Fluid Approximation for an Internet Congestion Control Model with Fair Bandwidth Sharing and General Document Size Distributions

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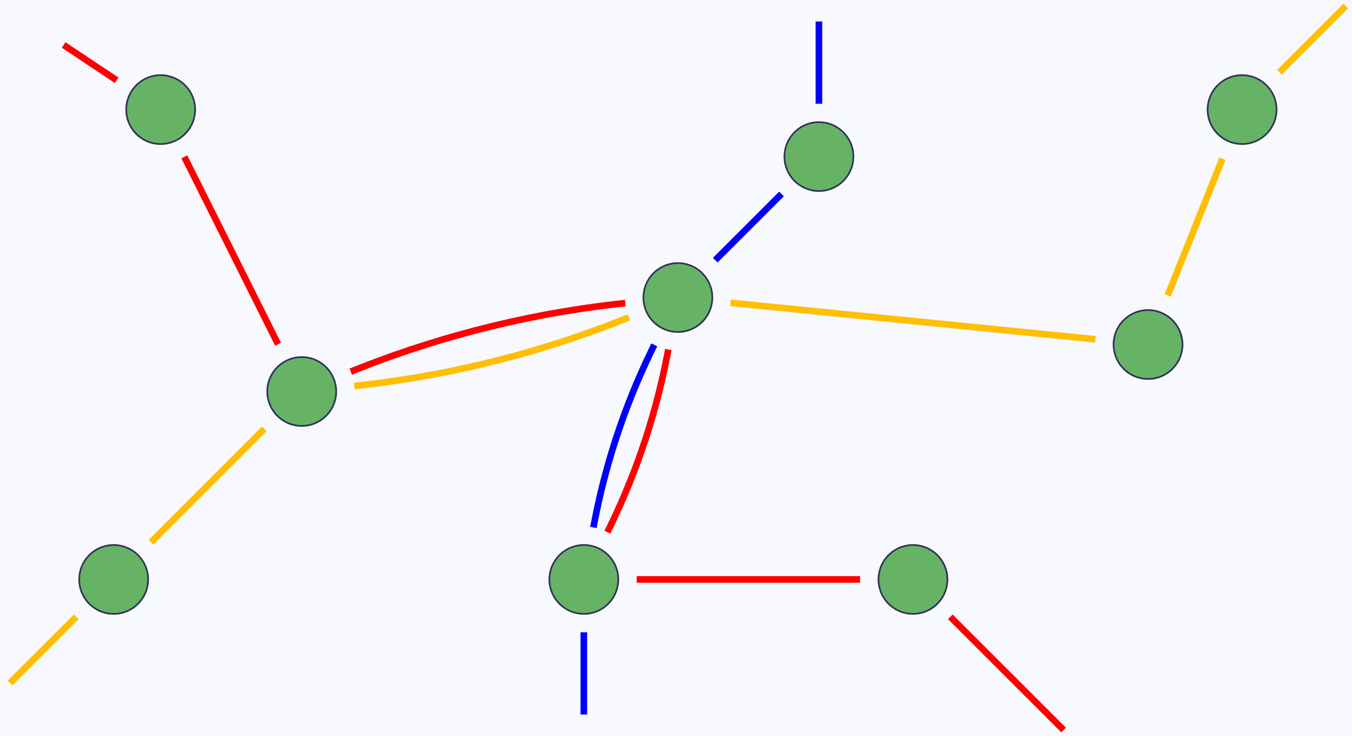
Joint work with H. C. Gromoll

Flow level data network model (Roberts & Massoulié '00)



Link resource or server

Route nonempty subset of links

Flow continuous transfer of document on a route

Simultaneous resource possession

Network structure

J links

I routes

$C_j > 0$ capacity of link j

$J \times I$ incidence matrix A

$$A_{ji} = \begin{cases} 1, & \text{if route } i \text{ uses link } j \\ 0, & \text{else} \end{cases}$$

Stochastic primitives

For each route i

- Renewal arrival process $E_i(\cdot)$ with rate ν_i
  kth document arrives at time τ_{ik}
- i.i.d. document sizes $\{v_{ik}\}_{k=1}^{\infty}$ with distribution ϑ_i, mean μ_i^{-1}, cumulative process $V_i(n) = \sum_{k=1}^{n} v_{ik}$
- Define traffic intensity $\rho_i = \nu_i/\mu_i$

Weighted α-fair bandwidth sharing (Mo & Walrand '00)

