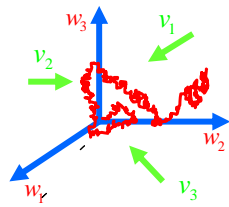
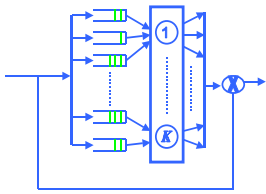


MULTICLASS QUEUEING NETWORKS AND SRBMS IN THE ORTHANT



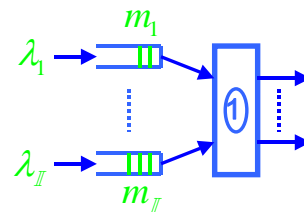
Ruth J. Williams
University of California, San Diego

Outline

- SIMPLE MULTICLASS EXAMPLE
- OPEN MULTICLASS HL NETWORK (CONJECTURES)
- HISTORY
- SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS IN THE ORTHANT
- FURTHER DEVELOPMENTS

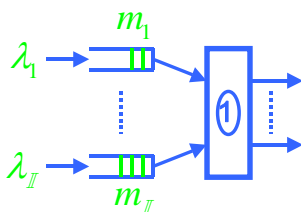
SIMPLE MULTICLASS EXAMPLE

Multiclass FIFO Station



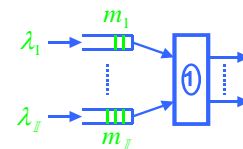
- Renewal arrivals to class i at rate λ_i
- i.i.d. service times for class i , mean m_i
- Service discipline: FIFO across all classes

Performance Processes



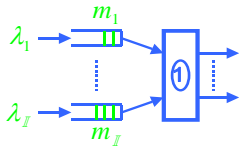
- Queue length for class i : Q_i
- Workload: W
- Idle time: Y

Stability



- Traffic Intensity $\rho_1 = \sum_{i=1}^I \lambda_i m_i$
- Stability iff $\rho_1 < 1$

Stability



•Traffic Intensity $\rho_1 = \sum_{i=1}^I \lambda_i m_i$

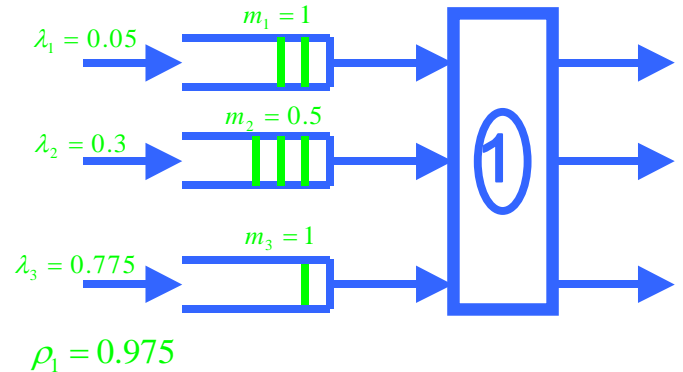
•Stability iff $\rho_1 < 1$

•Heavy traffic $\rho_1 \approx 1$

7

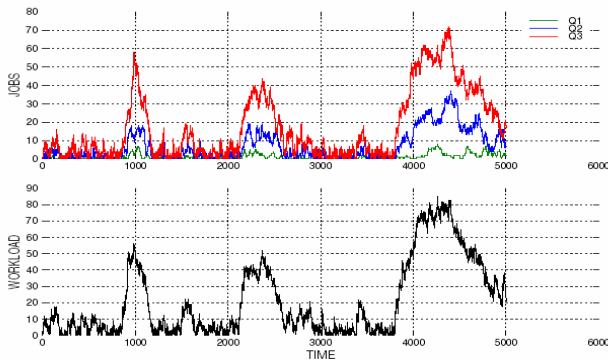
Simulation of a Multiclass FIFO queue

(Poisson arrivals, exponential service times)



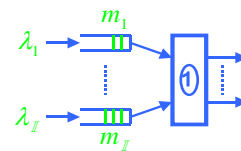
8

Simulation of a Multiclass FIFO queue



9

Stability



•Traffic Intensity $\rho_1 = \sum_{i=1}^I \lambda_i m_i$

•Stability iff $\rho_1 < 1$

•Heavy traffic $\rho_1 \approx 1$ (assume $\rho_1 = 1$ for simplicity)

