THE MATHEMATICS OF QUEUEING

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QUEUES

Photograph courtesy of Ilze Ziedins
EXAMPLES OF QUEUES

• Customers waiting in supermarket lines
• Cars waiting at a traffic light
• Patients waiting in an emergency department
• Clients waiting for service by a telephone agent
• Parts waiting to be assembled into finished products
• Packets waiting to be transmitted through a router in the Internet
A SIMPLE QUEUEING EXAMPLE
A SINGLE SERVER QUEUE

- Green bars: “jobs”
- Red disc: server
RELIABLE SERVER ASSUMPTION

- The server can process one “job” per minute
ARRIVALS: SCENARIO 1

- One job arrives at the beginning of each minute
- Balanced deterministic system: no queue will build up
ARRIVALS: SCENARIO 2

• At the beginning of each minute, either zero or two jobs arrive (equal probability for each)

• Random arrivals: simulate using coin flips

• Average # of arrivals per min = 1
A SAMPLE REALIZATION AFTER 10 MINUTES
AFTER 100 MINUTES

TIME

JOBS
AFTER 10,000 MINUTES
MATHEMATICAL ANALYSIS

Queue length

$Q(n) =$ number of jobs in the system after $n$ mins
MATHEMATICAL ANALYSIS

Queue length

\[ Q(n) = \text{number of jobs in the system after } n \text{ mins} \]

Average after \( n \) minutes

\[ A(n) = \frac{Q(1) + Q(2) + \cdots + Q(n)}{n} \]
MATHEMATICAL ANALYSIS

Queue length

\( Q(n) \) = number of jobs in the system after \( n \) mins

Average after \( n \) minutes

\[
A(n) = \frac{Q(1) + Q(2) + \cdots + Q(n)}{n}
\]

**THEOREM:** \( A(n) \to \infty \) as \( n \to \infty \)
SUBCRITICAL ARRIVALS: SCENARIO 3

- Fix $0 < \varepsilon < 0.5$
- At the beginning of each minute,
  - two jobs arrive with probability $\frac{1}{2} - \varepsilon$ and
  - zero jobs arrive with probability $\frac{1}{2} + \varepsilon$

- Average # of arrivals per min $= 1 - 2\varepsilon$
SUBCRITICAL ARRIVALS

Average after \( n \) mins: \( A(n) = \frac{Q(1) + Q(2) + \cdots + Q(n)}{n} \)

**THEOREM**

\[
A(n) \rightarrow \tilde{A} = \frac{1}{4\epsilon} - \frac{1}{2} \quad \text{as } n \rightarrow \infty
\]
SUBCRITICAL ARRIVALS

Average after $n$ mins: $A(n) = \frac{Q(1) + Q(2) + \cdots + Q(n)}{n}$

THEOREM

$A(n) \rightarrow \widetilde{A} = \frac{1}{4\epsilon} - \frac{1}{2}$ as $n \rightarrow \infty$

Example: $\epsilon = 0.05$
Average queue length (in long run) = 4.5
Associated average wait = 10 minutes
SIMULATION FOR 10,000 MINUTES

$\varepsilon = 0.05 \quad \text{Average queue length from simulation} = 4.13$
NETWORKS OF QUEUES

Analysis and control of networks of queues is a challenging mathematical problem with applications to

- manufacturing systems
- telecommunication networks
- transportation networks
- service networks
- biological networks
SEMICONDUCTOR MANUFACTURING

![Diagram of semiconductor manufacturing process](Image)

*Courtesy of P. R. Kumar*
TRANSPORTATION NETWORKS

http://www.highways.gov.uk/
CALL CENTERS

First Direct (branchless retail banking) Larreche et al., INSEAD
BIOLOGICAL NETWORKS
THANK YOU