Stochastic Processing Networks and SRBMs in Domains with Piecewise Smooth Boundaries

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OUTLINE

- Stochastic Processing Networks

- An Invariance Principle for SRBM s in Domains with Piecewise Smooth Boundary
STOCHASTIC PROCESSING NETWORKS
Stochastic Processing Networks (cf. Harrison ’00)

An activity consumes from certain classes, produces for certain (possibly different) classes, and uses certain servers.
Stochastic Processing Networks

*Activities are Very General*

Queueing network

Flexible servers, alternate routing

Simultaneous actions
Data Network (Roberts and Massoulie, ‘00)
SPN with Simultaneous Resource Possession
Diffusion Workload Cone for a 3-node Linear Network under a Fair Bandwidth Sharing Policy
NxN Input Queued Packet Switch: Prabhakar
2x2 Input Queued Packet Switch
Diffusion Workload Cone
(2 by 2 Switch using a Maximum Weight Matching algorithm)
SRBMs in Domains with Piecewise Smooth Boundaries

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Joint work with Weining Kang
Outline

1. Data for an SRBM
2. Definition of an SRBM
3. Assumptions on Data
4. Invariance Principle
5. Applications
6. Open Problems
$G$ a non-empty domain in $\mathbb{R}^d$ with piecewise smooth boundary:

$$G = \bigcap_{i \in \mathcal{I}} G_i,$$

where $\mathcal{I}$ is a finite index set and $G_i \neq \mathbb{R}^d$ is a domain with $C^1$ boundary, $i \in \mathcal{I}$.

- Denote the inward unit normal vector field on $\partial G_i$ by $n^i$, $i \in \mathcal{I}$
- $\gamma^i$ is a uniformly Lipschitz continuous unit length vector field on $\partial G_i$, $i \in \mathcal{I}$
- $\mu \in \mathbb{R}^d$, $\Gamma$ is a symmetric positive definite $d \times d$ matrix
- $\nu$ is a Borel probability measure on $\overline{G}$
SRBM with data \((G, \mu, \Gamma, \{\gamma^i\}, \nu)\)

An adapted, continuous \(d\)-dimensional process \(W\) defined on some filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) such that

(i) \(P\)-a.s., for all \(t \geq 0\), \(W(t) \in \overline{G}\) and
\[
W(t) = W(0) + X(t) + \sum_{i \in \mathcal{I}} \int_{(0,t]} \gamma^i(W(s))dY_i(s),
\]
and under \(P\), \(W(0)\) has distribution \(\nu\),

(ii) under \(P\), \(X\) is a \((\mu, \Gamma)\)-Brownian motion starting from the origin and \(\{X(t) - \mu t, \mathcal{F}_t, t \geq 0\}\) is a martingale,

(iii) for each \(i\), \(Y_i\) is a continuous, increasing adapted, one-dimensional process starting from zero, such that \(P\)-a.s.,
\[
Y_i(t) = \int_{(0,t]} 1\{W(s) \in \partial G_i \cap \partial G\}dY_i(s), \ t \geq 0.
\]
Assumptions on $G$

(A1) For each $\varepsilon \in (0, 1)$ there exists $R(\varepsilon) > 0$ such that for each $i \in \mathcal{I}$, $x \in \partial G_i \cap \partial G$ and $y \in \overline{G}_i \cap \overline{G}$ satisfying $\|x - y\| < R(\varepsilon)$, we have

$$\langle n^i(x), y - x \rangle \geq -\varepsilon \|y - x\|.$$

(A2) $D(r) \to 0$ as $r \to 0$ where

$$D(r) = \sup_{\emptyset \neq \mathcal{J} \subset \mathcal{I}} \sup \left\{ \text{dist} \left( x, \bigcap_{j \in \mathcal{J}} (\partial G_j \cap \partial G) \right) : \text{dist}(x, (\partial G_j \cap \partial G)) \leq r, \text{ for all } j \in \mathcal{J} \right\}.$$
1. $\mathcal{I}(x) = \{i \in \mathcal{I} : x \in \partial G_i\}$ is upper semi-continuous as a function of $x \in \partial G$.

2. If $G$ is bounded or convex, then (A1) holds.

3. If $G$ is bounded or a convex polyhedron, then (A2) holds.
Assumptions on \( \{ \gamma^i \} \)

(A3) There is \( a > 0 \) such that for each \( x \in \partial G \), there are convex combinations \( \gamma(x) \) of \( \gamma^i(x) \) and \( n(x) \) of \( n^i(x) \) for \( i \in I(x) \) such that

(i) \( \langle \gamma(x), n^i(x) \rangle > a \) for all \( i \in I(x) \),

(ii) \( \langle n(x), \gamma^i(x) \rangle > a \) for all \( i \in I(x) \).
Invariance Principle: Informally

Assume (A1)-(A3).

A sequence of processes that satisfies suitably perturbed versions of the SRBM conditions is $C$-tight.

In addition, if uniqueness in law holds for the SRBM, then the sequence of processes converges to the SRBM.

(Formal theorem — Kang-W ’07)
Applications: Existence

- Under (A1)-(A3), there exists an SRBM with the data $(G, \mu, \Gamma, \{\gamma^i\}, \nu)$. 
Applications: Approximation

A process that satisfies a perturbed version of the SRBM conditions is close in distribution to the SRBM under the following condition:

- $G$ is a convex polyhedron with minimal description. For each $i \in \mathcal{I}$, $\gamma^i$ is a constant vector field and $\{\gamma^i\}$ satisfies (A3).