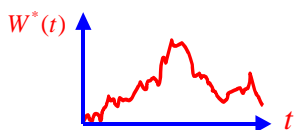
10

Heavy Traffic Diffusion Approximation

$$\hat{W}^r(t) = W(r^2 t) / r, \quad \hat{Y}^r(t) = Y(r^2 t) / r,$$

$$\hat{Q}_i^r(t) = Q_i(r^2 t) / r, \quad i = 1, \dots, I$$

Theorem (Whitt '71) $(\hat{W}^r, \hat{Y}^r, \hat{Q}^r) \Rightarrow (W^*, Y^*, Q^*)$
 where W^* is a one-dimensional reflecting Brownian motion with local time Y^* and $Q^* = \lambda W^*$ (state space collapse).



$$W^*(t) = X^*(t) + Y^*(t)$$

$$Y^*(t) = \sup\{-X^*(s) : 0 \leq s \leq t\}$$

$$X^* = \text{Brownian motion}$$

11

OPEN MULTICLASS

HL NETWORK

(CONJECTURES)

12

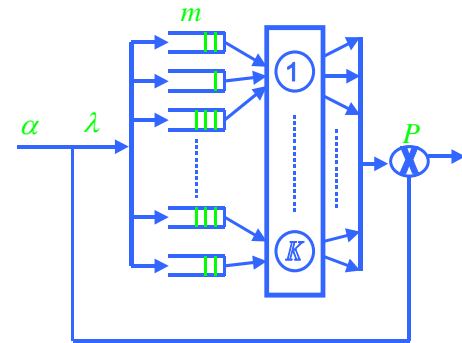
Assumptions

- **Open:** jobs enter the system from outside and eventually leave the network. Assume infinite capacity buffers.
- **HL:** jobs within a buffer are stored in the order in which they arrived and service is always given to the job at the head-of-the-line. Also, the discipline is non-idling.
- Primitive arrival, service and routing processes are assumed to satisfy functional central limit theorems.

13

Open Multiclass HL Queueing Network

First order parameters



$$\lambda = \alpha + P\lambda$$

$$\rho_k = \sum_{i \in k} \lambda_i m_i, \quad k = 1, \dots, K$$

14

Natural Conjectures

- **Stability:** Network is stable provided $\rho_k < 1$ for each $k = 1, \dots, K$
- **Heavy traffic diffusion approximation:** If $\rho_k \approx 1$, $k = 1, \dots, K$, then $(\hat{W}^r, \hat{Y}^r, \hat{Q}^r) \approx (W^*, Y^*, Q^*)$ where $Q^* = \Delta W^*$ for some $\mathbb{R}^{K \times K}$ lifting matrix Δ (that depends on the HL service discipline), and $W^* = X^* + RY^*$ is a reflecting Brownian motion (RBM) in the K -dimensional orthant.

15

HISTORY

16

Affirmative Answers

(Refs. are for diffusion approximations through early 1990s)

- **SINGLE CLASS (FIFO):**
 - Single station: Borovkov ('67), Iglehart-Whitt ('70)
 - Acyclic network: Iglehart-Whitt ('70), Tandem queue: Harrison ('78)
 - Network: Reiman ('84), Chen-Mandelbaum ('91)
- **MULTICLASS:**
 - Single station, priorities: Whitt ('71), Harrison ('73)
 - Network, priorities: Johnson ('83, SP), Peterson ('91, feedforward)
 - Single station, feedback, round robin & FIFO: Reiman ('88), Dai-Kurtz ('95)

Rely on continuous mapping construction of SRBM and do not cover multiclass networks with general feedback.

17

Counterexamples

(two-stations, deterministic routing)

- **STABILITY**
 - Kumar and Seidman ('90): clearing policy.
 - Lu and Kumar ('91): static priorities, deterministic interarrival and service times.
 - Rybko and Stolyar ('92): static priorities, exponential interarrival and service times. (See also Botvitch and Zamyatin ('92))
 - Seidman ('94): FIFO, deterministic interarrival and service times.
 - Bramson ('94): FIFO, exponential interarrival and service times.
- **DIFFUSION APPROXIMATION**
 - Dai-Wang ('93): FIFO, exponential interarrival and service times.

