Consider the space $C^d := C([0, \infty), \mathbb{R}^d) := \{ f : [0, \infty) \to \mathbb{R}^d \mid f \text{ is continuous } \}$. Then, we can endow $C^d$ with the metric:

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} (||x - y||_n \wedge 1)$$

(1.1)

where $||x||_n = \sup_{t \in [0, n]} |x(t)|$. We can prove that $(C^d, d)$ is a Polish space (metric, separable and complete).

In a similar way, consider $D^d := D([0, \infty), \mathbb{R}^d)$ as the space of functions $f : [0, \infty) \to \mathbb{R}^d$ such that $f$ is right-continuous with left limits. Then, we endow this set with the metric $m_{J_1}$, which induces the Skorohod-$J_1$ topology.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X^n\}_{n \geq 1}$ a sequence of stochastic processes with paths in $D^d$. We consider $D^d$ as the $\sigma$-algebra on $D^d$ associated with the $J_1$-topology. This is the smallest $\sigma$-algebra containing all open sets for the $J_1$-topology.

The fact that $X^n$ is measurable as a map to $D^d ((X^n)^{-1}(A) := \{ w \in \Omega \mid X^n(\omega) \in A \} \in \mathcal{F}$ for all $A \in D^d)$ is equivalent to: $\forall t > 0 \ X^n_t : \Omega \to \mathbb{R}^d$ is measurable and $t \mapsto X^n_t(\omega)$ is in $D^d$ for each $\omega \in \Omega$.

**Definition 1.1 (Law of $X^n$)** Define $\pi_n(A) := P((X^n)^{-1}(A))$ for $A \in D^d$. Then, $\pi_n$ is a probability measure on $(D^d, D^d)$ and it is called the law or distribution of $X^n$. In the same way, we consider $\pi(A) = P(X^{-1}(A))$ for all $A \in D^d$.

**1.1 Convergence in Distribution of $\{X^n\}$ to $X$**

Consider $\{X^n\}_{n \geq 1}$ and $X$ processes with paths in $D^d$.

**Definition 1.2** We will say that $\{X^n\}$ converges in distribution or in law to $X$ (denoted $X^n \Rightarrow X$) if for the corresponding laws $\{\pi_n\}_{n \geq 1}$ and $\pi$, we have that:

$$\pi_n \rightharpoonup \pi \quad \text{as } n \to \infty,$$

(1.2)

which means that: $\forall f : D^d \to \mathbb{R}$ continuous and bounded, we have that $\int_{D^d} f d\pi_n \to \int_{D^d} f d\pi$ as $n \to \infty$.

The set of probability measures on $D^d$ (or $C^d$) with the topology of weak convergence is a Polish space. In other words, there exists a metric on the space that makes it separable, complete and convergence in the metric is weak convergence.

We have other types of convergence for stochastic processes.
Definition 1.3 For \( d \)-dimensional stochastic processes \( \{X^n\}_{n \geq 1} \) and \( X \) defined on \((\Omega, \mathcal{F}, P)\), we will say that \( X^n \) converges to \( X \) almost surely if:

\[
X^n(\omega) \rightarrow X(\omega) \text{ in } D^d \text{ as } n \rightarrow \infty \tag{1.3}
\]

for \( P \)-a.e. \( \omega \in \Omega \). Also, we will say that \( X^n \) converges in probability to \( X \) if for each \( \epsilon > 0 \) we have that:

\[
P(m_{J_1}(X^n, X) > \epsilon) \rightarrow 0 \tag{1.4}
\]
as \( n \rightarrow \infty \).

1.2 Criteria for Weak Convergence

Definition 1.4 A set \( B \) of probability measures on \((D^d, D^d)\) is (weakly) relatively compact if each sequence \( \{\pi_n\}_{n=1}^\infty \) in \( B \) has a weakly convergent subsequence with limit that is a probability measure on \((D^d, D^d)\).

Definition 1.5 A set \( B \) of probability measures on \((D^d, D^d)\) is tight if \( \forall \epsilon > 0 \) there is a compact set \( A \) in \( D^d \) such that:

\[
\pi(A) > 1 - \epsilon \quad \forall \pi \in B. \tag{1.5}
\]

For a sequence of real random variables \( \{X^n\}_{n \geq 1} \), their laws will be tight if \( \forall \epsilon > 0, \exists K_\epsilon > 0 \) such that:

\[
P(|X^n| \leq K_\epsilon) > 1 - \epsilon \tag{1.6}
\]
for all \( n \geq 1 \).

The next theorem relates the previous concepts.

Theorem 1.6 (Prohorov’s Theorem) A set of probability measures \( B \) in \((D^d, D^d)\) is (weakly) relatively compact if and only if it is tight.