(Uniqueness in law of SRBM holds by Dai-W ’95)
A process that satisfies a perturbed version of the SRBM conditions is close in distribution to the SRBM under the following condition:

- $G$ is a bounded domain with piecewise smooth boundary. The vector fields $\gamma^i, i \in \mathcal{I}$ are continuously differentiable with locally Lipschitz first partial derivatives and there is $a \in (0, 1)$ such that for each $x \in \partial G$ there are non-negative $(b_i(x) : i \in \mathcal{I}(x))$ such that $\sum_{i \in \mathcal{I}(x)} b_i(x) = 1$ and for each $i \in \mathcal{I}(x)$:

  $$b_i(x) \langle n^i(x), \gamma^i(x) \rangle \geq a + \sum_{j \in \mathcal{I}(x) \setminus \{i\}} b_j(x) \langle n^j(x), \gamma^i(x) \rangle.$$  

(Pathwise uniqueness of SRBM holds by Dupuis-Ishii ’93)
Invariance Principle: Hypotheses

Suppose that \( \{\delta^n\}_{n=1}^\infty \) is a sequence of positive constants, and for each positive integer \( n \), \( d \)-dimensional processes \( W^n, X^n, \alpha^n \), and \( I \)-dimensional processes \( Y^n, \beta^n \) are all defined on some probability space \( (\Omega^n, \mathcal{F}^n, P^n) \) such that

(i) for \( \tilde{W}^n \equiv W^n + \alpha^n \), \( P^n \)-a.s.,
\[
\text{dist} \left( \tilde{W}^n(t), G \right) \leq \delta^n \text{ for all } t \geq 0,
\]

(ii) \( P^n \)-a.s.,
\[
W^n(t) = X^n(t) + \sum_{i \in \mathcal{I}} \int_{0}^{t} \gamma^{i,n}(W^n(s-), W^n(s))dY^n_i(s)
\]
for all \( t \geq 0 \), where \( \gamma^{i,n} : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is Borel measurable and \( \| \gamma^{i,n}(y, x) \| = 1 \) for all \( y, x \in \mathbb{R}^d \) and each \( i \in \mathcal{I} \),

(iii) \( X^n \) converges in distribution as \( n \rightarrow \infty \) to a \((\mu, \Gamma)\)-Brownian motion with initial distribution \( \nu \),
(iv) $\beta^n$ is locally of bounded variation and for $\tilde{Y}^n \equiv Y^n + \beta^n$, $P^n$-a.s., for each $i \in I$,

(a) $\tilde{Y}^n_i(0) = 0$,

(b) $\tilde{Y}^n_i$ is increasing and $\Delta \tilde{Y}^n_i(t) \leq \delta^n$ for all $t > 0$,

(c) $\tilde{Y}^n_i(t) = \int_{(0,t]} 1\{\text{dist}(\tilde{W}^n(s), \partial G_i \cap \partial G) \leq \delta^n\} d\tilde{Y}^n_i(s) \quad \forall \ t \geq 0$,

(v) $\delta^n \to 0$ as $n \to \infty$, and for each $\varepsilon > 0$, there is $\eta_\varepsilon > 0$ and $n_\varepsilon > 0$ such that for each $i \in I$, $\|\gamma^{i,n}(y, x) - \gamma^i(x)\| < \varepsilon$ whenever $\|x - y\| < \eta_\varepsilon$ and $n \geq n_\varepsilon$,

(vi) $\alpha^n \to 0$ and $\mathcal{V}(\beta^n) \to 0$ in probability as $n \to \infty$, where for each $t \geq 0$, $\mathcal{V}(\beta^n)(t)$ is the total variation of $\beta^n$ on $[0, t]$. 
Define $Z^n = (W^n, X^n, Y^n)$ for each $n$. The sequence of processes $\{Z^n\}_{n=1}^\infty$ is $C$-tight. Any (weak) limit point of this sequence is of the form $Z = (W, X, Y)$ where all properties of the SRBM Definition hold, except possibly the martingale property, with $\mathcal{F}_t = \sigma\{Z(s) : 0 \leq s \leq t\}, \; t \geq 0$.

Furthermore, if the following conditions (vii) and (viii) hold, then $W^n \Rightarrow W$ as $n \to \infty$ where $W$ is an SRBM.

(vii) For each (weak) limit point $Z = (W, X, Y)$ of $\{Z^n\}_{n=1}^\infty$, 

\[\{X(t) - \mu t, \mathcal{F}_t, t \geq 0\}\] is a martingale.

(viii) If a process $W$ satisfies the SRBM Definition, then the law of $W$ is unique.
Diffusion Workload Cone for a 3-node Linear Network under a Fair Bandwidth Sharing Policy
Diffusion Workload Cone
(2 by 2 Switch using a Maximum Weight Matching algorithm)
Open Problems

- More general sufficient conditions for (weak) uniqueness of SRBMs
- Treatment of domains with cusp-like boundary interfaces
- Treatment of domains with smooth meetings of boundaries
Diffusion Workload Cone for a 3-node Linear Network under another Fair Bandwidth Sharing Policy
Diffusion Workload Cone
(2 by 2 Switch using another Maximum Weight Matching algorithm)