18

HL MQN: Sufficient Conditions

- STABILITY
 - Subcritical fluid models
- PERFORMANCE ANALYSIS (in heavy traffic)
 - Reflecting diffusions and state space collapse via critical fluid models

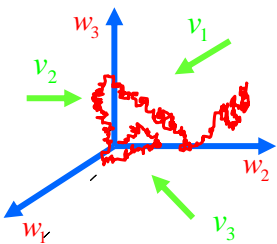
19

SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS (SRBMs)

20

SRBM DATA

- State space: $\mathbb{R}_+^{\mathbb{K}}$
- Brownian statistics: drift θ , covariance matrix Γ
- Reflection matrix: $R = (v_1, \dots, v_{\mathbb{K}})$



21

SRBM DEFINITION (w/starting point x_0)

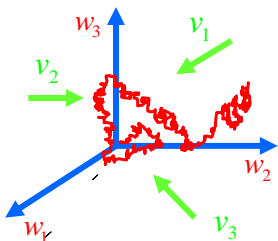
A continuous \mathbb{K} -dimensional process W such that

- (i) $W = X + RY$
- (ii) W has paths in $\mathbb{R}_+^{\mathbb{K}}$
- (iii) for $k=1, \dots, \mathbb{K}$, $Y_k(0) = 0$, Y_k is continuous, non-decreasing, and it can increase only when $W_k = 0$
- (iv) X is a (θ, Γ) BM s.t. $X(0) = x_0$, $\{X(t) - \theta t, t \geq 0\}$ is a martingale relative to the filtration generated by (W, X, Y)

22

Necessary Condition for Existence

Defn: R is completely-S iff for each principal submatrix \tilde{R} of R there is $\tilde{y} > 0: \tilde{R}\tilde{y} > 0$



$$R = (v_1, \dots, v_{\mathbb{K}})$$

23

Existence and Uniqueness in Law

Theorem (Reiman-W '88, Taylor-W '93)

There is an SRBM W starting from each point x_0 in $\mathbb{R}_+^{\mathbb{K}}$ iff R is completely-S. In this case, each such SRBM is unique in law and these laws define a continuous strong Markov process.

24

Oscillation Inequality

Assume that R is completely- S . There is a constant $C > 0$ such that whenever $\delta > 0$, $0 \leq t_1 < t_2 < \infty$, and w, x, y are r.c.l.l. satisfying

- (i) $w(t) = x(t) + Ry(t)$ for $t \in [t_1, t_2]$
- (ii) w lives in \mathbb{R}_+^K
- (iii) for $k=1, \dots, K$, $y_k(t_1) \geq 0$, y_k is continuous, non-decreasing, and can increase only when $w_k < \delta$,

Then

$$\text{Osc}(w, [t_1, t_2]) + \text{Osc}(y, [t_1, t_2]) \leq C(\text{Osc}(x, [t_1, t_2]) + \delta)$$

Cts case: Bernard-El Kharroubi '91, discts case: W '98

25

Analysis of multidimensional SRBMs

■ Sufficient conditions for positive recurrence

Dupuis-W '94, Chen '96, Budhiraja-Dupuis '99, El Kharroubi-Ben Tahar-Yaacoubi '00

■ Stationary distribution

- *Characterization*: Harrison-W '87, Dai-Harrison '92, Dai-Kurtz '98
- *Analytic solutions -two-dimensions*: Foddy '84, Trefethen-W '86, Harrison '06
- *product form*: Harrison-W '87
- *Numerical methods*: Dai-Harrison '91, '92, Shen-Chen-Dai-Dai '02, Schwerer '01

■ Large deviations

Majewski '98, '00, Avram-Dai-Hasenbein '01, Dupuis-Ramanan '02,

26

Some Related Work on RBMs & Queueing Networks

- Capacitated queues (convex polyhedral domains)
 - Dai-Williams '95, Dai-Dai '99
- HT limits that are not SRBMs (& have no state space collapse)
 - Single station-polling: Coffman-Puhalskii-Reiman '95
 - Dynamic HLPS: Dupuis-Ramanan '99, Ramanan-Reiman '03
- Non-HL service disciplines (Markovian state descriptor is typically infinite dimensional)
 - LIFO preemptive resume: Single station: Limic '00, '01
 - Processor sharing: Single station (Gromoll-Puha-W '01, Puha-W '03, Gromoll '03); network (stability: Bramson '04)
 - EDF: Single station (Doytchinov-Lehoczy-Shreve '01), acyclic network (Kruk-Lehoczy-Shreve-Yeung '03), network (stability: Bramson '01)

27

PERSPECTIVE

	MQN	SPN
HL	Sufficient conditions for stability and diffusion approximations	e.g., parallel server system, packet switch
Non-HL	e.g., LIFO, Processor Sharing (single station, PS: network stability)	e.g., Internet congestion control / bandwidth sharing model

28