$\alpha \in (0, \infty)$, κ_i weight for route i

$\Lambda_i(Z)$ dynamic bandwidth allocation for route i

Z_i number of flows on route i

Bandwidth Λ_i / Z_i provided to each flow on route i

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Bandwidth Λ_i/Z_i provided to each flow on route i

$$\Lambda(Z) = \operatorname{argmax}\{G_Z(\Lambda) : A\Lambda \leq C, \Lambda \geq 0\}$$

where $$G_Z(\Lambda) = \begin{cases} \sum_i \kappa_i Z_i^\alpha \frac{\Lambda_i{}^{1-\alpha}}{1-\alpha}, & \text{for } \alpha \neq 1 \\ \sum_i \kappa_i Z_i \log(\Lambda_i), & \text{for } \alpha = 1 \end{cases}$$

Questions

For the flow level model with *general* interarrival & document size distributions

- Stability when $A\rho < C$?
- Heavy traffic behavior

First step

- Propose fluid model
- Justify approximation via limit theorem
- Analyze fluid model

Literature

Roberts & Massoulié '00

Mo & Walrand '00

de Veciana, Lee & Konstantopoulos '01

Bonald & Massoulié '01

Ye '03

Kelly & W '04 Kang, Kelly, Lee & W '05

Key, Massoulié, Bain & Kelly '03, '05

Ye, Ou & Yuan '05

*Massoulié '06

*Bramson '05

*Lakshmikantha & Srikant '04

*Lin, Shroff & Srikant '06

Outline

- Stochastic model
- Fluid model
- Limit theorem
- Invariant States for Fluid Model
- Fluid Model Stability - Examples

Stochastic model

Performance processes

For each route i

- $Z_i(\cdot)$ queue length process (number of documents)
- $W_i(\cdot)$ workload process

For each link j

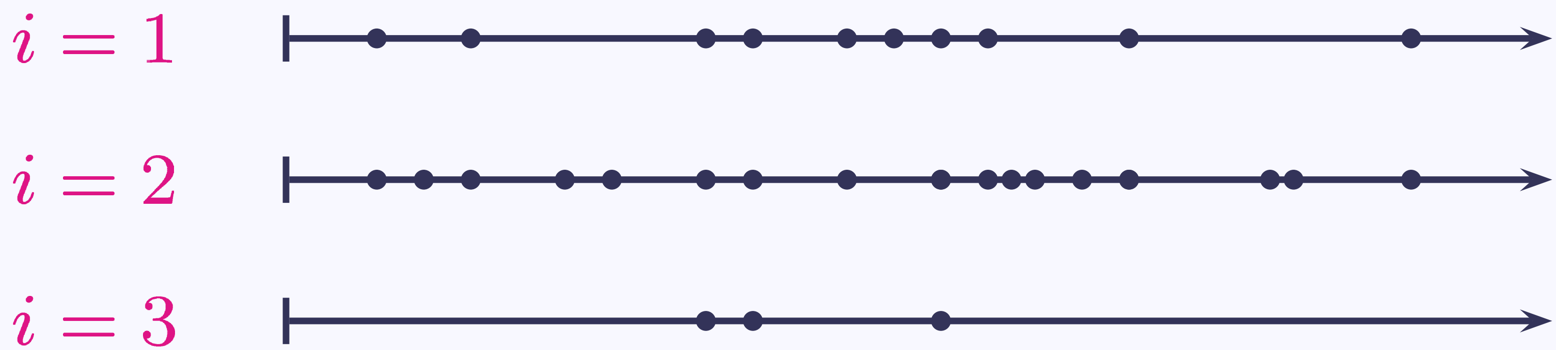
- $U_j(\cdot)$ cumulative unused capacity process for link j

Stochastic model

Measure valued process

For each route i:

- $\mathcal{Z}_i(\cdot)$ residual document size process



Stochastic model

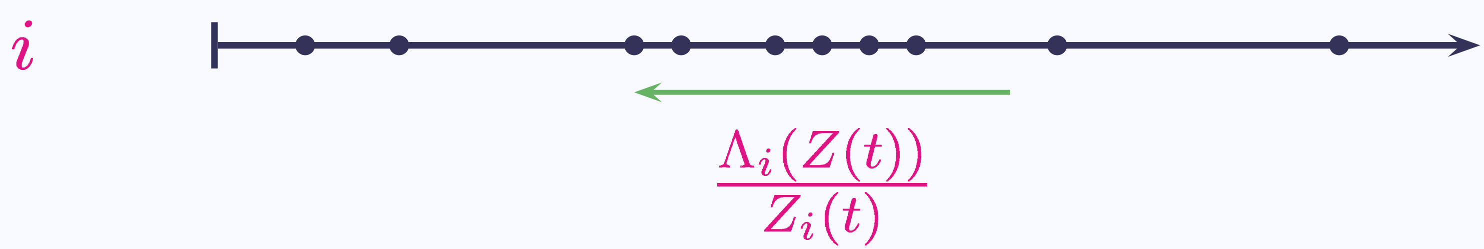
Measure valued process

For each route i:

- $\mathcal{Z}_i(\cdot)$ residual document size process

$$Z_i(t) = \langle 1, \mathcal{Z}_i(t) \rangle,$$

$$W_i(t) = \langle \chi, \mathcal{Z}_i(t) \rangle \text{ where } \chi(x) = x$$



Stochastic model

Dynamic equations

For the vector valued processes

$$W(t) = W(0) + V(E(t)) - T(t)$$

$$U(t) = Ct - AT(t)$$

where $T_i(t) = \int_0^t \Lambda_i(Z(u))du$

Stochastic model

Dynamic equations

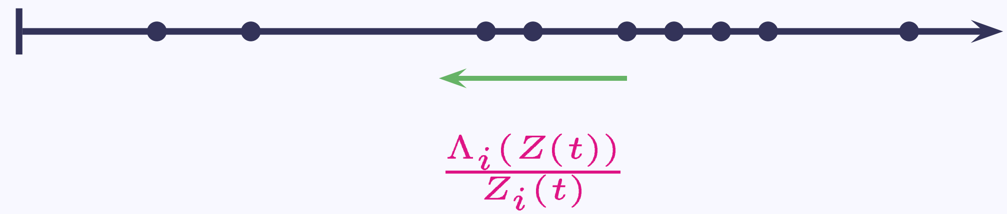
For the measure valued process

Consider projections $\langle g, \mathcal{Z}_i(\cdot) \rangle$ for all $g \in \mathcal{C}$

$$\mathcal{C} = \{g \in \mathbf{C}_b^1 : g(0) = g'(0) = 0\}$$

Stochastic model

Dynamic equations



Route i

$$\langle g, \mathcal{Z}_i(t)\rangle = \langle g(\cdot - S_i(t)), \mathcal{Z}_i(0)\rangle + \sum_{k=1}^{E_i(t)} g\left(v_{ik} - S_i(t) + S_i(\tau_{ik})\right)$$

where $\quad S_i(t) = \int_0^t \frac{\Lambda_i(Z(u))}{Z_i(u)} 1_{(0,\infty)}(Z_i(u))du$

Next. . .

Stochastic model

Fluid model

Limit theorem

Invariant States for Fluid Model

Fluid Model Stability - Examples

Functional law of large numbers approximation

Stochastic model

$$\mathscr{X} = (\mathcal{Z}, Z, W, U)$$

$\downarrow$ scaling limit

Fluid model

$$(\zeta, z, w, u)$$

Fluid model

Network (J, I, A, C) as before

Primitive parameters (ν, ϑ)

For each i

- $\nu_i > 0$
- ϑ_i probability measure on $(0, \infty)$ with mean μ_i^{-1}

State space $\mathbf{M}^I$, where $\mathbf{M}$ is the set of finite non-negative Borel measures on $\mathbb{R}_+$

Performance functions

(ζ, z, w, u)

Fluid model

A *fluid model solution* ζ is a continuous function $\zeta : [0,\infty) \to \mathbf{M}^I$ such that

(i) $\langle 1_{\{0\}}, \zeta_i(t)\rangle = 0$ for all $i \le I$, $t \ge 0$

(ii) For each $g \in \mathcal{C}$, $i \le I$, $t \ge 0$

$$\langle g, \zeta_i(t)\rangle = \langle g, \zeta_i(0)\rangle - \int_0^t \langle g', \zeta_i(u)\rangle \frac{\Lambda_i(z(u))}{z_i(u)} 1_{(0,\infty)}(z_i(u))du + \nu_i \langle g, \vartheta_i\rangle \int_0^t 1_{(0,\infty)}(z_i(u))du$$

(iii) For each $j \le J$

$$u_j(t) = C_j t - \sum_i A_{ji} \int_0^t \{\Lambda_i(z(u)) 1_{(0,\infty)}(z_i(u)) + \rho_i 1_{\{0\}}(z_i(u))\} du$$

is nondecreasing.

Here $z_i(\cdot) = \langle 1, \zeta_i(\cdot)\rangle$ for all $i \le I$

Fluid model

Workload (when finite)

For each $i \leq I$

$$w_i(t) = \langle \chi, \zeta_i(t) \rangle = w_i(0) + \int_0^t \big(\rho_i - \Lambda_i(z(u)\big)\, 1_{(0,\infty)}(z_i(u))du$$

Next. . .

