - y\|_m \wedge 1)$. $C^d$ with this metric is a complete separable metric space (Polish space).

Let $D^d = D([0, \infty), \mathbb{R}^d)$: let

$\Gamma = \{\gamma : [0, \infty) \to [0, \infty); \gamma \text{ is strictly increasing and continuous}, \gamma(0) = 0, \lim_{t \to \infty} \gamma(t) = \infty\}$.

For example,
A sequence \( \{x_n\} \) in \( \mathbb{D}^d \) converges to \( x \) in \( \mathbb{D}^d \) if for each \( T > 0 \), there is a sequence \( \{\gamma_n\}_{n=1}^{\infty} \) in \( \Gamma \) such that

(i) \( \sup_{t \in [0,T]} |\gamma_n(t) - t| \to 0 \) as \( n \to \infty \) and

(ii) \( \sup_{t \in [0,T]} |x_n(\gamma_n(t)) - x(t)| \to 0 \) as \( n \to \infty \).

**Example:**
From above, we can see it requires the location and size of the jumps in $x_n$’s to both be close to those of $x$. Metric associated with this convergence is called the Skorokhod $J_1$-metric. There is also an $M_1$-metric. The difference is that in $M_1$-metric, $x_n$ can have steps to approximate a big jump of $x$. For example, $x_n$ is as below and $x$ is as above.

**Ascoli-Arzela Theorem**

A sequence $\{x_n\}$ in $C([0,T], \mathbb{R}^d)$ is relatively compact

(i) there is finite $M$ such that $\|x_n\|_T \leq M < \infty$ for all $n$, and
(ii) for each $\epsilon > 0$, $\exists \delta > 0$ such that $|x_n(s) - x_n(t)| < \epsilon$ wherever $|s - t| < \delta$ for all $n$ (equicontinuity).

Equicontinuity $\iff \sup_n w(x_n, \delta, T) = \sup_n \sup_{s, t \in [0, T], |s - t| < \delta} |x_n(s) - x_n(t)| \to 0$
as $\delta \to 0$ where $w(x_n, \delta, T)$ is called the modulus of continuity of $x_n$.

For $D^d$, let

$$w'(x, \delta, T) = \inf_{\text{partition } \{t_i\}_{i=1}^m} \max_{i=1}^m \sup_{s, t \in [t_{i-1}, t_i)} |x(s) - x(t)|$$

where the inf is over all finite partitions $\{t_i\}_{i=0}^m$ of $[0, T]$ for which $\min(t_i - t_{i-1}) > \delta$ for $i = 1, \ldots, m$. Then $w'(x, \delta, T)$ is called the modified modulus of continuity of $x$.

**Proposition 1** A set $A \subset D^d$ is relatively compact (i.e. every sequence has a convergent subsequence) iff for each $T > 0$

(i) $\sup_{x \in A} \|x\|_T < \infty$

(ii) $\lim_{\delta \to 0} \sup_{x \in A} w'(x, \delta, T) = 0$. 

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