The next theorem gives us a very useful method to prove convergence.

Theorem 1.7 Suppose \( \{\pi_n\}_{n=1}^\infty \) is a tight sequence of probability measures on \((D^d, D^d)\). Assume each weakly convergent subsequence of \( \{\pi_n\}_{n=1}^\infty \) has limit \( \pi \). Then, \( \{\pi_n\}_{n=1}^\infty \) converges weakly to \( \pi \).

Proof We argue by contradiction. Suppose \( \{\pi_n\}_{n=1}^\infty \) does not converges weakly to \( \pi \). Then, there must be some \( \epsilon > 0 \) and some subsequence \( \{\pi_{n_k}\}_{k=1}^\infty \) such that:

\[
d(\pi_{n_k}, \pi) > \epsilon \quad \forall k \geq 1 \tag{1.7}
\]

where \( d \) is the metric for the space of probability measures. Notice that \( \{\pi_{n_k}\}_{k=1}^\infty \) is tight, and therefore, has a convergent subsequence \( \{\pi_{n_k}\}_{k=1}^\infty \). But, this limit should be \( \pi \), which contradicts (1.7).
We now look at some criteria for tightness in $\mathbb{D}^d$.

**Theorem 1.8** A sequence of probability measures $\{\pi_n\}_{n=1}^\infty$ on $(\mathbb{D}^d, \mathbb{D}^d)$ is tight if and only if $\forall T > 0, \epsilon > 0$ there exists $K_\epsilon, \delta_\epsilon > 0$ such that:

(i) $\limsup_{n \to \infty} \pi_n(x \in \mathbb{D}^d : ||x||_T \geq K_\epsilon) < \epsilon$

(ii) $\limsup_{n \to \infty} \pi_n(x \in \mathbb{D}^d : w'(x, \delta_\epsilon, T) \geq \epsilon) < \epsilon$

where $w'$ is the **modified modulus of continuity**, defined as:

$$w'(x, \delta, T) = \inf_{\{t_i\}} \max_{i=1,...,m} \sup_{t \in [t_{i-1}, t_i)} |x(s) - x(t)|$$

where $\{t_i\}$ ranges over all partitions of the form $t_0 = 0 < t_1 < \ldots < t_m = T$ where $\min(t_i - t_{i-1}) > \delta$ for $i = 1, \ldots, m$.

**Theorem 1.9 (Skorohod’s Representation Theorem)** Suppose $\{\pi_n\}_{n=1}^\infty$ and $\pi$ are probability measures on $(\mathbb{D}^d, \mathbb{D}^d)$ such that $\pi_n \rightharpoonup \pi$. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which stochastic processes $\{X_n\}_{n=1}^\infty$ and $X$ are defined in such a way that $X^n$ has law $\pi_n$ for each $n \geq 1$, $X$ has law $\pi$, and $X^n \rightharpoonup X$ almost surely as $n \to \infty$.

**Theorem 1.10 (Aldous)** Let $\{X^n\}_{n=1}^\infty$ be a sequence of stochastic processes with paths in $\mathbb{D}^d$. Then, the associated probability measures $\{\pi_n\}_{n=1}^\infty$ are tight if $\forall T > 0$:

(i) $\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P}(||X^n||_T \geq K) = 0$ (compact containment).

(ii) $\forall \epsilon, \eta > 0$ there exists $\delta_{\epsilon, \eta}, n_{\epsilon, \eta} > 0$ such that for all $0 < \delta \leq \delta_{\epsilon, \eta}$ and $n \geq n_{\epsilon, \eta}$,

$$\sup_{\tau_n \in \mathcal{T}^n[0,T]} \mathbb{P}(|X^n(\tau_n + \delta) - X^n(\tau_n)| \geq \epsilon) \leq \eta$$

where $\mathcal{T}^n[0,T]$ are all the finite-valued $X^n$-stopping times, i.e., an element of this set is a stopping time that only takes values in a finite set.

The next theorem includes the case where $X$ has paths in $\mathbb{C}^d$.

**Theorem 1.11** Let $\{X^n\}_{n=1}^\infty$ be processes with paths in $\mathbb{D}^d$. The associated probability measures $\{\pi_n\}_{n=1}^\infty$ will be tight and any limit point will be concentrated on $\mathbb{C}^d$ if and only if $\forall T, \epsilon > 0$:

(i) $\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P}(||X^n||_T \geq K) = 0$ (compact containment).

(ii) $\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}(w(X^n, \delta, T) \geq \epsilon) = 0$. 


where $w$ is the \textbf{modulus of continuity}:

\[
    w(x, \delta, T) = \sup_{s, t \in [0, T], |s - t| < \delta} |x(s) - x(t)|
\]  

(1.10)