Stochastic model

Fluid model

Limit theorem

Invariant States for Fluid Model

Fluid Model Stability - Examples

Sequence of systems

$\mathscr{X}^r = (\mathscr{Z}^r, Z^r, W^r, U^r)$

Fluid scaling $\bar{\mathscr{X}}^r(t) = \frac{1}{r}\mathscr{X}^r(rt)$

e.g. $\bar{\mathscr{Z}}_i^r(t) = \frac{1}{r}\mathscr{Z}_i^r(rt)$

Asymptotic assumptions

Arrivals

$\nu^r \to \nu$

$\bar{E}^r(\cdot) \Rightarrow \nu(\cdot)$

Document sizes

$\vartheta^r \xrightarrow{\mathrm{w}} \vartheta$

$\limsup_r \langle \chi^{1+\epsilon}, \vartheta^r \rangle < \infty$

($\Rightarrow$ $\mu^r \to \mu$)

Initial condition

$(\bar{\mathcal{Z}}^r(0), \bar{W}^r(0)) \Rightarrow (\mathcal{Z}^0, \langle \chi, \mathcal{Z}^0 \rangle) \in \mathbf{M}^I \times \mathbb{R}_+^I$

$E[\langle 1, \mathcal{Z}^0 \rangle] < \infty$, $E[\langle \chi, \mathcal{Z}^0 \rangle] < \infty$, $\langle 1_{\{x\}}, \mathcal{Z}^0 \rangle = 0 \ \forall x \in \mathbb{R}_+$

Limit theorem

Theorem (Gromoll-W) The sequence $\{\bar{\mathscr{X}}^r\}$ is **C**-tight; each weak limit point $\bar{\mathscr{X}} = (\bar{\mathscr{Z}}, \bar{Z}, \bar{W}, \bar{U})$ a.s. yields a fluid model solution $\bar{\mathscr{Z}}$ with associated queuelength $\bar{Z} = \langle 1, \bar{\mathscr{Z}} \rangle$, (finite) workload $\bar{W} = \langle \chi, \bar{\mathscr{Z}} \rangle$, and unused capacity $\bar{U}$.

Next. . .

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Invariant States for Fluid Model

For each $i \leq I$, let ϑ_i^e denote the excess lifetime probability measure associated with ϑ_i, i.e., ϑ_i^e has probability density $p_i^e(x) = \mu_i \langle 1_{(x,\infty)}, \vartheta_i \rangle$, $x \in \mathbb{R}_+$.

Theorem (Gromoll-W) There are no invariant states unless

$$\sum_{i \leq I} A_{ji} \rho_i \leq C_j \quad \text{for all } j \leq J.$$

When the above holds, $\xi \in \mathbf{M}^I$ is invariant if and only if $\xi_i = z_i \vartheta_i^e$ for all $i \leq I$, for some $z \in \mathbb{R}_+^I$ satisfying $\Lambda_i(z) = \rho_i$ for all i such that $z_i > 0$.

Next . . .

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Fluid Model Stability - Examples

Linear network

A linear network consists of J links and $I = J + 1$ routes where route j consists of link j alone for $j = 1, \ldots, J$ and route $J + 1$ consists of all of the J links.

A linear network with 3 links and 4 routes

Linear network

Theorem Consider a linear network and a fluid model solution ζ with finite initial workload $w(0) = \langle \chi, \zeta(0) \rangle$. Suppose that $\varepsilon = C - A\rho > 0$. Then

$$\zeta(t) = 0, \qquad \text{for all } t \geq \delta,$$

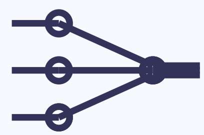
where $\delta = \max_{j \leq J}(w_j(0) + w_{J+1}(0))/\varepsilon$.

Lyapunov function

$$H(w) = \max_{j \leq J}(w_j + w_{J+1})$$

Tree network

A tree network (cf. Bonald and Proutière (2003)) consists of $J \geq 2$ links and $I = J - 1$ routes such that a single link (labeled J and referred to as the trunk) belongs to all routes and each of the other links (labeled by $1, \ldots, J-1$) belongs to a single route.



A tree network with 4 links and 3 routes

Tree network

Theorem Consider a tree network and a fluid model solution ζ with finite initial workload $w(0) = \langle \chi, \zeta(0) \rangle$. Suppose that $\varepsilon = C - A\rho > 0$. Then

$$\zeta(t) = 0, \qquad \text{for all } t \geq \delta,$$

where $\delta = \sum_{i \leq J-1} w_i(0)/\varepsilon$.

Lyapunov function

$$H(w) = \sum_{i \leq J-1} w_i$$